ECE171A: Linear Control System Theory
Lecture 10: Frequency Response

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Frequency Response

- Consider a control system with input $R(s)$, output $Y(s)$, and transfer function $G(s)$
- Consider an exponential test signal defined by $s_0 = \sigma + j\omega$:

$$R(s) = \frac{1}{s - s_0} \qquad r(t) = \mathcal{L}^{-1}\left\{\frac{1}{s - s_0}\right\} = e^{s_0 t}, \quad t \geq 0$$

- The system response is:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{G(s)R(s)\} = (g * r)(t) &= \int_0^t g(\tau) r(t - \tau) d\tau \\ &= e^{s_0 t}\left[\int_0^t g(\tau) e^{-s_0 \tau} d\tau\right] \end{aligned}$$

- As $t \to \infty$, $\left[\int_0^t g(\tau) e^{-s_0 \tau} d\tau\right] \to \mathcal{L}\{g(t)\} = G(s_0)$
- The steady-state response to $r(t) = e^{s_0 t}$ is:

$$y_{ss}(t) = G(s_0) e^{s_0 t}$$

Frequency Response

- Applying the test signal $r(t) = e^{s_0 t}$ for different s_0 gives us a way to identify the transfer function $G(s)$ of an unknown system using the steady-state response:

$$y_{ss}(t) = G(s_0)e^{s_0 t}$$

- How do we apply $r(t) = e^{s_0 t}$ in practice?

- If $\text{Re}(s_0) > 0$ or $\text{Re}(s_0) < 0$, the system response either blows up or decays very quickly.

- Consider $s_0 = j\omega$ and recall that $\sin(\omega t) = \text{Im}(e^{j\omega t}) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$

- By superposition, the steady-state response to $r(t) = \sin(\omega t)$ is:

$$\begin{aligned} y_{ss}(t) &= \frac{1}{2j}G(j\omega)e^{j\omega t} - \frac{1}{2j}G(j\omega)e^{-j\omega t} \\ &= |G(j\omega)|e^{j\angle G(j\omega)}\frac{e^{j\omega t} - e^{-j\omega t}}{2j} = |G(j\omega)|\sin\left(\omega t + \underline{/G(j\omega)}\right) \end{aligned}$$

Frequency Response

- The **frequency response** is the steady-state output of a system with sinusoidal input
- The frequency response of a system with transfer function $T(s)$ to a reference signal $r(t) = \sin(\omega t)$ is a sinusoid, scaled by $|T(j\omega)|$ and phase-shifted by $\angle T(j\omega)$:

$$y_{ss}(t) = |T(j\omega)| \sin\left(\omega t + \angle T(j\omega)\right)$$

- The experimental determination of the frequency response is often easily accomplished due to the ready availability of sinusoidal test signals for various frequency ranges

Frequency Domain Plots

- Plotting the magnitude and phase of $T(j\omega)$ provides insight into the analysis and design of linear control systems

- The following frequency-domain plots of the transfer function are used:
  - **Polar plot**: a plot of $\mathrm{Im}(T(j\omega))$ versus $\mathrm{Re}(T(j\omega))$ of a transfer function $T(j\omega)$ as ω varies from 0 to ∞
  - **Magnitude-phase plot**: a plot of the log-magnitude $20\log_{10}|T(j\omega)|$ in decibels (dB) versus the phase $\angle T(j\omega)$ as ω varies from 0 to ∞
  - **Bode plot**: a plot of the log-magnitude $20\log_{10}|T(j\omega)|$ in decibels (dB) and the phase $\angle T(j\omega)$ versus $\log_{10}\omega$ as ω varies from 0 to ∞

Log-scale Units

- **Bel**: a relative measurement unit of the log-ratio of measured power P to reference power P_0

$$\text{Log-power ratio} = \log_{10}\left(\frac{P}{P_0}\right) \text{ Bels}$$

- **Decibel**: ten Bels:

$$\text{Log-power ratio} = 10\log_{10}\left(\frac{P}{P_0}\right) \text{ dB}$$

- The input-output power spectral density relationship for a linear time-invariant system with input $R(s)$, transfer function $T(s)$, and output $Y(s)$ is:

$$S_Y(\omega) = |T(j\omega)|^2 S_R(\omega)$$

- The log-power ratio at ω in dB is:

$$10\log_{10}\left(\frac{S_Y(\omega)}{S_R(\omega)}\right) = 10\log_{10}|T(j\omega)|^2 = 20\log_{10}|T(j\omega)|$$

Log-scale Units

- **Bode plot**: the magnitude $20\log_{10}|T(j\omega)|$ in dB and phase $\underline{/T(j\omega)}$ in radians of a transfer function $T(s)$ are plotted versus $\log_{10}\omega$

- The intervals on a log scale are known as decades (base 10) or octaves (base 2):
  - The number of **decades** between ω_1 and ω_2 is $\log_{10}\frac{\omega_2}{\omega_1}$
  - The number of **octaves** between ω_1 and ω_2 is $\log_2\frac{\omega_2}{\omega_1}$
  - There are $\log_2(10) \approx 3.32$ octaves in one decade
  - A slope of 20 dB/decade is the same as $\frac{20 \text{ dB/decade}}{\log_2(10) \text{ octave/decade}} \approx 6$ dB/octave

Transfer Function in Bode Form

- **Transfer function in Bode form**: a transfer function with m_1 real zeros, m_2 complex conjugate zero pairs, n_0 poles at the origin, n_1 real poles, and n_2 complex conjugate pole pairs:

$$T(s) = \kappa \frac{\prod_{i=1}^{m_1} \left(\frac{s}{z_i} + 1\right) \prod_{l=1}^{m_2} \left(\left(\frac{s}{\omega_{n_l}}\right)^2 + 2\zeta_l \left(\frac{s}{\omega_{n_l}}\right) + 1\right)}{s^{n_0} \prod_{i=1}^{n_1} \left(\frac{s}{p_i} + 1\right) \prod_{k=1}^{n_2} \left(\left(\frac{s}{\omega_{n_k}}\right)^2 + 2\zeta_k \left(\frac{s}{\omega_{n_k}}\right) + 1\right)}$$

- The magnitude and phase of $T(j\omega)$ are needed to draw a Bode plot

Magnitude and Phase of $T(j\omega)$

- Magnitude of $T(j\omega)$ in decibels (dB):

$$
\begin{aligned}
&20\log|T(j\omega)| \\
&\quad = 20\log|\kappa| + \sum_{i=1}^{m_1} 20\log\left|j\frac{\omega}{z_i}+1\right| + \sum_{l=1}^{m_2} 20\log\left|\left(\frac{j\omega}{\omega_{n_l}}\right)^2 + 2\zeta_l\left(\frac{j\omega}{\omega_{n_l}}\right)+1\right| \\
&\quad - 20\log|(j\omega)^{n_0}| - \sum_{i=1}^{n_1} 20\log\left|j\frac{\omega}{p_i}+1\right| - \sum_{k=1}^{n_2} 20\log\left|\left(\frac{j\omega}{\omega_{n_k}}\right)^2 + 2\zeta_k\left(\frac{j\omega}{\omega_{n_k}}\right)+1\right|
\end{aligned}
$$

- Phase of $T(j\omega)$ in radians:

