ECE171A: Linear Control System Theory
Lecture 1: Introduction

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Electrical and Computer Engineering

Course Overview

- **ECE 171A**: Linear Control System Theory focuses on modeling and analysis of single-input single-output linear control systems emphasizing frequency domain techniques:
  - **Modeling**: ordinary differential equations, transfer functions, block diagrams, signal flow graphs
  - **Charcteristics**: feedback, disturbances, sensitivty, transient and steady-state response
  - **Stability**: Routh-Hurwitz stability criterion, relative stability
  - **Frequency domain behavior**: root locus, Bode diagrams, Nyquist plots, Nichols charts
- **Textbook**: Modern Control Systems: Dorf & Bishop
- Other references:
  - Feedback Control of Dynamic Systems: Franklin, Powell & Emami-Naeini
  - Automatic Control Systems: Kuo & Golnaraghi
  - Feedback Systems: Astrom & Murray
  - Control System Design: Goodwin, Graebe & Salgado

Logistics

- Course website: `https://natanaso.github.io/ece171a`
- Includes links to:
  - **Canvas**: **course password**, discussion Zoom schedule, lecture recordings
  - **Gradescope**: homework submission and grades
  - **Piazza**: discussion and class announcements (please check regularly)
- Assignments:
  - 6 homework sets (48% of grade)
  - midterm exam (26% of grade)
  - final exam (26% of grade)
- Grading:
  - A standard grade scale (e.g., 93%+ = A) will be used with a curve based on the class performance (e.g., if the top students have grades in the 83%-86% range, then this will correspond to letter grade A)
  - **no late policy**: homework submitted past the deadline will receive 0 credit
- Prerequisites: ECE45: Circuits and Systems or MAE 140: Linear Circuits

Office Hours and Discussion Session

- Office hours:
  - **Nikolay**: Monday, 3:00 pm - 4:00 pm, on Zoom (links on Canvas)
  - **Chenfeng**: Thursday, 3:00 pm - 4:00 pm, on Zoom (links on Canvas).
- Discussion session:
  - There is no distinction between a discussion session and office hours.
  - No new material will be covered during the discussion session/office hours.
  - We will use the time to go over homework solutions and answer questions.
  - If you think that two sessions per week are insufficient, I will be happy to add more.

Course Schedule (Tentative)

Date	Lecture	Material	Assignment
Sep 27	Intro, ODEs	Dorf-Bishop Ch. 1, Ch. 2.1-2.3	
Sep 29	Laplace Transform, Transfer Function	Dorf-Bishop Ch. 2.4-2.5	HW1
Oct 04	Block Diagram, Signal Flow Graph	Dorf-Bishop Ch. 2.6-2.7	
Oct 06	Sensitivity, Transient and Steady-state Response	Dorf-Bishop Ch. 4.2-4.7	HW2
Oct 11	Test Signals, Second-order System	Dorf-Bishop Ch. 5.2-5.4	
Oct 13	Root Location, Steady-state Error, Performance Indices	Dorf-Bishop Ch. 5.5-5.7	HW3
Oct 18	Routh-Hurwitz Stability, Relative Stability	Dorf-Bishop Ch. 6.2-6.4	
Oct 20	Catch up		
Oct 25	Midterm Exam		
Oct 27	Root Locus	Dorf-Bishop Ch. 7.2-7.3	
Nov 01	Parameter Design, Sensitivity	Dorf-Bishop Ch. 7.4-7.5	
Nov 03	PID Control, Negative Gain Root Locus	Dorf-Bishop Ch. 7.6-7.7	HW4
Nov 08	Frequency Response	Dorf-Bishop Ch. 8.2-8.3	
Nov 10	Frequency Performance Specifications	Dorf-Bishop Ch. 8.4	HW5
Nov 15	Log-Magnitude and Phase Diagrams	Dorf-Bishop Ch. 8.5	
Nov 17	Catch up		
Nov 22	s-Plane Contours, Nyquist Criterion	Dorf-Bishop Ch. 9.2-9.3	
Nov 24	Nyquist Relative Stability	Dorf-Bishop Ch. 9.4	HW6
Nov 29	Performance Criteria, System Bandwidth	Dorf-Bishop Ch. 9.5-9.6	
Dec 01	Review		
Dec 09	Final Exam		

