ECE171A: Linear Control System Theory
Lecture 11: Nyquist Stability

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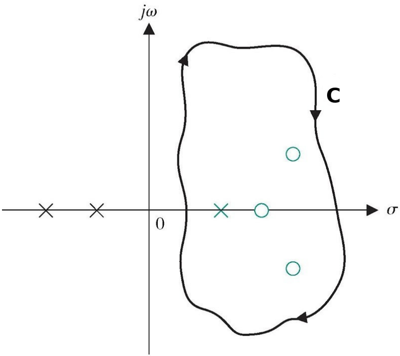
Contours in the Complex Plane

- Nyquist plots complement Bode plots to provide us with frequency response techniques to determine the stability of a closed-loop system
- Nyquist's stability criterion utilizes contours in the complex plane to relate the locations of the open-loop and closed-loop poles
- A **contour** is a piecewise smooth path in the complex plane
- A contour is **closed** if it starts and ends at the same point
- A contour is **simple** if it does not cross itself at any point
- A parameterization $z(t) \in \mathbb{C}$ of a contour has direction indicated by increasing the parameter $t \in \mathbb{R}$



Open-loop Transfer Function

- Consider a control system with open-loop transfer function:

$$G(s) = \kappa \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- At each s, $G(s)$ is a complex number with magnitude and phase:

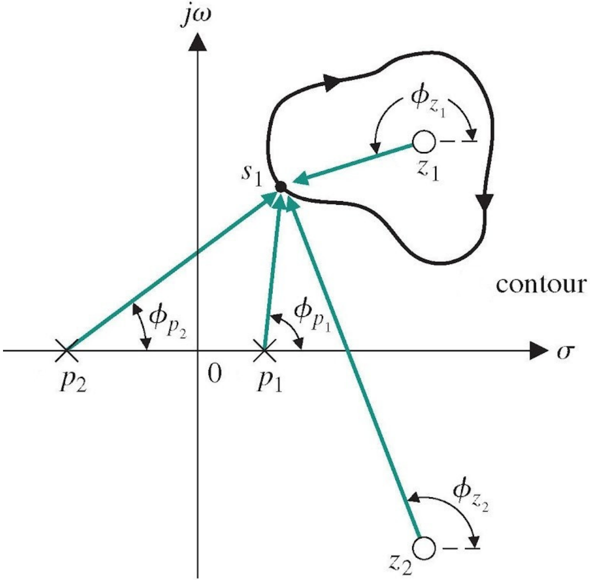
$$|G(s)| = |\kappa| \frac{\prod_{i=1}^m |s - z_i|}{\prod_{i=1}^n |s - p_i|} \qquad \angle G(s) = \angle \kappa + \sum_{i=1}^m \angle (s - z_i) - \sum_{i=1}^n \angle (s - p_i)$$

- Graphical evaluation of the magnitude and phase:
  - $|s - z_i|$ is the length of the vector from z_i to s
  - $|s - p_i|$ is the length of the vector from p_i to s
  - $\angle (s - z_i)$ is the angle from the real axis to the vector from z_i to s
  - $\angle (s - p_i)$ is the angle from the real axis to the vector from p_i to s

Evaluating $G(s)$ along a Contour

- Let C be a simple closed clockwise contour C in the complex plane
- Evaluating $G(s)$ at all points on C produces a new closed contour $G(C)$
- **Assumption**: C does not pass through the origin or any of the poles or zeros of $G(s)$ (otherwise $\angle G(s)$ is undefined)
- A zero z_i outside the contour C:
  - As s moves around the contour C, the vector $s - z_i$ swings up and down but not all the way around
  - The net change in $\angle(s - z_i)$ is 0
- A zero z_i inside the contour C:
  - As s moves around the contour C, the vector $s - z_i$ turns all the way around
  - The net change in $\angle(s - z_i)$ is -2π
- A pole p_i outside the contour C: the net change in $\angle(s - p_i)$ is 0
- A pole p_i inside the contour C: the net change in $\angle(s - p_i)$ is -2π