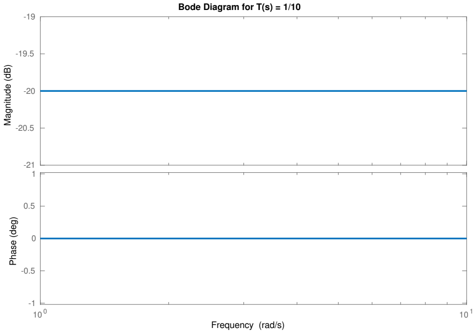
$$
\begin{aligned}
\angle T(j\omega) = \angle\kappa &+ \sum_{i=1}^{m_1}\tan^{-1}\left(\frac{\omega}{z_i}\right) + \sum_{l=1}^{m_2}\tan^{-1}\left(\frac{2\zeta_l\omega_{n_l}\omega}{\omega_{n_l}^2-\omega^2}\right) \\
&- n_0\frac{\pi}{2} - \sum_{i=1}^{n_1}\tan^{-1}\left(\frac{\omega}{p_i}\right) - \sum_{k=1}^{n_2}\tan^{-1}\left(\frac{2\zeta_k\omega_{n_k}\omega}{\omega_{n_k}^2-\omega^2}\right)
\end{aligned}
$$

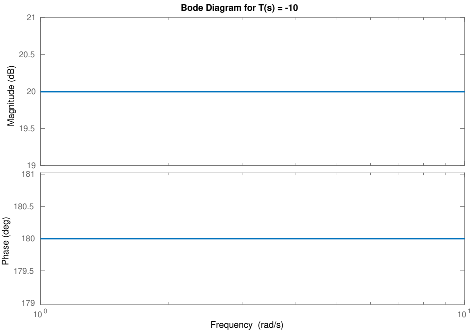
Drawing Bode Plots

- Instead of computing the magnitude and phase of $T(j\omega)$ directly, it is preferable to obtain general rules for drawing Bode plots
- A transfer function may contain only four kinds of factors:
  - Constant terms: κ
  - Poles or zeros at the origin: s^q
  - Real poles or zeros: $\left(\frac{s}{p}+1\right)^{-1}$ or $\left(\frac{s}{z}+1\right)$
  - Complex conjugate poles or zeros: $\left(\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1\right)^{\pm 1}$
- If we determine the magnitude and phase plots for these four factors, then we can add them together graphically to obtain a Bode plot for any transfer function

Bode Plot for a Constant Term κ

- **Magnitude**: $20\log|\kappa|$
- **Phase**: $\angle\kappa = \begin{cases} 0 & \text{if } \kappa > 0 \\ \pi & \text{if } \kappa < 0 \end{cases}$
- Example: Bode plot for $T(s) = \frac{1}{10}$ and $T(s) = -10$





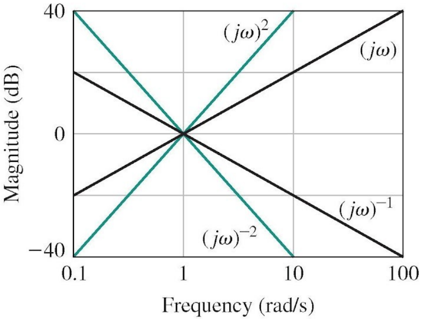
Bode Plot for Pole or Zero at the Origin: s^q

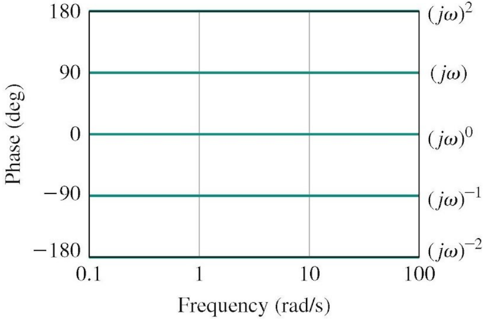
- **Magnitude**: a straight line on a log scale going through the origin with slope $20q$:

$$20 \log |(j\omega)^q| = 20q \log |\omega|$$

- **Phase**: a horizontal line at $q\frac{\pi}{2}$:

$$\underline{/(j\omega)^q} = q\underline{/(j\omega)} = q\frac{\pi}{2}$$





Bode Plot for Real Zero $\left(\frac{s}{z}+1\right)$

- **Magnitude**: $20\log\left|j\frac{\omega}{z}+1\right| = 20\log\sqrt{1+\left(\frac{\omega}{z}\right)^2}$
- **Phase**: $\angle\left(j\frac{\omega}{z}+1\right) = \tan^{-1}\frac{\omega}{z}$
- Extreme ω values:
  - **Case 1**: $\omega \ll z$: horizontal line at 0:

$$20\log\left|j\frac{\omega}{z}+1\right| \approx 0 \qquad \angle\left(j\frac{\omega}{z}+1\right) \approx 0$$

  - **Case 2**: $\omega \gg z$: log-scale line of slope 20 going through 0 when $\omega = z$:

$$20\log\left|j\frac{\omega}{z}+1\right| \approx 20\log\frac{1}{z} + 20\log\omega \qquad \angle\left(j\frac{\omega}{z}+1\right) \approx \frac{\pi}{2}$$

  - **Case 3**: $\omega = z$ (**corner frequency**):

$$20\log\left|j\frac{\omega}{z}+1\right| \approx 3dB \qquad \angle\left(j\frac{\omega}{z}+1\right) = \frac{\pi}{4}$$

Bode Plot for Real Pole $\left(\frac{s}{p}+1\right)^{-1}$

- **Magnitude**: $20\log\left|\left(j\frac{\omega}{p}+1\right)^{-1}\right| = -20\log\sqrt{1+\left(\frac{\omega}{p}\right)^2}$
- **Phase**: $\angle\left(j\frac{\omega}{p}+1\right)^{-1} = -\tan^{-1}\frac{\omega}{p}$
- Extreme ω values:
  - **Case 1**: $\omega \ll p$: horizontal line at 0:

$$-20\log\left|j\frac{\omega}{p}+1\right| \approx 0 \qquad \angle\left(j\frac{\omega}{p}+1\right)^{-1} \approx 0$$

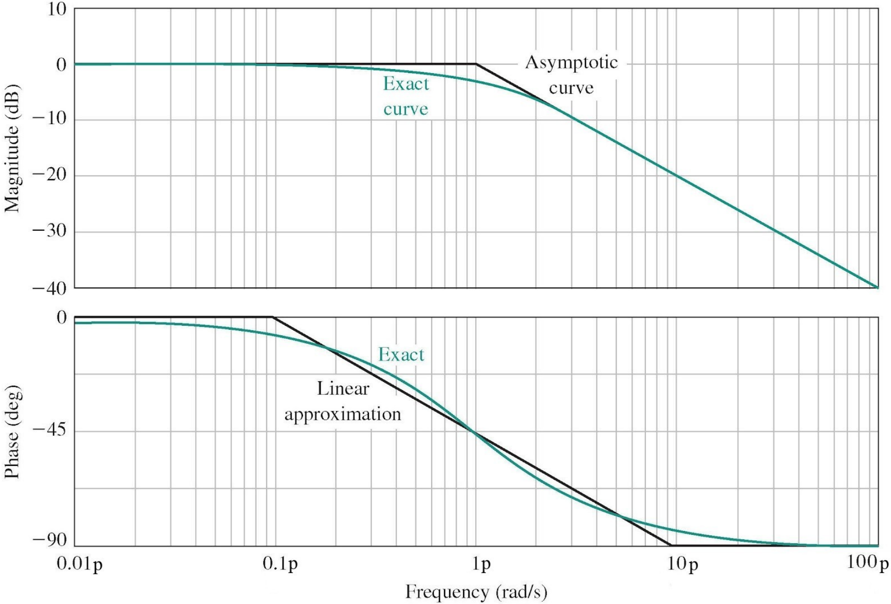
  - **Case 2**: $\omega \gg p$: log-scale line of slope -20 going through 0 when $\omega = p$:

$$-20\log\left|j\frac{\omega}{p}+1\right| \approx -20\log\frac{1}{p} - 20\log\omega \qquad \angle\left(j\frac{\omega}{p}+1\right)^{-1} \approx -\frac{\pi}{2}$$