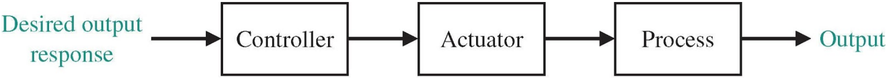
- Check the course website for updates

Control System

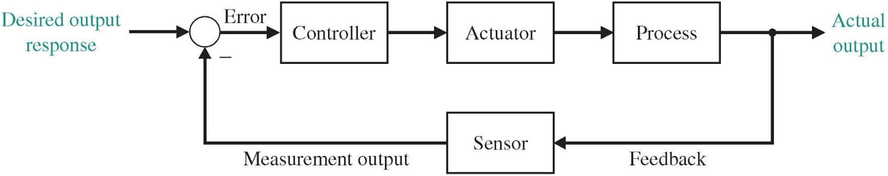
- A **control system** is an interconnection of components that provides a desired response
- Modern control systems include physical and cyber components
- A **physical component** is a mechanical, electrical, fluid, or thermal device acting as a sensor, actuator, or embedded system component
  - A **sensor** is a device that provides measurements of a signal of interest
  - An **actuator** is a device that alters the configuration of the system or its environment
- A **cyber component** is a software node that executes a specific function
- **Control system engineering** focuses on:
  - modeling cyberphysical systems
  - designing controllers that achieve desired system performance characteristics, such as stability, transient and steady-state tracking, rejection of external disturbances and robustness to modeling uncertainties

Open-loop vs Closed-loop Control Systems

- An **open-loop control system** utilizes a controller without measurement feedback of the system output

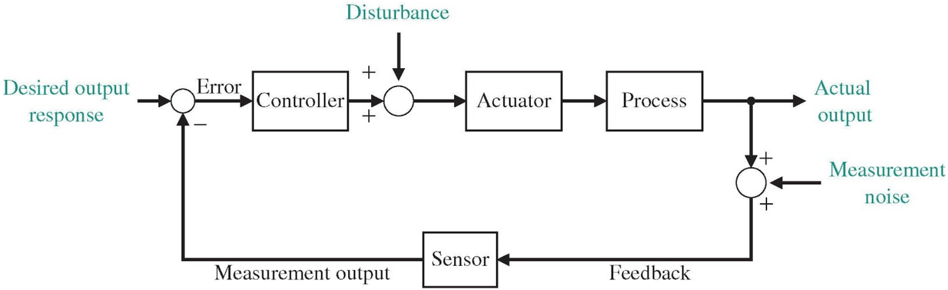


- A **closed-loop control system** utilizes a controller with measurement feedback of the system output



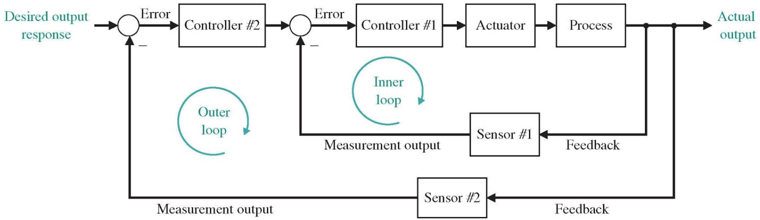
Disturbances

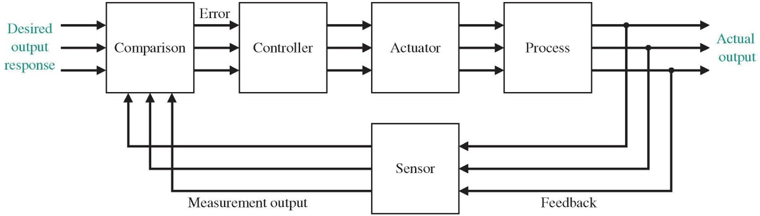
- A closed-loop control system controls the actuators to reduce the **error** between the desired system output and the measured system output
- Unlike open-loop control systems, closed-loop control systems may attenuate the effects of **process noise** (disturbance), **measurement noise**, and **modeling errors**



Multi-loop Multi-variable Control Systems

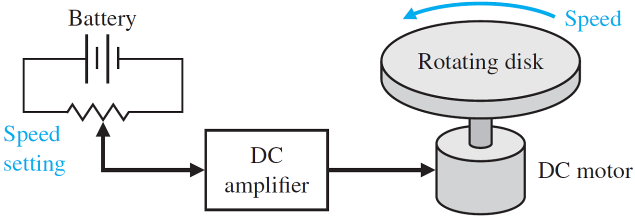
- Modern control systems involve multiple measurement and control variables and multiple feedback loops



