ECE171A: Linear Control System Theory
Lecture 2: Transfer Function

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LTI ODE Control Systems

- ECE 171A will focus on systems with components modeled as linear time-invariant (LTI) ordinary differential equations (ODEs):

$$a_n \frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = u(t)$$

  with forcing function $u(t)$ of the form:

$$u(t) = b_{n-1} \frac{d^{n-1}}{dt^{n-1}} r(t) + b_{n-2} \frac{d^{n-2}}{dt^{n-2}} r(t) + \ldots + b_0 r(t)$$

- When clear from the context, we may use short-hand derivative notation:

$$\frac{d}{dt} y(t) \equiv \dot{y}(t) \qquad \frac{d^2}{dt^2} y(t) \equiv \ddot{y}(t)$$
$$\frac{d^3}{dt^3} y(t) \equiv \dddot{y}(t) \qquad \frac{d^n}{dt^n} y(t) \equiv y^{(n)}(t)$$

- LTI ODEs can be analyzed using a Laplace transform

Laplace Transform

- The Laplace transform $\mathcal{L}$ converts an LTI ODE in the time domain into a linear algebraic equation in the complex domain
- Example:

$$\begin{array}{ccc} \ddot{y}(t) + y(t) = 0 & \xrightarrow{\mathcal{L}} & s^2 Y(s) - sy(0) - \dot{y}(0) + Y(s) = 0 \\ & & \downarrow \\ y(t) = y(0)\cos(t) + \dot{y}(0)\sin(t) & \xleftarrow{\mathcal{L}^{-1}} & Y(s) = \dfrac{sy(0) + \dot{y}(0)}{s^2 + 1} \end{array}$$

- Advantage: instead of an ODE, we get an algebraic equation (easier to solve), e.g., differentiation in t becomes multiplication by s, integration in t becomes division by s, convolution becomes multiplication
- Drawback: instead of a scalar variable t, we need to work with a complex variable $s = \sigma + j\omega$

Complex Numbers $\mathbb{C}$

- A **complex number** is a number of the form $s = \sigma + j\omega$, where σ and ω are real numbers and $j = \sqrt{-1}$

- The **space of complex numbers** is denoted by $\mathbb{C}$

- Euclidean coordinates:
  - The **real part** of $s = \sigma + j\omega$ is $\text{Re}(s) = \sigma$
  - The **imaginary part** of $s = \sigma + j\omega$ is $\text{Im}(s) = \omega$

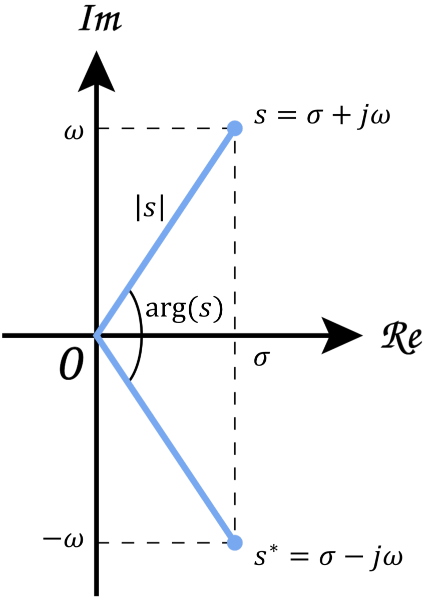
- Polar coordinates:
  - The **magnitude** of $s = \sigma + j\omega$ is $|s| = \sqrt{\sigma^2 + \omega^2}$
  - The **phase** of $s = \sigma + j\omega$ is $\arg(s) = \text{atan2}(\text{Im}(s), \text{Re}(s))$

- The **complex conjugate** of $s = \sigma + j\omega$ is $s^* = \sigma - j\omega$

- Example:

$$\frac{1}{s} = \frac{s^*}{ss^*} = \frac{s^*}{|s|^2} = \frac{\sigma}{\sigma^2 + \omega^2} - j\frac{\omega}{\sigma^2 + \omega^2}$$

Complex Numbers $\mathbb{C}$



Complex Polynomial

- A **complex polynomial** of order n is a function $a : \mathbb{C} \mapsto \mathbb{C}$:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_2 s^2 + a_1 s + a_0$$

  where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ are constants.

- A **root** of a complex polynomial $a(s)$ is a number $\lambda \in \mathbb{C}$ such that:

$$a(\lambda) = 0$$

- A root λ of **multiplicity** m of a complex polynomial $a(s)$ satisfies:

$$\lim_{s \to \lambda} \frac{a(s)}{(s - \lambda)^m} < \infty$$

Complex Polynomial

- **Fundamental theorem of algebra**: a polynomial of degree n has exactly n roots, counting multiplicities

- A polynomial $a(s)$ can be expressed in **factored form**:

$$a(s) = a_n s^n + \ldots + a_0 = a_n(s - \lambda_1) \cdots (s - \lambda_n)$$

  where $\lambda_1, \ldots, \lambda_n$ are the n roots of $a(s)$

- The roots of a complex polynomial with real coefficients are either real or come in complex conjugate pairs

- **Vieta's formulas** relate the polynomial coefficients a_i to its roots λ_i:

$$\sum_{i=1}^{n} \lambda_i = -\frac{a_{n-1}}{a_n} \qquad \prod_{i=1}^{n} \lambda_i = (-1)^n \frac{a_0}{a_n} \qquad \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j} = (-1)^k \frac{a_{n-k}}{a_n}$$