  - **Case 3**: $\omega = p$ (**corner frequency**):

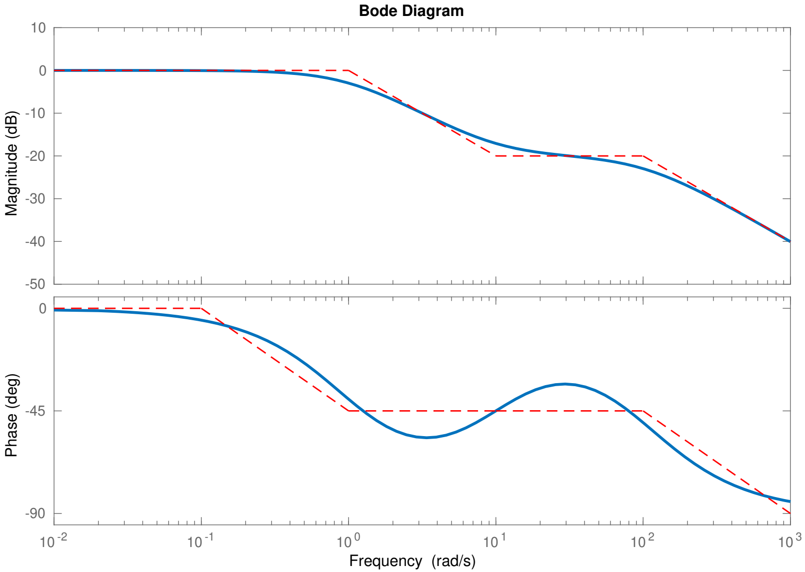
$$-20\log\left|j\frac{\omega}{p}+1\right| \approx -3dB \qquad \angle\left(j\frac{\omega}{p}+1\right)^{-1} \approx -\frac{\pi}{4}$$

Bode Plot for Real Pole $\left(\frac{s}{p}+1\right)^{-1}$



Bode Plot Example 1

- Draw a Bode plot for $T(s) = 10\frac{s+10}{(s+1)(s+100)} = \frac{(s/10+1)}{(s+1)(s/100+1)}$



Bode Plot for Complex Conjugate Zeros

- Consider $T(s) = \left(\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right)$
- **Magnitude**:

$$|T(j\omega)| = \left|-\frac{\omega^2}{\omega_n^2} + 2\zeta\frac{\omega}{\omega_n}j + 1\right| = \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}$$

- **Phase**:

$$\angle T(j\omega) = \angle -\frac{\omega^2}{\omega_n^2} + 2\zeta\frac{\omega}{\omega_n}j + 1 = \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Bode Plot for Complex Conjugate Zeros

$$|T(j\omega)| = \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2 \left(\frac{\omega}{\omega_n}\right)^2} \qquad \angle T(j\omega) = \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

- Extreme ω values:
  - **Case 1**: $\omega \ll \omega_n$: horizontal line at 0:

    $$20 \log |T(j\omega)| \approx 0 \qquad \angle T(j\omega) \approx 0$$

  - **Case 2**: $\omega \gg \omega_n$: log-scale line of slope 40 going through 0 when $\omega = \omega_n$:

    $$20 \log |T(j\omega)| \approx 20 \log \sqrt{\left(\frac{\omega}{\omega_n}\right)^4} = 40 \log \omega - 40 \log \omega_n \qquad \angle T(j\omega) \approx \pi$$

  - **Case 3**: $\omega = \omega_n$:

    $$20 \log |T(j\omega)| = 20 \log(2\zeta) \qquad \angle T(j\omega) = \frac{\pi}{2}$$

Bode Plot for Complex Conjugate Poles

- Consider $T(s) = \left(\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right)^{-1}$

$$|T(j\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} \qquad \angle T(j\omega) = -\tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

- Extreme ω values:
  - **Case 1**: $\omega \ll \omega_n$: horizontal line at 0:

$$20\log|T(j\omega)| \approx 0 \qquad \angle T(j\omega) \approx 0$$

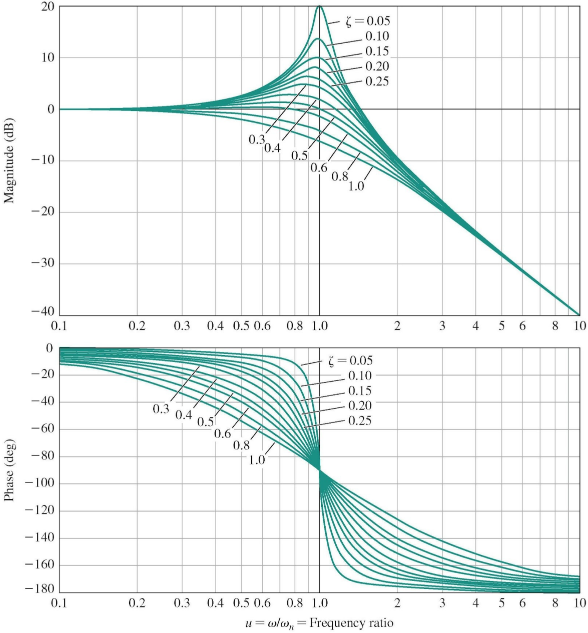
  - **Case 2**: $\omega \gg \omega_n$: log-scale line of slope -40 going through 0 when $\omega = \omega_n$

$$20\log|T(j\omega)| \approx -20\log\sqrt{\left(\frac{\omega}{\omega_n}\right)^4} = -40\log\omega + 40\log\omega_n \qquad \angle T(j\omega) \approx -\pi$$

  - **Case 3**: $\omega = \omega_n$:

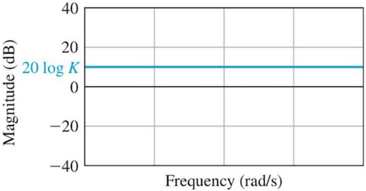
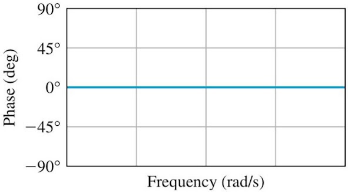
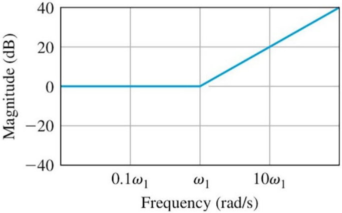
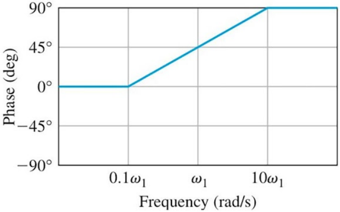
$$20\log|T(j\omega)| = -20\log(2\zeta) \qquad \angle T(j\omega) = -\frac{\pi}{2}$$