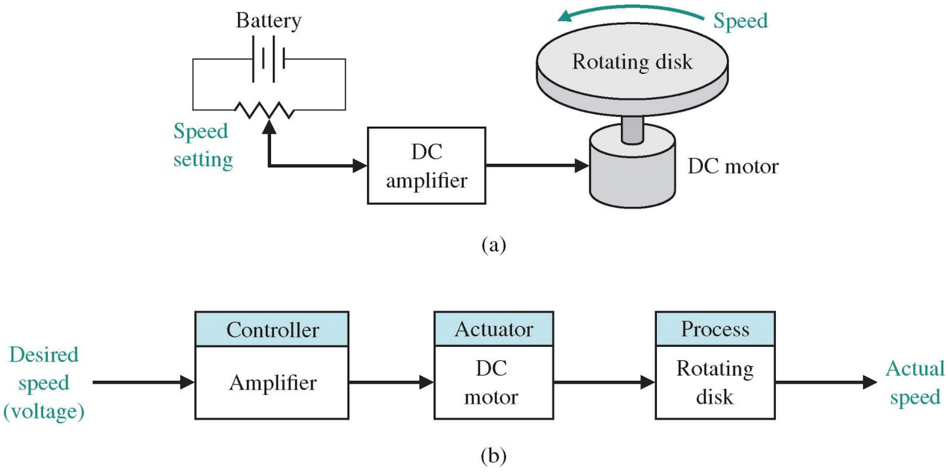


Example: Rotating Disk Speed Control

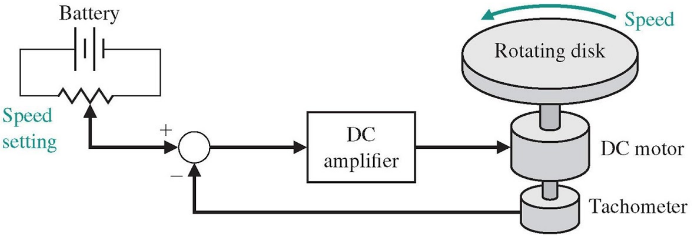
- Line-cell imaging in biomedical applications use spinning disk conformal microscopes
- Objective: design a controller for a rotating disk system to ensure the speed of rotation is within a specified percentage of a desired speed
- System components:
  - DC motor actuator: provides speed proportional to the applied voltage
  - Battery source: provides voltage proprotional to the desired speed
  - DC amplifier: amplifies the battery voltage to meet the motor volatage requirements
  - Tachometer: provides output voltage proprotional to the speed of its shaft



Open-Loop Rotating Disk System



Closed-Loop Rotating Disk System



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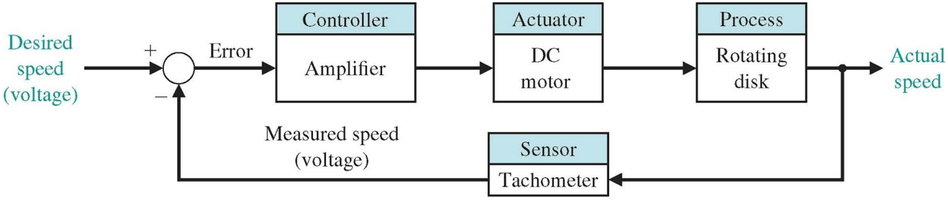

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Control System Analysis

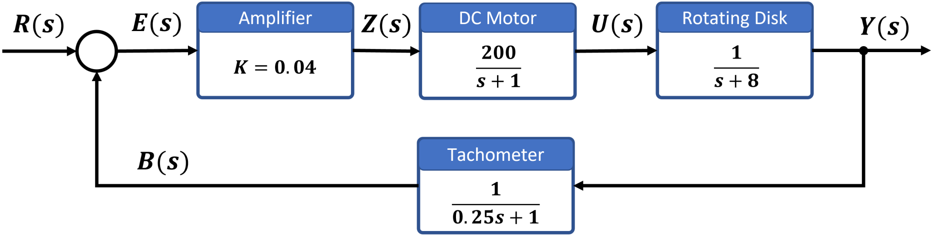
- System elements will be described using linear constant coefficient ordinary differential equations
- Instead of solving the differential equations in the time domain, we will use Laplace transform to study the system behavior in the complex plane