Evaluating $G(s)$ along a Contour



Principle of the Argument

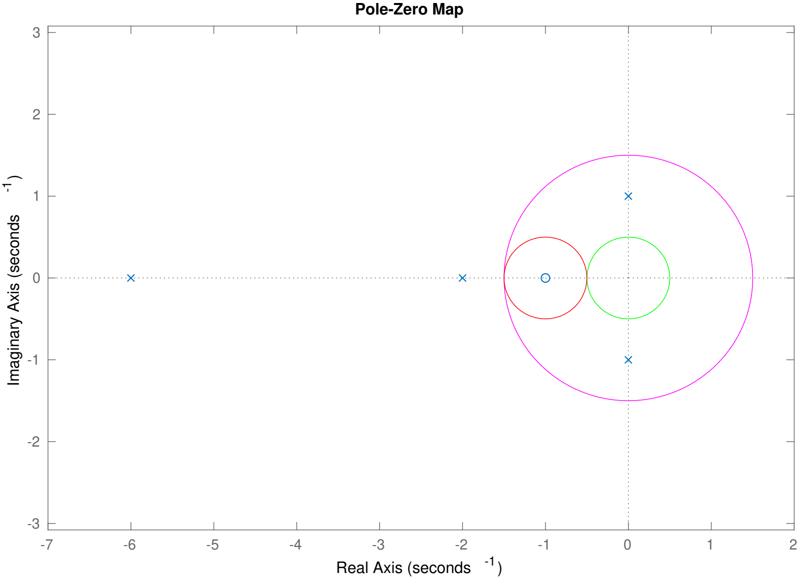
- Let Z and P be the number of zeros and poles of $G(s)$ inside C
- As s moves around C, $\angle G(s)$ undergoes a net change of $-(Z-P)2\pi$
- A net change of -2π means that the vector from 0 to $G(s)$ swings clockwise around the origin one full rotation
- A net change of $-(Z-P)2\pi$ means that the vector from 0 to $G(s)$ must encircle the origin in clockwise direction $(Z-P)$ times

Cauchy's Principle of the Argument

Consider a transfer function $G(s)$ and a simple closed clockwise contour C. Let Z and P be the number of zeros and poles of $G(s)$ inside C. Then, the contour generated by evaluating $G(s)$ along C will encircle the origin in a clockwise direction $Z-P$ times.

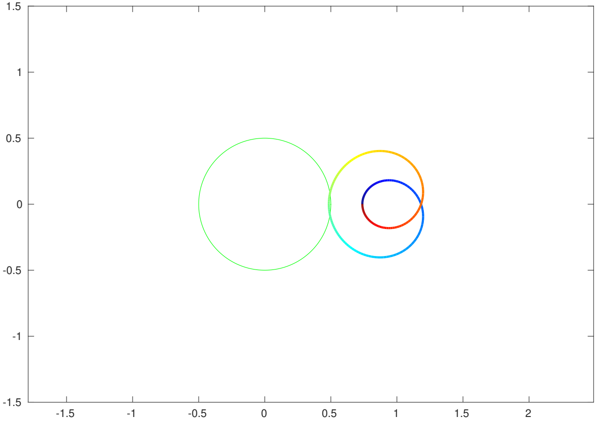
Principle of the Argument: Example

- Pole-zero map for $G(s) = \frac{10(s+1)}{(s+2)(s^2+1)(s+6)}$



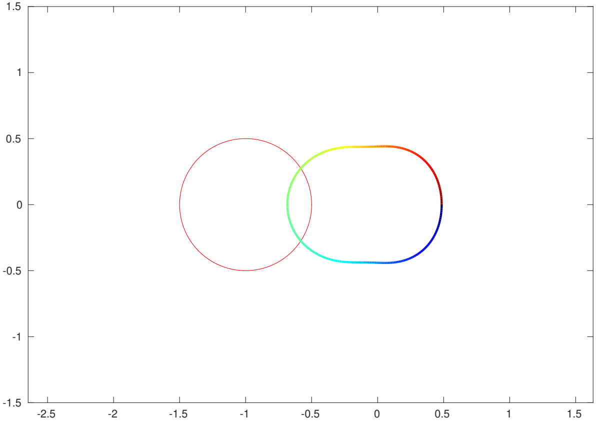
Principle of the Argument: Example

- A circle contour C centered at the origin with radius 0.5 (green)
- The contour may be parameterized by $z(t) = 0.5e^{-jt}$ for $t \in [0, 2\pi]$
- The contour C is mapped by $G(s)$ to a new contour (from blue to red), e.g., parameterized by $G(z(t))$ for $t \in [0, 2\pi]$