Bode Plot for Complex Conjugate Poles



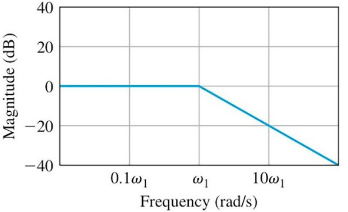
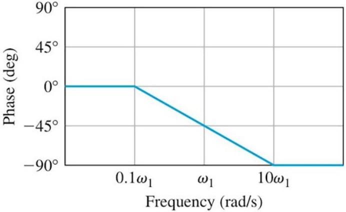
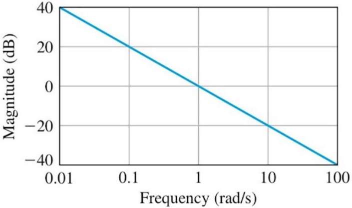
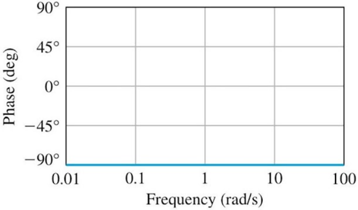
Bode Plot Approximations for Basic Transfer Function Terms

Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude $20 \log_{10}\lvert G(j\omega)\rvert$	Phase $\phi(\omega)$
1. Gain, $G(j\omega) = K$		
2. Zero, $G(j\omega) = 1 + j\omega/\omega_1$		

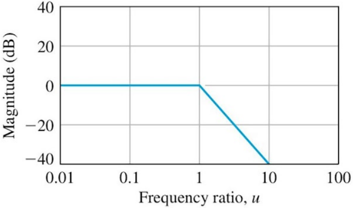
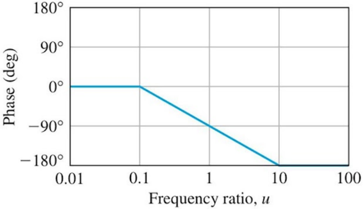
Bode Plot Approximations for Basic Transfer Function Terms

Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude $20 \log_{10}\lvert G(j\omega)\rvert$	Phase $\phi(\omega)$
3. Pole, $G(j\omega) = (1 + j\omega/\omega_1)^{-1}$		
4. Pole at the origin, $G(j\omega) = 1/j\omega$		

Bode Plot Approximations for Basic Transfer Function Terms

Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude $20 \log_{10}\lvert G(j\omega)\rvert$	Phase $\phi(\omega)$
5. Two complex poles, $0.1 < \zeta < 1$, $G(j\omega) = (1 + j2\zeta u - u^2)^{-1}$ $u = \omega/\omega_n$		

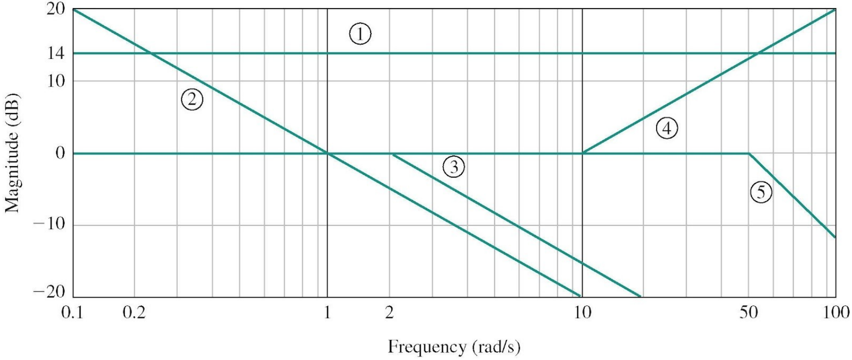
Bode Plot Example 2

- Draw a Bode plot for $T(s) = \frac{4(1+0.1s)}{s(1+0.5s)(1+0.6(s/50)+(s/50)^2)}$
- Factors in order of their occurrence as $s = j\omega$ increases:
  1. A constant gain $\kappa = 4$
  2. A pole at the origin
  3. A pole at $\omega = 2$
  4. A zero at $\omega = 10$
  5. A pair of complex poles at $\omega = \omega_n = 50$

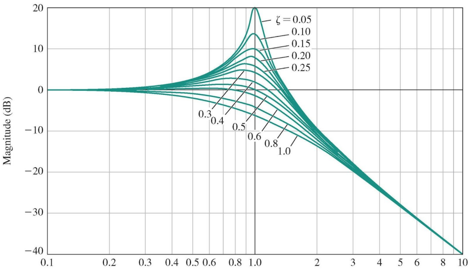
Bode Plot Example 2

- Consider the approximate magnitude plots:
  1. **Constant gain**: $20 \log |\kappa| = 14$ dB
  2. **Pole at the origin**: a line with slope -20 dB/decade going 0 when $\omega = 1$
  3. **Pole at $\omega = 2$**: horizontal line at 0 dB until the corner frequency at $\omega = 2$ and a line with slope -20 dB/decade after
  4. **Zero at $\omega = 10$**: horizontal line at 0 dB until the corner frequency at $\omega = 10$ and a line with slope 20 dB/decade after
  5. **Complex pole pair at $\omega = \omega_n = 50$**: horizontal line at 0 dB until the corner frequency at $\omega = 50$ and a line with slope -40 dB/decade after
- The approximations must be corrected at the corner frequencies:
  - Real zero/pole: $\pm 3dB$
  - Complex pair of zeros/poles: based on ζ

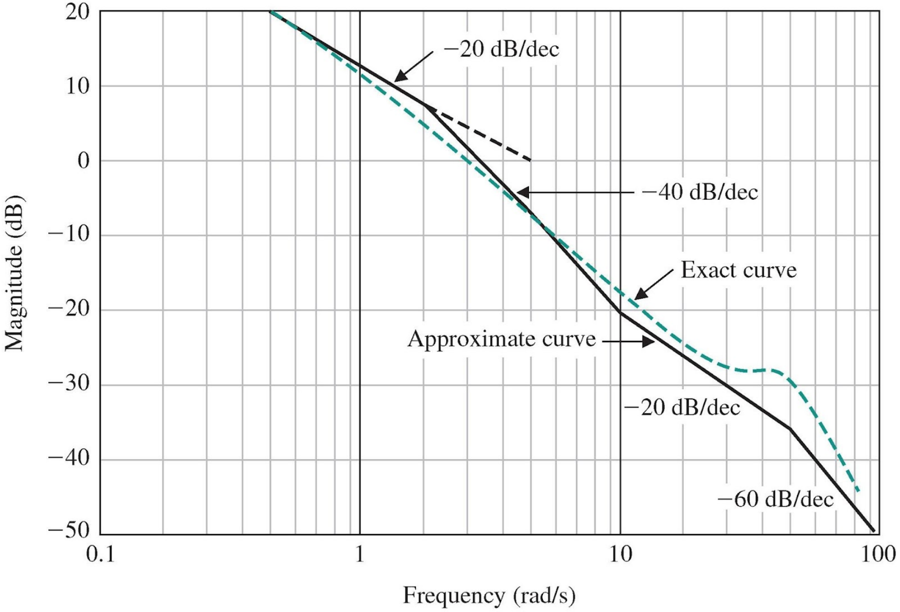
Bode Plot Example 2



- Complex pole pair correction:



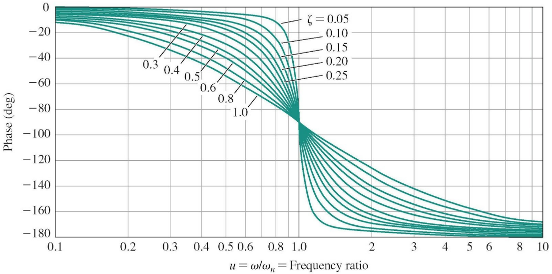
Bode Plot Example 2



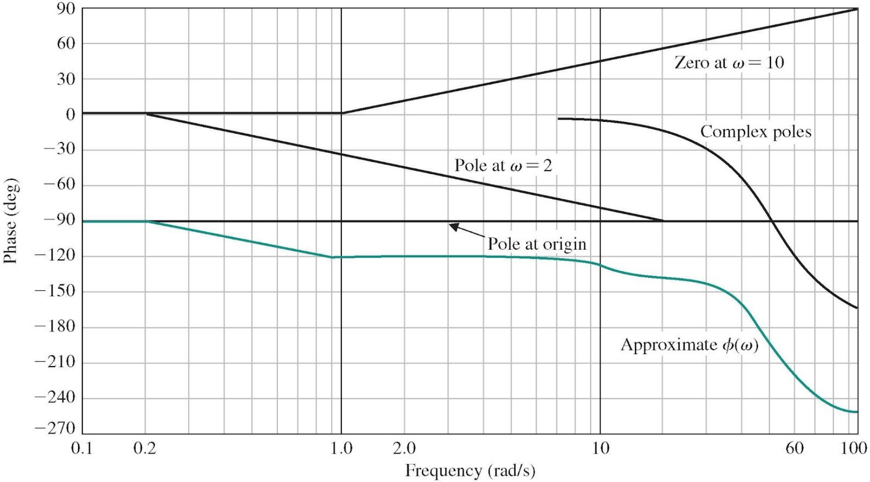
Bode Plot Example 2

- Consider the approximate phase plots:
  1. **Constant gain**: $\angle K = 0^\circ$
  2. **Pole at the origin**: -90°
  3. **Pole at** $\omega = 2$: a line with slope -45 deg/decade from $\omega = 0.2$ to $\omega = 20$
  4. **Zero at** $\omega = 10$: a line with slope 45 deg/decade from $\omega = 1$ to $\omega = 100$
  5. **Complex pole pair at** $\omega = \omega_n = 50$: phase shift of -90 deg/decade from $\omega = 5$ to $\omega = 500$

- The actual phase characteristic for the complex pole pair should be obtained from:



Bode Plot Example 2



- The exact phase shift can be evaluated at important frequencies:

$$\underline{/T(j\omega)} = \underline{/K} + \sum_{i=1}^{m_1} \tan^{-1}\left(\frac{\omega}{z_i}\right) + \sum_{l=1}^{m_2} \tan^{-1}\left(\frac{2\zeta_l \omega_{n_l} \omega}{\omega_{n_l}^2 - \omega^2}\right) - n_0 \frac{\pi}{2} - \sum_{i=1}^{n_1} \tan^{-1}\left(\frac{\omega}{p_i}\right) - \sum_{k=1}^{n_2} \tan^{-1}\left(\frac{2\zeta_k \omega_{n_k} \omega}{\omega_{n_k}^2 - \omega^2}\right)$$

Bode Plot Example 3

- Draw a Bode plot for

$$T(s) = \frac{(s+1)(s^2+3s+100)}{s^2(s+10)(s+100)} = \frac{(s+1)((s/10)^2+2(0.15)(s/10)+1)}{10s^2(s/10+1)(s/100+1)}$$

- Magnitude and phase at $\omega = 0.1$:

$$20\log|T(j\omega)| \approx 20dB \qquad \angle T(j\omega) \approx -\pi$$

- Magnitude slope in dB/decade:

ω	Zero at -1	Zeros with $\omega_n = 10$	Double pole at 0	Pole at -10	Pole at -100
0.1 - 1	0	0	-40	0	0
1 - 10	20	0	-40	0	0
10 - 100	20	40	-40	-20	0
100 - 1000	20	40	-40	-20	-20

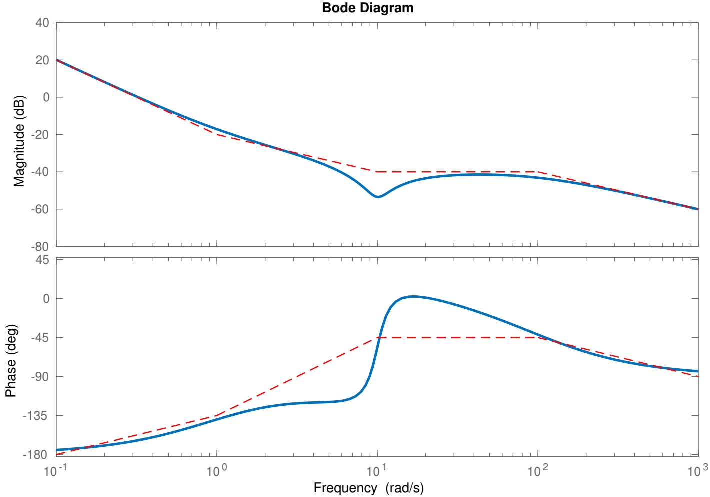
- Phase slope in degrees/decade:

ω	Zero at -1	Zeros with $\omega_n = 10$	Double pole at 0	Pole at -10	Pole at -100
0.1 - 1	45	0	0	0	0
1 - 10	45	90	0	-45	0
10 - 100	0	90	0	-45	-45
100 - 1000	0	0	0	0	-45

Bode Plot Example 3

- Draw a Bode plot for

$$T(s) = \frac{(s+1)(s^2+3s+100)}{s^2(s+10)(s+100)} = \frac{(s+1)((s/10)^2+2(0.15)(s/10)+1)}{10s^2(s/10+1)(s/100+1)}$$



Bode Plot Example 4

- Draw a Bode plot for $T(s) = \frac{4(s/2+1)}{s(1+2s)(1+0.05s+(s/8)^2)}$
- Magnitude at $\omega = 2^{-2}$:

$$20 \log |T(j\omega)| \approx 20 \log \left| \frac{4}{j\omega} \right| = 20 \log 16 \approx 24 dB$$

- Magnitude slope in dB/octave:

ω	Zero at -2	Pole at 0	Pole at -2^{-1}	Poles with $\omega_n = 2^3$
2^{-2} - 2^{-1}	0	-6	0	0
2^{-1} - 2^{1}	0	-6	-6	0
2^{1} - 2^{3}	6	-6	-6	0
2^{3} - 2^{4}	6	-6	-6	-12

- Phase slope in degrees/decade

ω	Zero at -2	Pole at 0	Pole at -2^{-1}	Poles with $\omega_n = 2^3$
0.2 - 0.8	45	0	-45	0
0.8 - 5	45	0	-45	-90
5 - 20	45	0	0	-90
20 - 80	0	0	0	-90

Bode Plot Example 4

- Draw a Bode plot for $T(s) = \frac{4(s/2+1)}{s(1+2s)(1+0.05s+(s/8)^2)}$

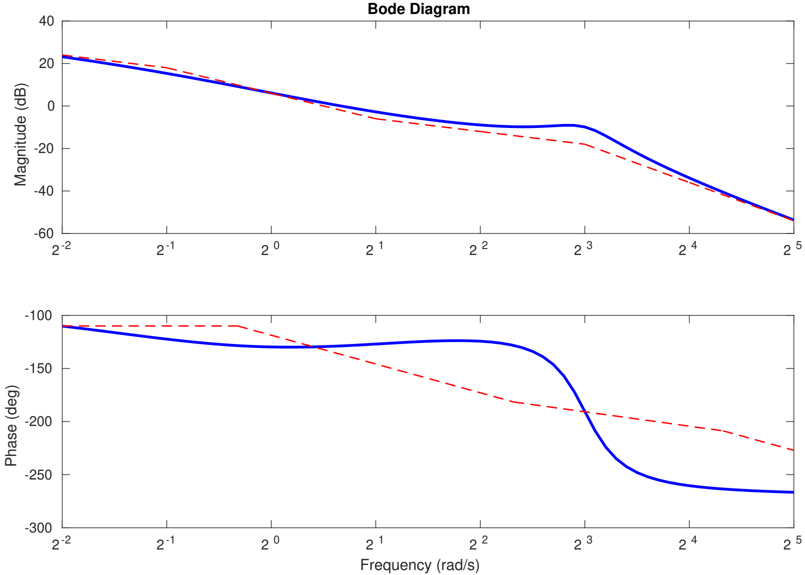
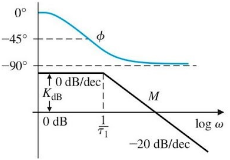