- Time domain:
  - Desired Speed: $r(t)$
  - Amplifier: $z(t) = Kr(t)$
  - DC Motor: $\dot{u}(t) + u(t) = 200z(t)$
  - Rotating Disk: $\dot{y}(t) + 8y(t) = u(t)$

- Laplace domain:
  - Desired Speed: $R(s)$
  - Amplifier: $Z(s) = KR(s)$
  - DC Motor: $U(s) = \frac{200}{s+1}Z(s)$
  - Rotating Disk: $Y(s) = \frac{1}{s+8}U(s)$

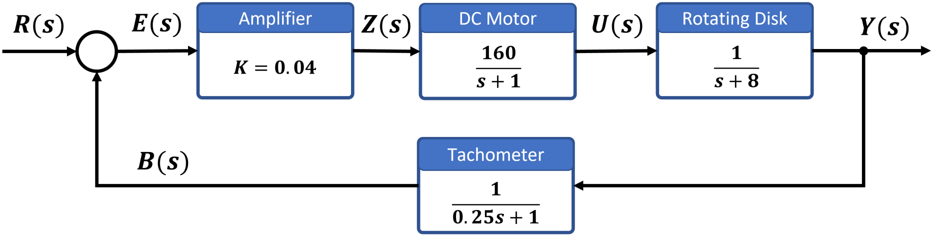
- We will study how to choose the amplifier gain K to ensure that system output $y(t)$ tracks the desired reference input $r(t)$

Nominal Rotating Disk System



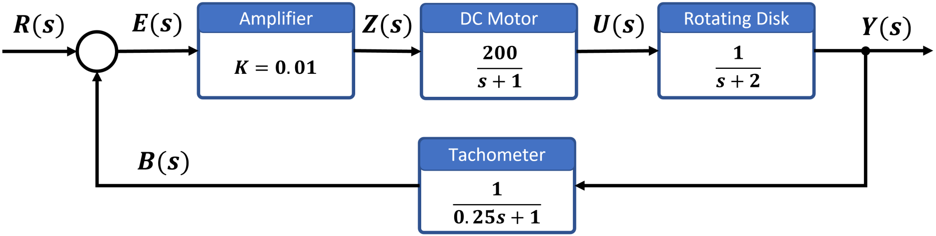
- A nominal model aims to capture the system behavior accurately but parameter errors or disturbances might be present
- Closed-loop/feedback control becomes important when there are parameter errors and disturbances

Low Gain Rotating Disk System



- The DC motor gain might be different in the real system (e.g., 160) compared to the nominal model (e.g., 200)

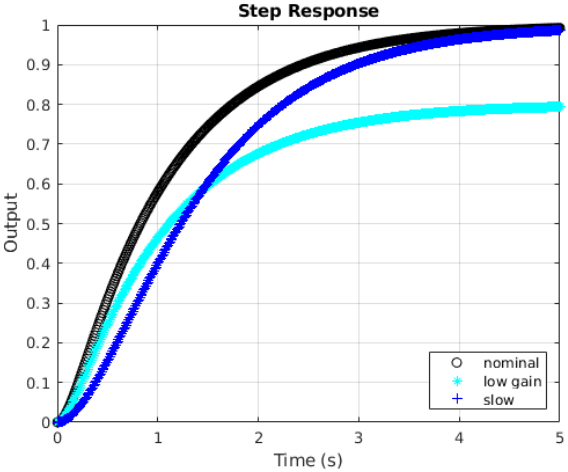
Slow Rotating Disk System



- The disk might rotate slower in the real system (e.g., $\dot{y}(t) + 2y(t) = u(t)$) compared to the nominal model (e.g., $\dot{y}(t) + 8y(t) = u(t)$)

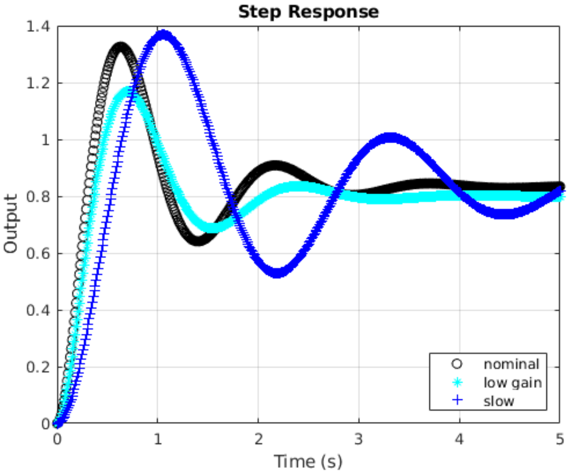
Open-loop Step Response

- Without feedback, the real system response might be different than what was planned