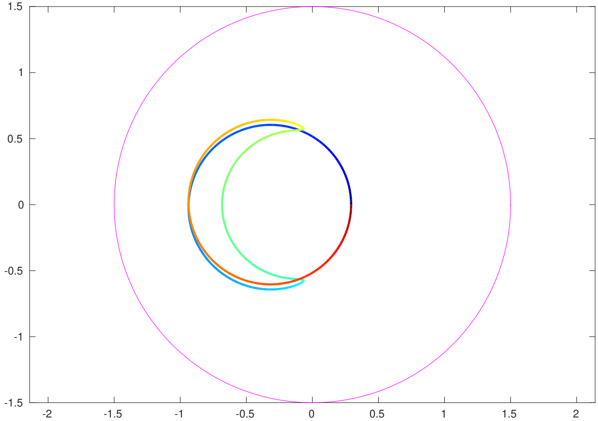
Principle of the Argument: Example

- A circle contour C centered at $(-1, 0)$ with radius 1 (red)
- The contour C is mapped by $G(s)$ to a new contour (from blue to red)

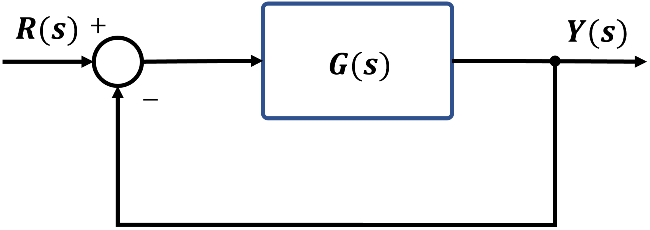


Principle of the Argument: Example

- A circle contour C centered at the origin with radius 1.5 (magenta)
- The contour C is mapped by $G(s)$ to a new contour (from blue to red)

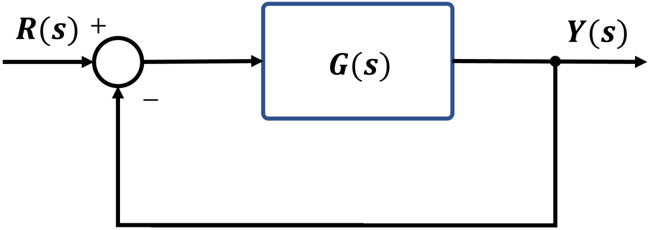


Frequency-domain Stability



- Consider a feedback control system
- **Root locus**: analyzes the poles of the closed-loop transfer function $T(s)$ based on the poles and zeros of the open-loop transfer function $G(s)$
- Given a **Bode plot** of the open-loop transfer function $G(s)$, we would like to analyze the properties of the closed-loop transfer function
- The principle of the argument can be used to study the stability of the closed-loop system

Frequency-domain Stability



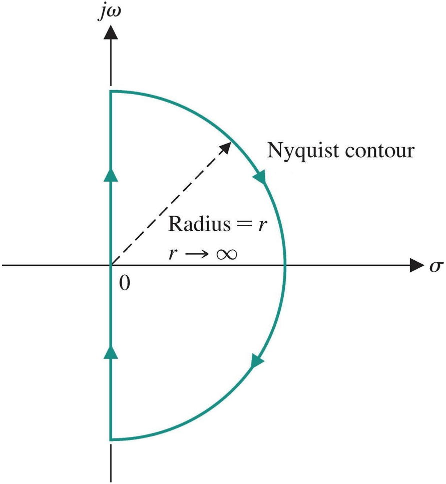
- Closed-loop transfer function: $T(s) = \dfrac{G(s)}{1 + G(s)}$
- The closed-loop poles are all s such that $\Delta(s) = 1 + G(s) = 0$
- The poles of $\Delta(s)$ are the open-loop poles:

$$\Delta(s) = 1 + G(s) = 1 + \frac{b(s)}{a(s)} = \frac{a(s) + b(s)}{a(s)}$$

- The zeros of $\Delta(s)$ are the closed-loop poles

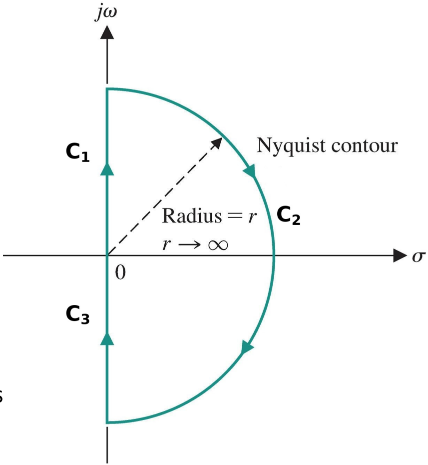
Frequency-domain Stability

- To determine how many closed-loop poles lie in the closed right half-plane, we will apply the Principle of the Argument to $\Delta(s)$
- Define a contour that covers the closed right half-plane



