Discussion Hour: ODEs and State space

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ECE 171A

Problem Statement 1

Consider the non-homogeneous first-order linear ODE:

$$\dot{y}(t) + 2y(t) = 4e^{-t}, \quad y(0) = 1$$

Solve this ODE using the method of integrating factors.

Solution

Step 1: Rewrite the Equation

The given ODE is already in the form:

$$\dot{y}(t) + 2y(t) = 4e^{-t}$$

Step 2: Find the Integrating Factor

The integrating factor $\mu(t)$ is given by:

$$\mu(t) = e^{\int 2\,dt} = e^{2t}$$

Step 3: Multiply Both Sides by the Integrating Factor

Multiply the ODE by the integrating factor:

$$e^{2t}\dot{y}(t) + 2e^{2t}y(t) = 4e^{2t}e^{-t}$$

Simplifying:

$$e^{2t}\dot{y}(t) + 2e^{2t}y(t) = 4e^{t}$$

Step 4: Recognize the Left-Hand Side as a Product Derivative

The left-hand side can be written as:

$$\frac{d}{dt}\left(e^{2t}y(t)\right) = 4e^{t}$$

Step 5: Integrate Both Sides

Integrate both sides with respect to t:

$$e^{2t}y(t) = \int 4e^{t}\,dt$$

Step 6: Solve the Integral

The integral of $4e^t$ is:

$$\int 4e^t \, dt = 4e^t + C$$

Thus:

$$e^{2t}y(t) = 4e^t + C$$

Step 7: Solve for $y(t)$

Divide both sides by e^{2t}:

$$y(t) = \frac{4e^t + C}{e^{2t}} = 4e^{-t} + Ce^{-2t}$$

Step 8: Apply the Initial Condition

Using the initial condition $y(0) = 1$:

$$1 = 4e^0 + Ce^0 = 4 + C$$

Therefore, $C = 1 - 4 = -3$.

Final Solution

The solution to the ODE is:

$$y(t) = 4e^{-t} - 3e^{-2t}$$

MATLAB Solution

To verify our solution, we can solve the ODE numerically using MATLAB's `ode45` function. Below is the MATLAB code used to solve and plot the solution.

```
% MATLAB Code to Solve the ODE using ode45
% The ODE: dy/dt + 2*y = 4*exp(-t), with the initial condition y(0) = 1.

% Define the ODE as an anonymous function
dydt = @(t, y) 4 * exp(-t) - 2 * y;  % ODE in the form of dy/dt = 4*exp(-t) - 2*y

% Initial condition
y0 = 1;  % y(0) = 1

% Time span for the solution
tspan = [0 10];  % Solve from t = 0 to t = 10

% Solve the ODE using ode45, a versatile ODE solver in MATLAB
[t, y] = ode45(dydt, tspan, y0);

% Plot the solution
figure;  % Create a new figure
plot(t, y, 'b-', 'LineWidth', 2);  % Plot the numerical solution with a blue line

% Add labels and title to the plot
xlabel('Time t');  % Label for x-axis
ylabel('y(t)');  % Label for y-axis
title('Solution of First-Order ODE');  % Plot title
```

```
grid on;
legend('y(t) (Numerical Solution)', 'Location', 'best');  % Add legend
set(gca, 'FontSize', 12);  % Set font size for readability

% Optional: Display results in command window
disp('Time values:');
disp(t);
disp('Solution y(t):');
disp(y);
```

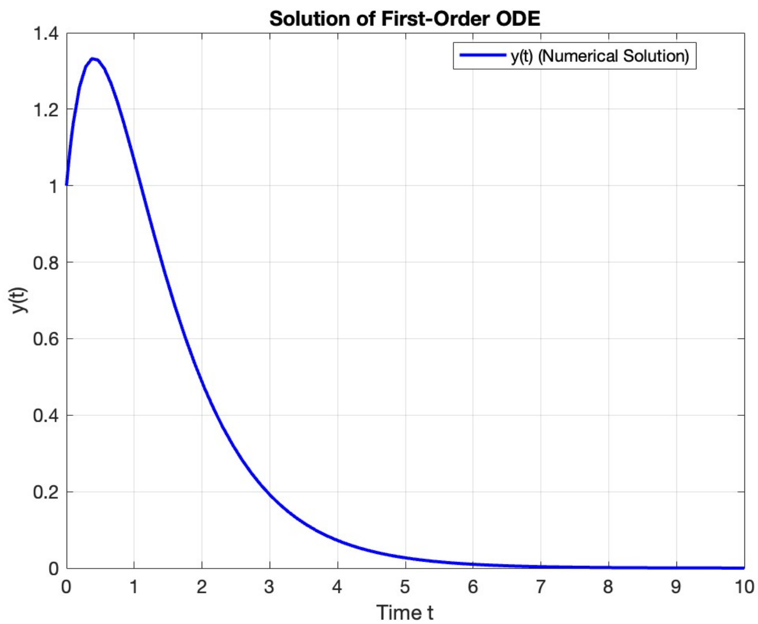

Figure 1: Numerical solution of the ODE using MATLAB.

Conclusion

We have solved the first-order non-homogeneous ODE both analytically and numerically using MATLAB. The analytical solution is:

$$y(t) = 4e^{-t} - 3e^{-2t}$$

The MATLAB solution confirms the accuracy of our analytical solution.

Problem Statement 2

Consider the following second-order linear time-invariant (LTI) system:

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = u(t)$$

1. Convert the above second-order ODE into a state-space representation. 2. Define appropriate state variables and write the state-space equations in the form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

3. Identify the matrices A, B, C, and D.

Solution

Step 1: Define State Variables

Define the state variables:

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

Step 2: Write State Equations

Rewrite the second-order ODE in terms of the state variables:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -4x_2(t) - 3x_1(t) + u(t)$$

Step 3: Write in State-Space Form

The state-space model can be written in matrix form as:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Final State-Space Equations

The state-space equations are:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Application Problem 3: RLC Circuit

Consider an RLC circuit with a resistor (R), inductor (L), and capacitor (C) connected in series. The governing differential equation for the current $i(t)$ is given by:

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = V_s(t)$$

where $V_s(t)$ is the source voltage applied to the circuit. Assume $L = 1\,H$, $R = 2\,\Omega$, $C = 0.5\,F$, and $V_s(t) = 10e^{-t}$. The initial conditions are $i(0) = 0$ and $\dot{i}(0) = 0$.

Problem Statement

1. Convert the second-order ODE into a state-space representation. 2. Define appropriate state variables and write the state-space equations in the form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

3. Identify the matrices A, B, C, and D.

Solution

Step 1: Define State Variables

Define the state variables:

$$x_1(t) = i(t)$$

$$x_2(t) = \frac{di(t)}{dt}$$

Step 2: Write State Equations

Rewrite the second-order ODE in terms of the state variables:

$$\dot{x}_1(t) = x_2(t)$$

$$L\dot{x}_2(t) = V_s(t) - Rx_2(t) - \frac{1}{C}x_1(t)$$

Substituting $L = 1$, $R = 2$, and $C = 0.5$:

$$\dot{x}_2(t) = 10e^{-t} - 2x_2(t) - 2x_1(t)$$

Step 3: Write in State-Space Form

The state-space model can be written in matrix form as:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Final State-Space Equations

The state-space equations are:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$