Closed-loop Step Response

- Feedback improves the sensitivity to parameter errors and disturbances
- Despite the advantages, feedback architectures need to be designed carefully to avoid oscillations and steady-state error



Overview of Control System Modeling

- Mathematical models of physical systems are key elements in the design and analysis of control systems
- Dynamic behavior is described by **ordinary differential equations** (ODEs)
- Linearization approximation of a nonlinear system is used to simplify the analysis of the system behavior
- **Laplace transform** methods describe the input-output relationship of a **linear time-invariant** (LTI) system in the form of a **transfer function**
- A transfer functions can represented as a **block diagram** or **signal-flow graph** to graphically depict the system interconnections

Differential Equations of Physical Systems

- Physical systems from different domains (electrical, mechanical, fluid, thermal) contain elements that share similar roles

Variable	Electrical	Mechanical	Fluid	Thermal
Through	Current	Force, Torque	Flow rate	Flow rate
Across	Voltage	Velocity	Pressure	Temperature
Inductive	Inductance	Inverse Stiffness	Inertia	–
Capacitive	Capacitance	Mass, Moment of Inertia	Capacitance	Capacitance
Resistive	Resistance	Friction	Resistance	Resistance

- Dynamic behavior is described by physical laws, such as Kirchhoff's laws or Newton's laws, enabling an ODE description of the system

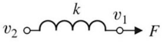
Through and Across Element Variables

Table 2.1 Summary of Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

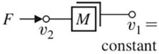
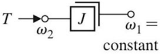
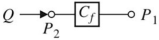
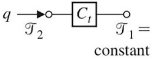
Inductive Elements

Table 2.2 Summary of Governing Differential Equations for Ideal Elements

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	

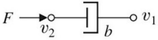
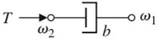
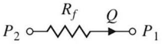
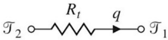
Capacitive Elements

Table 2.2 Summary of Governing Differential Equations for Ideal Elements

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Capacitive storage	Electrical capacitance	$i = C\dfrac{dv_{21}}{dt}$	$E = \dfrac{1}{2}Cv_{21}{}^2$	
	Translational mass	$F = M\dfrac{dv_2}{dt}$	$E = \dfrac{1}{2}Mv_2{}^2$	
	Rotational mass	$T = J\dfrac{d\omega_2}{dt}$	$E = \dfrac{1}{2}J\omega_2{}^2$	
	Fluid capacitance	$Q = C_f\dfrac{dP_{21}}{dt}$	$E = \dfrac{1}{2}C_fP_{21}{}^2$	
	Thermal capacitance	$q = C_t\dfrac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	

Resistive Elements

Table 2.2 Summary of Governing Differential Equations for Ideal Elements

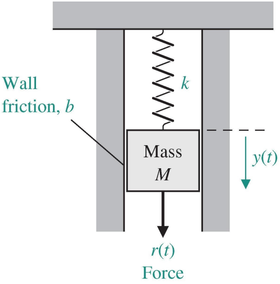
Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Energy dissipators	Electrical resistance	$i = \frac{1}{R} v_{21}$	$\mathcal{P} = \frac{1}{R} {v_{21}}^2$	
	Translational damper	$F = b v_{21}$	$\mathcal{P} = b {v_{21}}^2$	
	Rotational damper	$T = b \omega_{21}$	$\mathcal{P} = b {\omega_{21}}^2$	
	Fluid resistance	$Q = \frac{1}{R_f} P_{21}$	$\mathcal{P} = \frac{1}{R_f} {P_{21}}^2$	
	Thermal resistance	$q = \frac{1}{R_t} \mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t} \mathcal{T}_{21}$	

Spring-Mass-Damper Example

- The behavior of a spring-mass-damper system is described by Newton's second law:

$$M\frac{d^2y(t)}{dt^2} + \underbrace{b\frac{dy(t)}{dt}}_{\text{viscous damper}} + \underbrace{ky(t)}_{\text{spring force}} = \underbrace{r(t)}_{\text{input force}}$$

- The mass displacement $y(t)$ satisfies a second-order linear time-invariant (LTI) ordinary differential equation (ODE)