Nyquist Contour

- The Nyquist contour is made up of three parts:
  - **Contour** C_1: points $s = j\omega$ on the positive imaginary axis, as ω ranges from 0 to ∞
  - **Contour** C_2: points $s = re^{j\theta}$ on a semi-circle as $r \to \infty$ and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
  - **Contour** C_3: points $s = j\omega$ on the negative imaginary axis, as ω ranges from $-\infty$ to 0

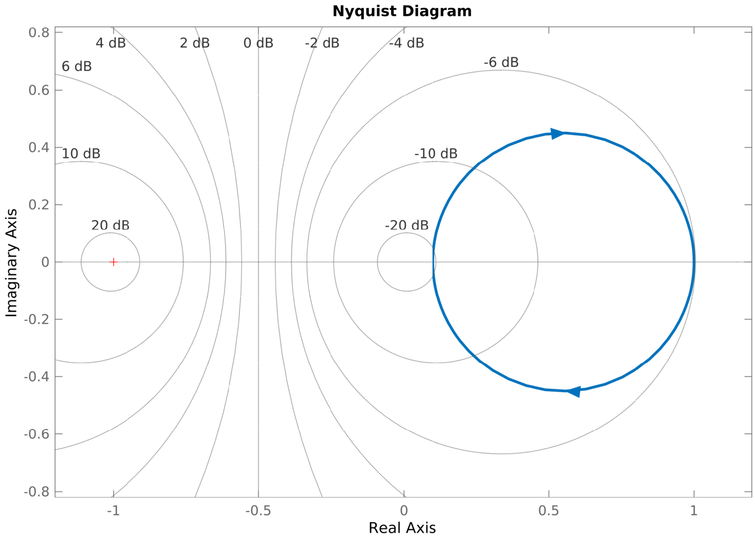


Nyquist Plot

- A **Nyquist plot** evaluates $\Delta(s) = 1 + G(s)$ over the Nyquist contour C
- The contour $\Delta(C)$ may be obtained by shifting the contour $G(C)$ by one unit to the right
- The contour $G(C)$ is obtained by combining $G(C_1)$, $G(C_2)$, and $G(C_3)$:
  - **Contour** C_1:
    - plot $G(j\omega)$ for $\omega \in (0, \infty)$ in the complex plane
    - equivalent to a **polar plot** for $G(s)$
  - **Contour** C_2:
    - plot $G(re^{j\theta})$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
    - as $r \to \infty$, $s = re^{j\theta}$ dominates every factor it appears in
    - if $G(s)$ is strictly proper, then $G(re^{j\theta}) \to 0$
    - if $G(s)$ is non-strictly proper, then $G(re^{j\theta}) \to$ const
  - **Contour** C_3:
    - plot $G(j\omega)$ for $\omega \in (-\infty, 0)$ in the complex plane
    - $G(-jb)$ is the complex conjugate of $G(jb)$
    - $G(-jb)$ and $G(jb)$ have the same magnitude but opposite phases
    - $G(C_3)$ is a reflected version of $G(C_1)$ about the real axis

Nyquist Plot: Example 1

- Draw a Nyquist plot for $G(s) = \frac{s+1}{s+10}$
- Type 0 system as on Slide 42 of Lecture 10 with $\lim_{r \to \infty} G(re^{j\theta}) = 1$



Nyquist Plot: Example 1

- Draw a Nyquist plot for $G(s) = \frac{s+1}{s+10}$
- **Contour** C_1:
  - $\omega = 0$ and $\omega \to \infty$:

$$G(j0) = \frac{1}{10}\angle 0^\circ \qquad G(j\infty) = 1\angle 0^\circ$$

  - for $0 < \omega < \infty$:

$$|G(j\omega)| = \frac{1}{10}\frac{\sqrt{1+\omega^2}}{\sqrt{1+(\omega/10)^2}} \qquad \angle G(j\omega) = \tan^{-1}(\omega) - \tan^{-1}(\omega/10)$$