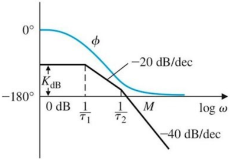
Table 8.2 Bode Plots for Typical Transfer Functions

$G(s)$	Bode Plot

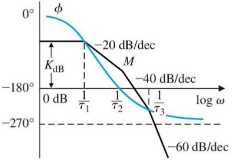
1. $\dfrac{K}{s\tau_1 + 1}$



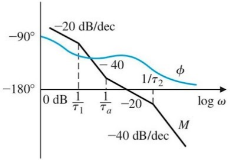
2. $\dfrac{K}{(s\tau_1 + 1)(s\tau_2 + 1)}$



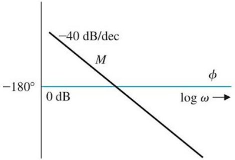
3. $\dfrac{K}{(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)}$



7. $\dfrac{K(s\tau_a + 1)}{s(s\tau_1 + 1)(s\tau_2 + 1)}$



8. $\dfrac{K}{s^2}$



9. $\dfrac{K}{s^2(s\tau_1 + 1)}$

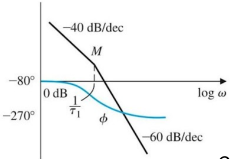
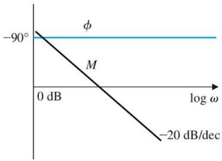
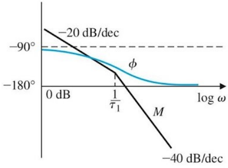
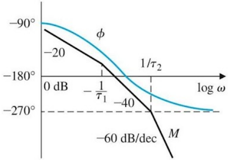
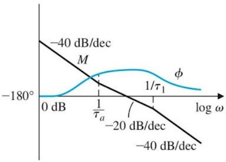
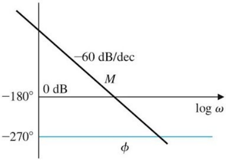
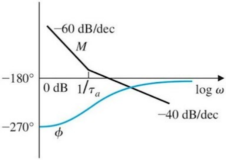


Table 8.2 Bode Plots for Typical Transfer Functions

$G(s)$	Bode Plot
4. $\dfrac{K}{s}$	
5. $\dfrac{K}{s(s\tau_1 + 1)}$	
6. $\dfrac{K}{s(s\tau_1 + 1)(s\tau_2 + 1)}$	
10. $\dfrac{K(s\tau_a + 1)}{s^2(s\tau_1 + 1)}$, $\tau_a > \tau_1$	
11. $\dfrac{K}{s^3}$	
12. $\dfrac{K(s\tau_a + 1)}{s^3}$	

Nonminimum Phase Systems

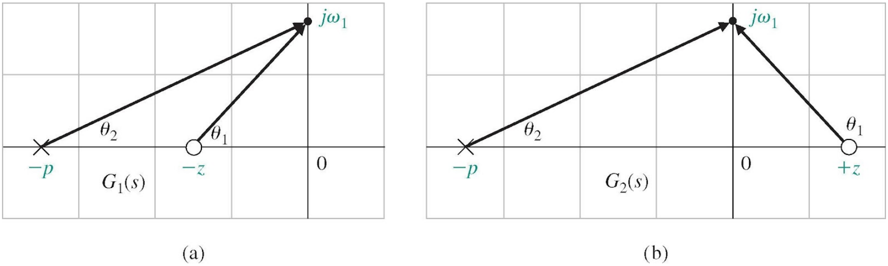
- **Minimum phase system**: a system whose transfer function poles and zeros are in the closed left half-plane
- **Nonminimum phase system**: a system whose transfer function has zeros in the right half-plane
- Bode plots can also be drawn for nonminimum phase systems
- The magnitude of a transfer function does not depend on whether the zeros are in the left or right half-plane
- The phase contribution of a zero in the right half-plane is always at least as large as the phase contribution of a zero in the left half-plane

Nonminimum Phase Systems

- To understand the difference between minimum and nonminimum phase systems compare the transfer functions:

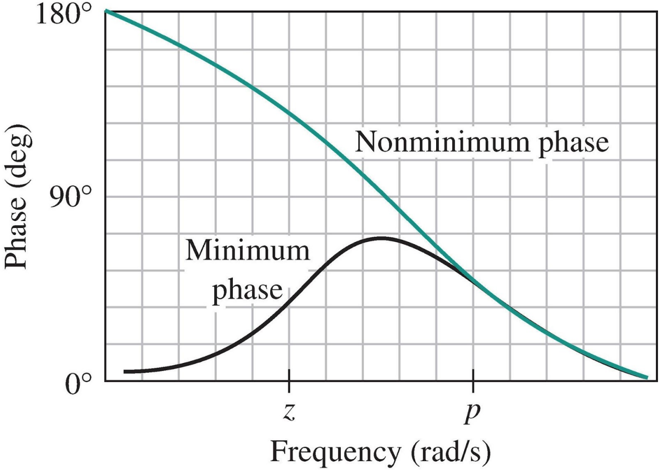
$$G_1(s) = \frac{s+z}{s+p} \qquad G_2(s) = \frac{s-z}{s+p}$$

- Magnitude: $|G_1(j\omega)| = |G_2(j\omega)| = \frac{\sqrt{\omega^2+z^2}}{\sqrt{\omega^2+p^2}}$
- Phase: $\underline{/G_1(j\omega_1)}$ vs $\underline{/G_2(j\omega_1)}$



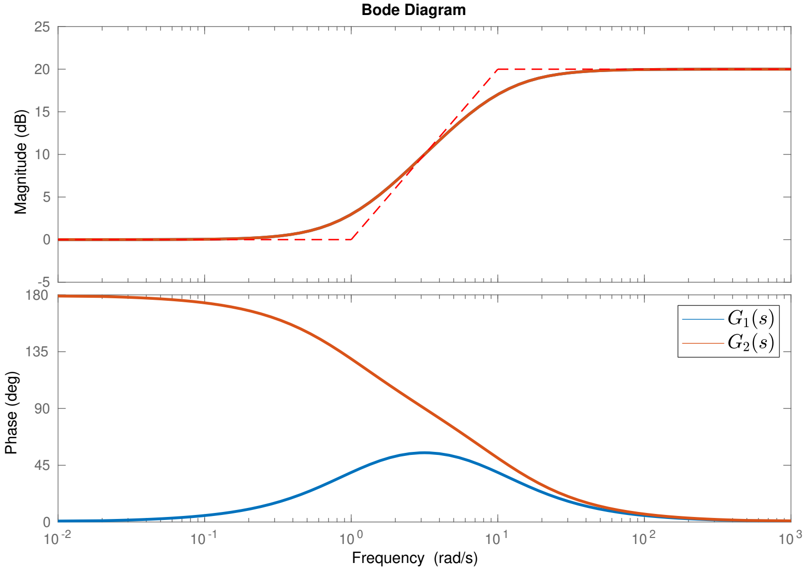
Nonminimum Phase Systems

- The range of phase shifts for a minimum phase transfer function is the least possible for a given magnitude curve