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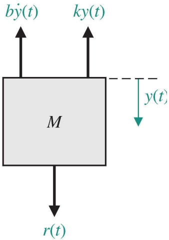

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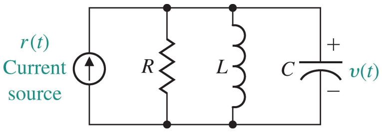
Parallel RLC Circuit Example

- The behavior of an electrical RLC circuit is described by Kirchhoff's current law:

$$r(t) = i_R(t) + i_L(t) + i_C(t)$$



- Parallel devices have the same voltage $v(t)$:
  - Resistor: $v(t) = Ri_R(t)$
  - Inductor: $v(t) = L\frac{di_L(t)}{dt}$
  - Capacitor: $i_C(t) = C\frac{dv(t)}{dt}$

- The inductor current $i_L(t)$ satisfies a second-order LTI ODE:

$$CL\frac{d^2 i_L(t)}{dt^2} + \frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = r(t)$$

Ordinary Differential Equations

- A **differential equation** is any equation involving a function and its derivatives
- A **solution to a differential equation** is any function that satisfies the equation
- An ***n*th-order linear ordinary differential equation** is:

$$a_n(t)\frac{d^n}{dt^n}y(t) + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1(t)\frac{d}{dt}y(t) + a_0(t)y(t) = u(t)$$

- If $u(t) = 0$, then the nth-order linear ODE is called **homogeneous**
- A solution $y(t)$ of an nth-order ODE that contains n arbitrary constants is called a **general solution**
- A solution $y_p(t)$ of an ODE that contains no arbitrary constants is called a **particular solution**

Existence and Uniqueness of Solutions

- An **initial value problem** is an ODE:

$$a_n(t)\frac{d^n}{dt^n}y(t)+a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t)+\ldots+a_1(t)\frac{d}{dt}y(t)+a_0(t)y(t)=u(t)$$

  together with initial value constraints:

$$y(t_0)=y_0, \quad \dot{y}(t_0)=y_1, \quad \ldots, \quad y^{(n-1)}(t_0)=y_{n-1}.$$

Theorem

Let $a_n(t)$, $a_{n-1}(t)$, …, $a_1(t)$, $a_0(t)$, and $u(t)$ be continuous on an interval $\mathcal{I} \subseteq \mathbb{R}$. Let $a_n(t) \neq 0$ for all $t \in \mathcal{I}$. Then, for any $t_0 \in \mathcal{I}$, a solution $y(t)$ of the initial value problem exists on $\mathcal{I}$ and is unique.

Superposition Principle for Homogeneous Linear ODEs

Let y_1, y_2, ..., y_k be solutions to a homogeneous nth-order linear ODE on an interval $\mathcal{I}$. Then, any linear combination:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_k y_k(t)$$

is also a solution, where c_1, c_2, ..., c_k are constants.

Superposition Principle for Nonhomogeneous Linear ODEs

For $i = 1, \ldots, k$, let $y_{p_i}(t)$ denote particular solutions to the linear ODEs:

$$a_n(t)\frac{d^n}{dt^n}y(t) + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1(t)\frac{d}{dt}y(t) + a_0(t)y(t) = u_i(t).$$

Then, $y_p(t) = c_1 y_{p_1}(t) + c_2 y_{p_2}(t) + \ldots + c_k y_{p_k}(t)$ is a particular solution of:

$$\begin{aligned} a_n(t)\frac{d^n}{dt^n}y(t) + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1(t)\frac{d}{dt}y(t) + a_0(t)y(t) \\ = c_1 u_1(t) + c_2 u_2(t) + \ldots + c_k u_k(t), \end{aligned}$$

where c_1, c_2, ..., c_k are constants.

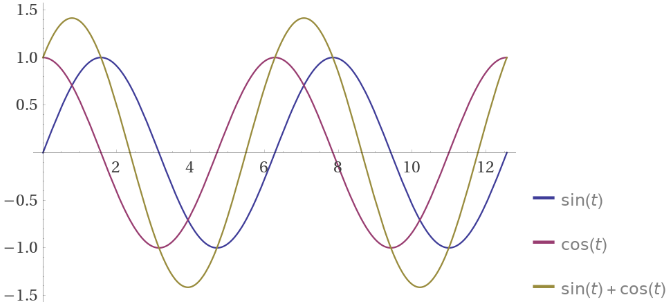
Superposition Example

- Consider the homogeneous linear ODE: $\frac{d^2}{dt^2}y(t) + y(t) = 0$
- Two particular solutions are:

$$y_1(t) = \cos(t) \qquad \frac{d^2}{dt^2}\cos(t) = -\cos(t)$$

$$y_2(t) = \sin(t) \qquad \frac{d^2}{dt^2}\sin(t) = -\sin(t)$$

- Then, any linear combination $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is also a solution