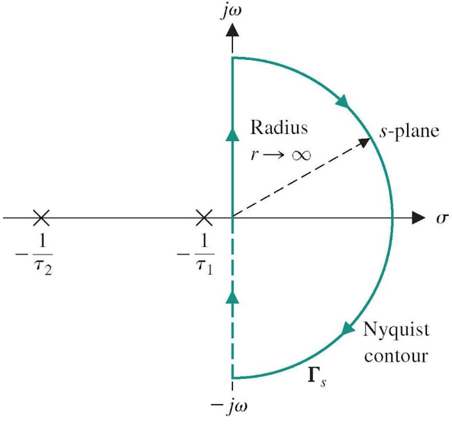
- **Contour** C_2 with $s = re^{j\theta}$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r\to\infty} G(re^{j\theta}) = \lim_{r\to\infty} \frac{re^{j\theta}+1}{re^{j\theta}+10} = 1\angle 0^\circ$$

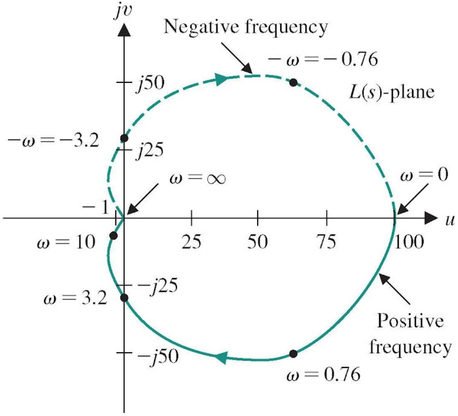
- **Contour** C_3 with $\omega \in (-\infty, 0)$:
  - $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

Nyquist Plot: Example 2

- Draw a Nyquist plot for $G(s) = \frac{\kappa}{(1+\tau_1 s)(1+\tau_2 s)} = \frac{100}{(1+s)(1+s/10)}$
- Contour C_1: $G(j0) = \kappa\underline{/0^\circ}$, $G(j\infty) = 0\underline{/-180^\circ}$
- Contour C_2: $\lim_{r\to\infty} G(re^{j\theta}) = 0$



(a)



(b)

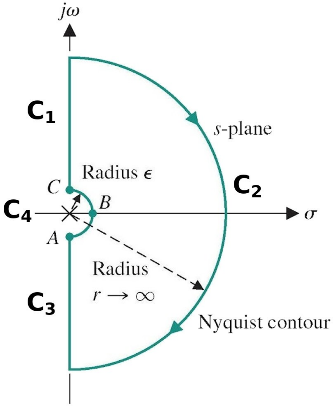
Nyquist Plot: Pole/Zero on the Imaginary Axis

- The Principle of the Argument assumes that C does not pass through any zeros or poles

- There might be poles or zeros of $G(s)$ on the imaginary axis

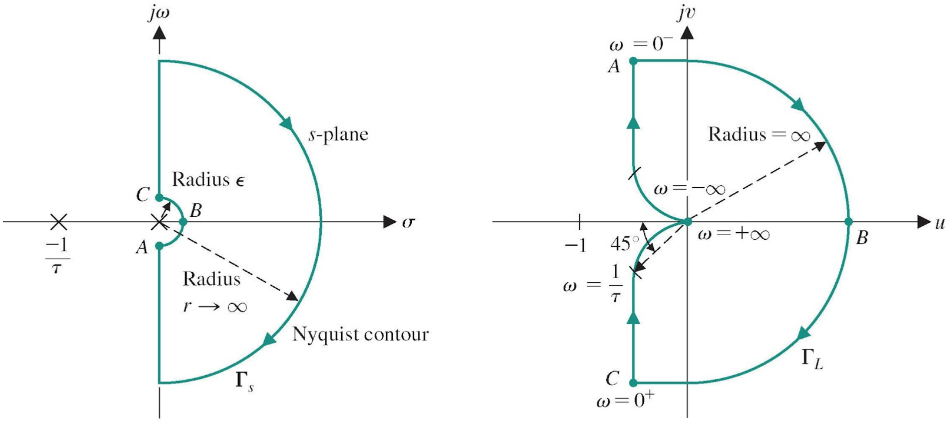
- The Nyquist contour needs to be modified to take a small detour around such poles or zeros

- **Contour** C_4:
  - plot $G(\epsilon e^{j\theta})$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$
  - substitute $s = \epsilon e^{j\theta}$ into $G(s)$ and examine what happens as $\epsilon \to 0$



Nyquist Plot: Example 3

- Draw a Nyquist plot for a type 1 system: $G(s) = \frac{K}{s(1+\tau s)}$
- Since there is a pole at the origin, we need to use a modified Nyquist contour



(a) (b)