Nonminimum Phase Systems: Example

- Draw a Bode plot for $G_1(s) = 10\frac{s+1}{s+10}$ and $G_2(s) = 10\frac{s-1}{s+10}$

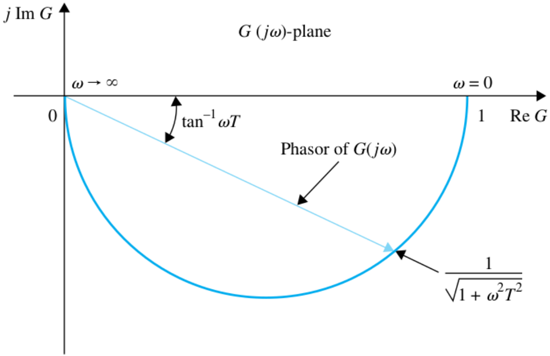


Polar Plot

- **Polar plot**: a plot of $\text{Im}(G(j\omega))$ versus $\text{Re}(G(j\omega))$ of a transfer function $G(j\omega)$ as ω varies from 0 to ∞
- A polar plot contains less information than a Bode plot because the frequency values ω are not captured
- The general shape of the polar plot can be determined from:
  - Magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$ at $\omega = 0$ and $\omega = \infty$
  - Intersection of the polar plot with the real and imaginary axes

Polar Plot: Type 0 System

- Draw a polar plot for $G(s) = \frac{1}{1+Ts}$
- Magnitude: $|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2 T^2}}$
- Phase: $\angle G(j\omega) = -\tan^{-1}(\omega T)$
- Polar plot: $|G(j0)| = 1$, $\angle G(j0) = 0$; $|G(j\infty)| = 0$, $\angle G(j\infty) = -\frac{\pi}{2}$



Polar Plot: Type 0 System

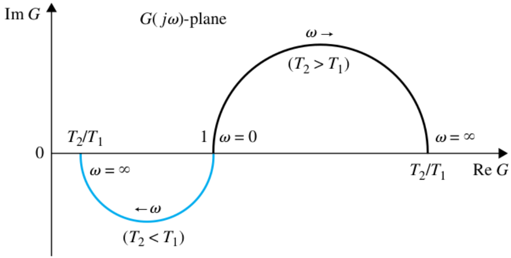
- Draw a polar plot for $G(s) = \frac{1+T_2 s}{1+T_1 s}$
- Magnitude: $|G(j\omega)| = \frac{\sqrt{1+\omega^2 T_2^2}}{\sqrt{1+\omega^2 T_1^2}}$
- Phase: $\angle G(j\omega) = \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_1)$
- The polar plot depends on the relative magnitudes of T_1 and T_2

- If $T_2 > T_1$:

  $|G(j\omega)| \geq 1 \qquad \angle G(j\omega) \geq 0$

- If $T_1 > T_2$:

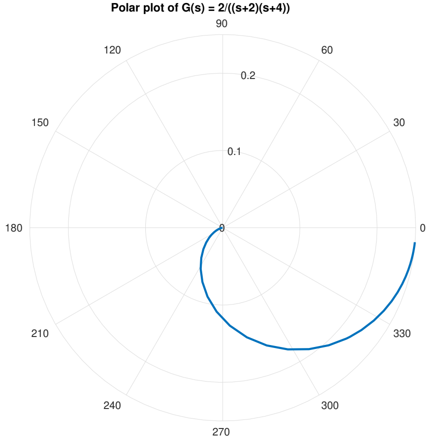
  $|G(j\omega)| \leq 1 \qquad \angle G(j\omega) \leq 0$



Polar Plot: Type 0 System

- Draw a polar plot for $G(s) = \frac{\kappa}{(1+T_1 s)(1+T_2 s)}$
- Magnitude $|G(j\omega)|$ and phase $\underline{/G(j\omega)}$ at $\omega = 0$ and $\omega = \infty$:

$$G(j0) = \kappa\underline{/0^\circ} \qquad G(j\infty) = 0\underline{/-180^\circ}$$



Polar Plot: Type 1 System

- Draw a polar plot for $G(s) = \frac{\kappa}{s(1+\tau s)}$
- Magnitude $|G(j\omega)|$ and phase $\underline{/G(j\omega)}$:

$$|G(j\omega)| = \frac{\kappa}{\sqrt{\omega^2 + \omega^4\tau^2}}$$

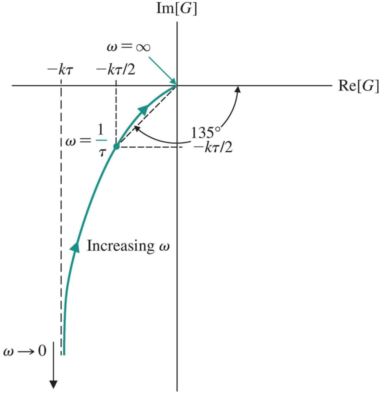
$$\underline{/G(j\omega)} = -\frac{\pi}{2} - \tan^{-1}(\omega\tau)$$

- Values at $\omega = 0$, $\omega = 1/\tau$, $\omega = \infty$:

$$G(j0) = \infty\underline{/-90^\circ}$$

$$G(j\frac{1}{\tau}) = \frac{\kappa\tau}{\sqrt{2}}\underline{/-135^\circ}$$

$$G(j\infty) = 0\underline{/-180^\circ}$$



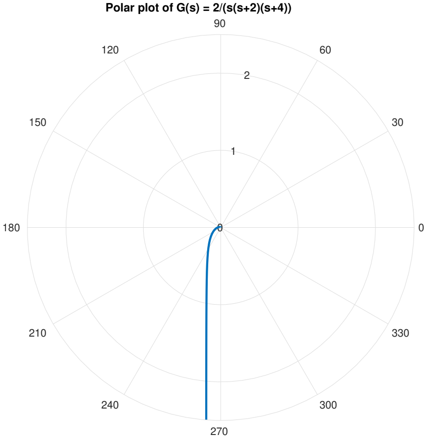
- Asymptote as $\omega \to 0$:

$$G(j\omega) = \frac{\kappa}{j\omega(1+\tau j\omega)} \overset{\text{small } \omega}{\approx} \frac{\kappa}{j\omega}(1 - j\tau\omega) = -\kappa\tau - j\frac{\kappa}{\omega}$$

Polar Plot: Type 1 System

- Draw a polar plot for $G(s) = \frac{K}{s(1+T_1 s)(1+T_2 s)}$
- Magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$ at $\omega = 0$ and $\omega = \infty$:

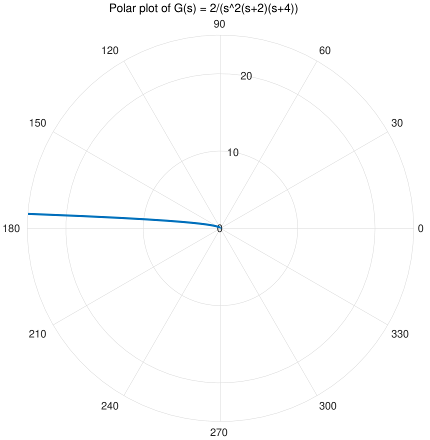
$$G(j0) = \infty\angle{-90^\circ} \qquad G(j\infty) = 0\angle{-270^\circ}$$