State Space Model

- Define variables:

$$x_1(t) = y(t), \qquad x_2(t) = \frac{d}{dt}y(t), \qquad \ldots, \qquad x_n(t) = \frac{d^{n-1}}{dt^{n-1}}y(t)$$

- The linear ODE specifies the following relationships:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
&\vdots \\
\dot{x}_{n-1}(t) &= x_n(t) \\
\dot{x}_n(t) &= -\frac{a_0(t)}{a_n(t)}x_1(t) - \frac{a_1(t)}{a_n(t)}x_2(t) - \cdots - \frac{a_{n-1}(t)}{a_n(t)}x_n(t) + \frac{1}{a_n(t)}u(t)
\end{aligned}$$

State Space Model

- Let $\mathbf{x}(t) := \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}^\top$ be a vector called **system state**

- A **state space model** of the linear ODE is obtained by re-writing the equations in vector-matrix form:

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{a_0(t)}{a_n(t)} & -\frac{a_1(t)}{a_n(t)} & \cdots & -\frac{a_{n-1}(t)}{a_n(t)} \end{bmatrix}}_{\mathbf{A}(t)} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{a_n(t)} \end{bmatrix}}_{\mathbf{b}(t)} u(t)$$

- ECE 171B will focus on time-domain analysis of state space models $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t)$

Linearization

- In practice, many systems may be described by a **nonlinear ODE**:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t), t)$$

- A nonlinear system can be approximated with a linear one by modeling its behavior in a restricted operational domain
- Linearization is based on a Taylor series expansion around a nominal state-input trajectory
- The **Taylor series expansion** of an infinitely differentiable function $f(x)$ around a nominal point $\bar{x}$ is:

$$f(x) = f(\bar{x}) + \frac{1}{1!}f'(\bar{x})(x-\bar{x}) + \frac{1}{2!}f''(\bar{x})(x-\bar{x})^2 + \frac{1}{3!}f'''(\bar{x})(x-\bar{x})^3 + \cdots$$

Linearization

- Linearization of $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t), t)$
- A nominal trajectory $\bar{\mathbf{x}}(t)$ is obtained from a nominal initial state $\bar{\mathbf{x}}_0$ with nominal reference input $\bar{u}(t)$:

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{f}(\bar{\mathbf{x}}(t), \bar{u}(t), t), \qquad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0$$

- An operational domain is specified as the deviation $(\tilde{\mathbf{x}}(t), \tilde{u}(t))$ around the nominal state-input trajectory $(\bar{\mathbf{x}}(t), \bar{u}(t))$:

$$\mathbf{x}(t) = \bar{\mathbf{x}}(t) + \tilde{\mathbf{x}}(t) \qquad u(t) = \bar{u}(t) + \tilde{u}(t)$$

- The nonlinear function $\mathbf{f}$ is linearized around $(\bar{\mathbf{x}}(t), \bar{u}(t))$ using the first two terms from its **Taylor series expansion**:

$$\underbrace{\mathbf{f}(\mathbf{x}, u, t)}_{\dot{\bar{\mathbf{x}}}+\dot{\tilde{\mathbf{x}}}} \approx \underbrace{\mathbf{f}(\bar{\mathbf{x}}, \bar{u}, t)}_{\dot{\bar{\mathbf{x}}}} + \underbrace{\left[\frac{d}{d\mathbf{x}}\mathbf{f}(\bar{\mathbf{x}}, \bar{u}, t)\right]}_{\mathbf{A}(t)} \underbrace{(\mathbf{x} - \bar{\mathbf{x}})}_{\tilde{\mathbf{x}}} + \underbrace{\left[\frac{d}{du}\mathbf{f}(\bar{\mathbf{x}}, \bar{u}, t)\right]}_{\mathbf{b}(t)} \underbrace{(u - \bar{u})}_{\tilde{u}}$$

- Linearized system: $\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}(t)\tilde{\mathbf{x}}(t) + \mathbf{b}(t)\tilde{u}(t)$