Nyquist Plot: Example 3

- **Contour** C_4 with $s = \epsilon e^{j\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon \to 0} G(\epsilon e^{j\theta}) = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon e^{j\theta}} = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon} e^{-j\theta} = \infty \underline{/-\theta}$$

  - The phase of $G(s)$ changes from $\frac{\pi}{2}$ at $\omega = 0^-$ to $-\frac{\pi}{2}$ at $\omega = 0^+$
  - Asymptote as $\omega \to 0$:

$$\lim_{\omega \to 0} G(j\omega) = \lim_{\omega \to 0} \frac{\kappa}{j\omega(1 + j\omega\tau)} = \lim_{\omega \to 0} \frac{\kappa}{j\omega}(1 - j\omega\tau) = \lim_{\omega \to 0} -\kappa\tau - j\frac{\kappa}{\omega}$$

Nyquist Plot: Example 3

- **Contour** C_1 with $\omega \in (0, \infty)$: polar plot as on Slide 44 of Lecture 10:

$$G(j0^+) = \infty \underline{/-90^\circ}$$

$$G(j\infty) = \lim_{\omega \to \infty} \frac{\kappa}{j\omega(1 + j\omega\tau)} = \lim_{\omega \to \infty} \left| \frac{\kappa}{\tau\omega^2} \right| \underline{/-\pi/2 - \tan^{-1}(\omega\tau)}$$

$$= 0\underline{/-180^\circ}$$

- **Contour** C_2 with $s = re^{j\theta}$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r \to \infty} G(re^{j\theta}) = \lim_{r \to \infty} \left| \frac{\kappa}{\tau r^2} \right| e^{-2j\theta} = 0\underline{/-2\theta}$$

  - The phase of $G(s)$ changes from $-\pi$ at $\omega = \infty$ to π at $\omega = -\infty$

- **Contour** C_3 with $\omega \in (-\infty, 0)$:
  - $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

Nyquist Plot: Example 4

- Draw a Nyquist plot for a type 1 system: $G(s) = \frac{\kappa}{s(1+\tau_1 s)(1+\tau_2 s)}$
- **Contour** C_4 with $s = \epsilon e^{j\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:
  - C_4 maps into a semicircle with infinite radius as in Example 3:

$$G(j0) = \infty\underline{/-\theta}$$

- **Contour** C_2 with $s = re^{j\theta}$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:
  - C_2 maps into a point at 0 with phase $\underline{/-3\theta}$
- **Contour** C_3: $G(C_3)$ is a reflection of $G(C_1)$ about the real axis
- **Contour** C_1 with $\omega \in (0, \infty)$: polar plot as on Slide 45 of Lecture 10:

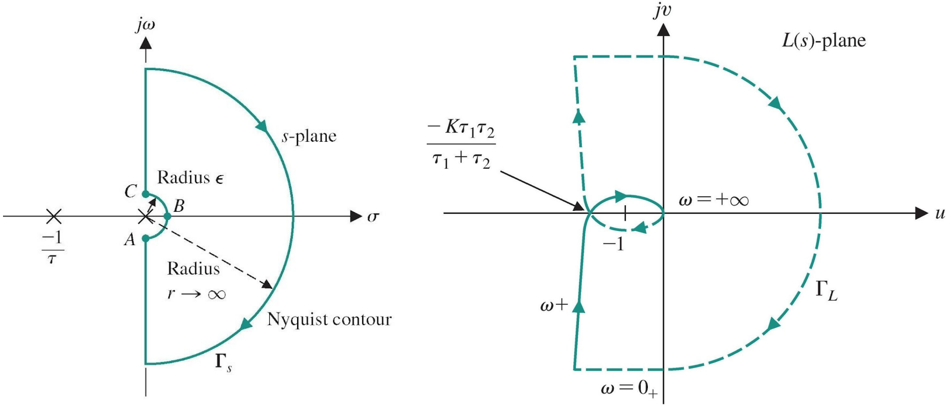
$$G(j\infty) = 0\underline{/-270^\circ}$$

Nyquist Plot: Example 4

- **Contour** C_1 with $\omega \in (0, \infty)$:

$$G(j\omega) = \frac{\kappa}{j\omega(1 + j\omega\tau_1)(1 + j\omega\tau_2)} = \frac{-\kappa(\tau_1 + \tau_2) - j\kappa(1 - \omega^2\tau_1\tau_2)\omega}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4\tau_1^2\tau_2^2}$$

$$= \frac{\kappa}{\sqrt{\omega^4(\tau_1 + \tau_2)^2 + \omega^2(1 - \omega^2\tau_1\tau_2)^2}} \angle -(\pi/2) - \tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)$$