Polar Plot: Type 2 System

- Draw a polar plot for $G(s) = \frac{\kappa}{s^2(1+T_1 s)(1+T_2 s)}$
- Magnitude $|G(j\omega)|$ and phase $\underline{/G(j\omega)}$ at $\omega = 0$ and $\omega = \infty$:

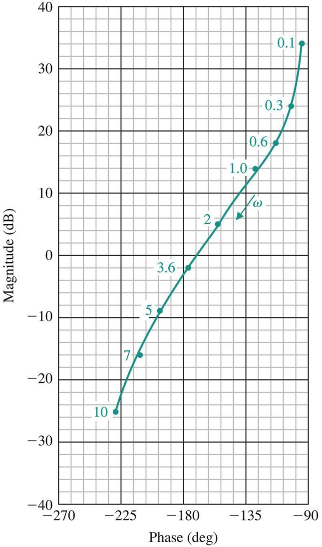
$$G(j0) = \infty\underline{/-180^\circ} \qquad G(j\infty) = 0\underline{/-360^\circ}$$

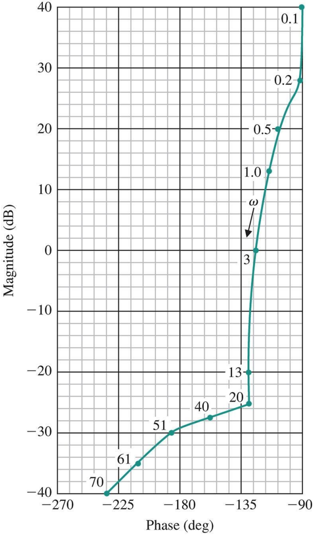


Magnitude-Phase Plot

- **Magnitude-phase plot**: a plot of the magnitude $20\log_{10}|G(j\omega)|$ in decibels (dB) versus the phase $\underline{/G(j\omega)}$ as ω varies from 0 to ∞
- A magnitude-phase plot can be obtained from the information on a Bode plot
- A magnitude-phase plot is shifted up or down when the gain factor κ varies
- The Bode plot property of adding plots of individual components does not carry over
- The magnitude-phase plot of the forward-path transfer function $G(s)$ of a unity-feedback system can be superposed on a Nichols chart to give information about the system's relative stability and frequency response

Magnitude-Phase Plot





(a) $G_1(s) = \frac{5}{s(s/2+1)(s/6+2)}$ (b) $G_2(s) = \frac{5(s/10+1)}{s(s/2+1)(1+0.6(s/50)+(s/50)^2)}$

Magnitude-Phase Plot

- The shape of the magnitude-phase plot is important where the magnitude approaches 0 dB and the phase approaches 180°
- **Gain-Crossover frequency**: ω at which $20\log|G(j\omega)| = 0$ dB
- **Phase-Crossover frequency**: ω at which $\angle G(j\omega) = -\pi$
- Unity-feedback closed-loop transfer function:

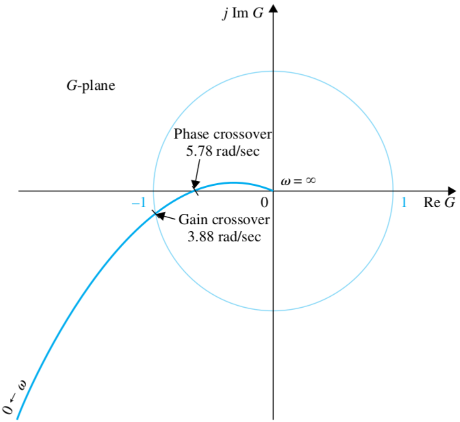
$$T(s) = \frac{G(s)}{1 + G(s)}$$

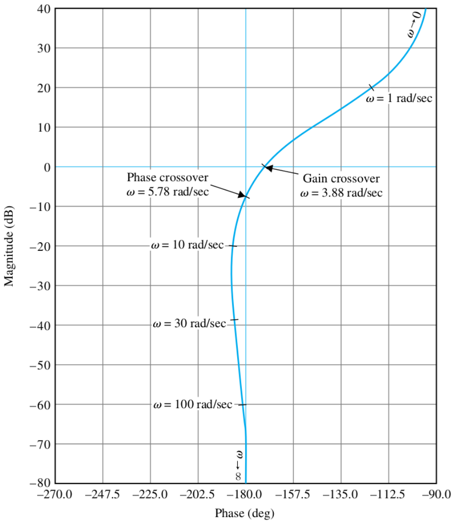
- Instability occurs when $1 + G(s) = 0$:

$$|G(s)| = 1 \qquad\qquad \angle G(s) = -\pi$$

Magnitude-Phase Plot

- Draw a polar plot and a magnitude-phase plot for $G(s) = \frac{10(s+10)}{s(s+2)(s+5)}$

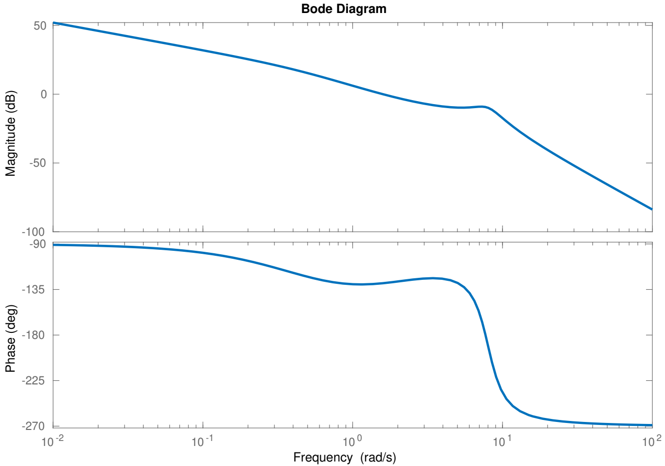




Frequency Domain Plots in Matlab

- Bode plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$

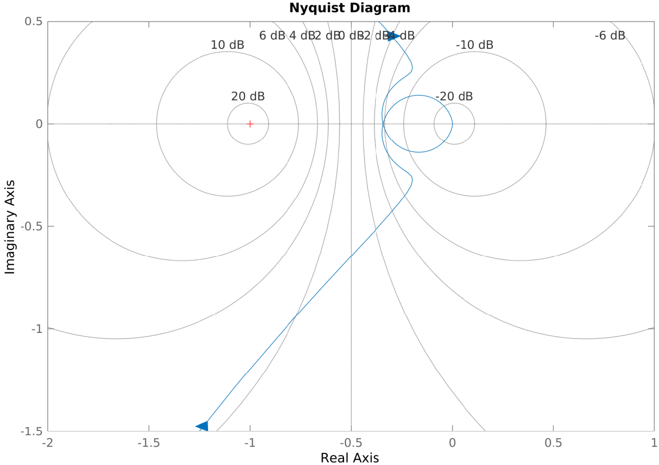
```
s = tf('s');
G = 4*(s/2+1)/s/(1+2*s)/(1+0.4*(s/8) + (s/8)^2);
bodeplot(G);
```



Frequency Domain Plots in Matlab

- Nyquist plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$

```
s = tf('s');
G = 4*(s/2+1)/s/(1+2*s)/(1+0.4*(s/8) + (s/8)^2);
nyquistplot(G);
```



Frequency Domain Plots in Matlab

- Nichols plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$

```
s = tf('s');
G = 4*(s/2+1)/s/(1+2*s)/(1+0.4*(s/8) + (s/8)^2);
nicholsplot(G);
```