Nyquist Plot: Example 5

- Draw a Nyquist plot for a type 2 system: $G(s) = \frac{\kappa}{s^2(1+\tau s)}$
- Two poles at the origin $\Rightarrow$ need to use a modified Nyquist contour
- Magnitude and phase:

$$G(j\omega) = \frac{\kappa}{(j\omega)^2(1+j\omega\tau)} = \frac{|\kappa|}{\sqrt{\omega^4+\omega^6\tau^2}}\underline{/-\pi-\tan^{-1}(\omega\tau)}$$

- **Contour** C_4 with $s = \epsilon e^{j\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon\to 0} G(s) = \lim_{\epsilon\to 0} \frac{\kappa}{s^2} = \lim_{\epsilon\to 0} \frac{\kappa}{\epsilon^2} e^{-2j\theta} = \infty\underline{/-2\theta}$$

  - The phase of $G(s)$ changes from π at $\omega = 0^-$ to $-\pi$ at $\omega = 0^+$

Nyquist Plot: Example 5

- **Contour** C_1 with $\omega \in (0, \infty)$:

$$G(j0^+) = \infty \underline{/-180^\circ}$$

$$G(j\infty) = \lim_{\omega \to \infty} \frac{\kappa}{(j\omega)^2(1 + j\omega\tau)} = \lim_{\omega \to \infty} \left| \frac{\kappa}{\tau\omega^3} \right| \underline{/-\pi - \tan^{-1}(\omega\tau)}$$

$$= 0\underline{/-270^\circ}$$

- **Contour** C_2 with $s = re^{j\theta}$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

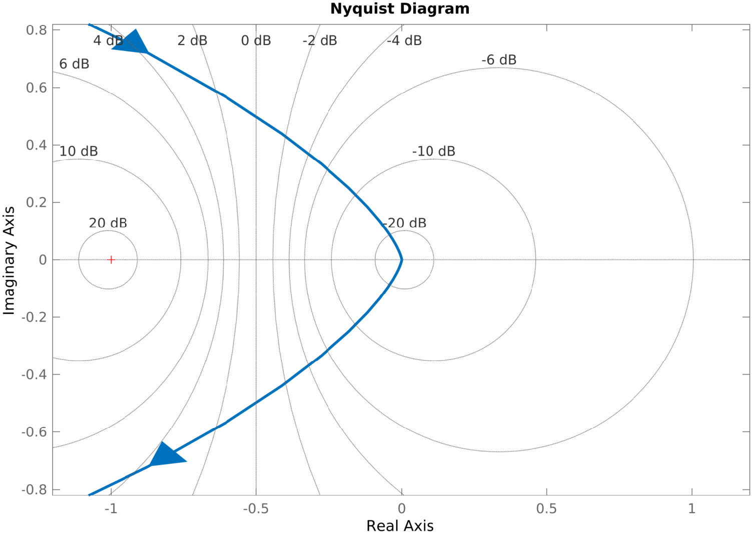
$$\lim_{r \to \infty} G(s) = \lim_{r \to \infty} \frac{\kappa}{\tau s^3} = \lim_{r \to \infty} \left| \frac{\kappa}{\tau r^3} \right| e^{-3j\theta} = 0\underline{/-3\theta}$$

  - The phase of $G(s)$ changes from $-\frac{3\pi}{2}$ at $\omega = \infty$ to $\frac{3\pi}{2}$ at $\omega = -\infty$

- **Contour** C_3 with $\omega \in (-\infty, 0)$:
  - $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

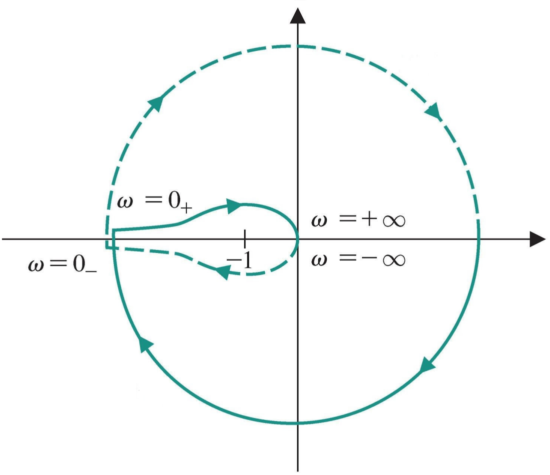
Nyquist Plot: Example 5

- Draw a Nyquist plot for a type 2 system: $G(s) = \frac{\kappa}{s^2(1+\tau s)} = \frac{1}{s^2(s+1)}$



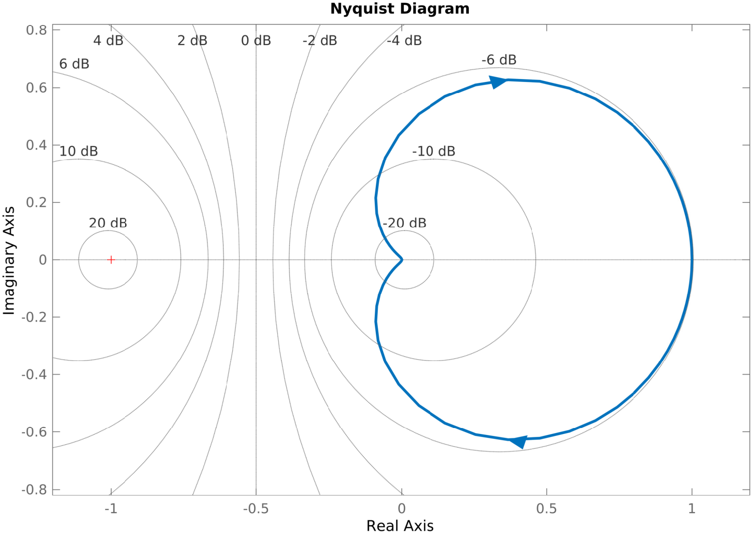
Nyquist Plot: Example 5

- Draw a Nyquist plot for a type 2 system: $G(s) = \frac{\kappa}{s^2(1+\tau s)}$



Nyquist Plot: Example 6

- Draw a Nyquist plot for $G(s) = \frac{s(s+1)}{(s+10)^2}$



Nyquist's Stability Criterion

- Consider the stability of the closed-loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{G(s)}{\Delta(s)}$$

- The poles of $\Delta(s)$ are the poles of $G(s)$ (**open-loop poles**)
- The zeros of $\Delta(s)$ are the poles of $T(s)$ (**closed-loop poles**)
- Principle of the Argument applied to $\Delta(s) = 1 + G(s)$:
  - Let C be a Nyquist contour.
  - Let Z be the number of zeros of $\Delta(s)$ (closed-loop poles) inside C.
  - Let P be the number of poles of $\Delta(s)$ (open-loop poles) inside C.
  - Then, $\Delta(C)$ encircles the origin in clockwise direction $N = Z - P$ times.

Nyquist's Stability Criterion

- From the Principle of the Argument applied to $\Delta(s)$, the number of closed-loop poles in the closed right half-plane is:

$$Z = N + P$$

where:

  - N: the clockwise encirclements of the origin by $\Delta(C)$ correspond to the clockwise encirclements of $-1 + j0$ by $G(C)$ and can be determined from a Nyquist plot of $G(s)$
  - P: the number of poles of $\Delta(s)$ inside C corresponds to the number of poles of $G(s)$ inside C and can be determined from $G(s)$ or its Bode plot

Nyquist's Stability Criterion

Consider a unity feedback control system with open-loop transfer function $G(s)$. Let C be a Nyquist contour. The system is stable if and only if the number of counterclockwise encirclements of $-1 + j0$ by $G(C)$ is equal to the number of poles of $G(s)$ inside C.

Nyquist Stability: Example 4

- Determine the closed-loop stability of $G(s) = \frac{\kappa}{s(1+\tau_1 s)(1+\tau_2 s)} = \frac{\kappa}{s(1+s)^2}$
- $G(C_1)$ crosses the real axis when:

$$G(j\omega) = \frac{-\kappa(\tau_1 + \tau_2) - j\kappa(1 - \omega^2\tau_1\tau_2)\omega}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4\tau_1^2\tau_2^2} = \alpha + j0$$

$$\Rightarrow \omega = \frac{1}{\sqrt{\tau_1\tau_2}} \qquad \alpha = -\frac{\kappa\tau_1\tau_2}{\tau_1 + \tau_2}$$

- The system is stable when $\alpha = -\frac{\kappa\tau_1\tau_2}{\tau_1+\tau_2} \geq -1$