Rational Function

- A **rational function** $F : \mathbb{C} \mapsto \mathbb{C}$ is a ratio of two polynomials:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

- Rational functions are closed under addition, subtraction, multiplication, division (except by 0)

- The **characteristic equation** of a rational function $F(s)$ is:

$$a(s) = 0$$

- A **zero** $z \in \mathbb{C}$ of a rational function $F(s)$ is a root of the numerator: $b(z) = 0$

- A **pole** $p \in \mathbb{C}$ of a rational function $F(s)$ is a root of the characteristic equation: $a(p) = 0$

Pole-Zero Map

- The **pole-zero form** of a rational function $F(s)$ is:

$$F(s) = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0} = k\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

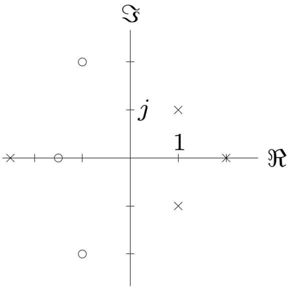
  where $k = b_m/a_n$, $z_1, \ldots, z_m$ are the zeros of $F(s)$, and $p_1, \ldots, p_n$ are the poles of $F(s)$

- A **pole-zero map** is a plot of the poles and zeros of a rational function $F(s)$ in the s-domain:

  - Example:

$$F(s) = k\frac{(s + 1.5)(s + 1 + 2j)(s + 1 - 2j)}{(s + 2.5)(s - 2)(s - 1 - j)(s - 1 + j)}$$

  - $\times$ = pole; $\circ$ = zero; k = not available



Partial Fraction Expansion

- Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

  is **strictly proper** ($m < n$) and has no repeated poles (all roots of $a(s)$ have multiplicity one)

- The **partial fraction expansion** of $F(s)$ is:

$$F(s) = \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n}$$

  where $\lambda_1, \ldots, \lambda_n$ and $r_1, \ldots, r_n$ are the poles and residues of $F(s)$

- The **residue** r_i associated with pole p_i is:

$$r_i = \lim_{s \to p_i} (s - p_i) F(s)$$

Partial Fraction Expansion (repeated poles)

- Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n (s-p_1)^{m_1} \cdots (s-p_k)^{m_k}}$$

  is **strictly proper** and has poles $p_1, \ldots, p_k$ with multiplicities $m_1, \ldots, m_k$

- The **partial fraction expansion** of $F(s)$ is:

$$\begin{aligned} F(s) =& \frac{r_{1,m_1}}{(s-p_1)^{m_1}} + \frac{r_{1,m_1-1}}{(s-p_1)^{m_1-1}} + \cdots + \frac{r_{1,1}}{s-p_1} \\ &+ \frac{r_{2,m_2}}{(s-p_2)^{m_2}} + \frac{r_{2,m_2-1}}{(s-p_2)^{m_2-1}} + \cdots + \frac{r_{2,1}}{s-p_2} \\ &+ \cdots \\ &+ \frac{r_{k,m_k}}{(s-p_k)^{m_k}} + \frac{r_{k,m_k-1}}{(s-p_k)^{m_k-1}} + \cdots + \frac{r_{k,1}}{s-p_k} \end{aligned}$$

- The **residue** r_{i,m_i-j} associated with pole p_i is:

$$r_{i,m_i-j} = \lim_{s \to p_i} \frac{1}{j!} \frac{d^j}{ds^j} \left[(s-p_i)^{m_i} F(s) \right]$$

Partial Fraction Expansion (nonproper rational function)

- Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

  is proper $m \leq n$ or nonproper $m > n$

- The numerator $b(s)$ can be divied by the denominator $a(s)$ to obtain:

$$F(s) = \frac{b(s)}{a(s)} = c(s) + \frac{d(s)}{a(s)}$$

  where $c(s)$ is of order $m - n$ and $d(s)$ is of order $k < n$

- $d(s)/a(s)$ is now strictly proper and has a partial fraction expansion

Example

- Consider $F(s) = \frac{2s+1}{3s^2+2s+1}$
- $F(s)$ has one zero: $z = -\frac{1}{2}$
- The roots of a quadratic polynomial $a(s) = a_2 s^2 + a_1 s + a_0$ are:

$$s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

- $F(s)$ has two conjugate poles: $p_1 = -\frac{1}{3} + j\frac{\sqrt{2}}{3}$ and $p_2 = -\frac{1}{3} - j\frac{\sqrt{2}}{3}$:

$$F(s) = \frac{2(s-z)}{3(s-p_1)(s-p_2)}$$

Complex Rational Function Example

- The residue associated with p_1 is:

$$r_1 = \lim_{s \to p_1} (s - p_1) F(s) = \lim_{s \to p_1} \frac{2(s - z)}{3(s - p_2)} = \frac{2(p_1 + 1/2)}{3(p_1 - p_2)}$$

$$= \frac{2(p_1 + 1/2)}{j2\sqrt{2}} = -j\frac{\sqrt{2}}{2}\left(\frac{1}{6} + j\frac{\sqrt{2}}{3}\right) = \frac{1}{3} - j\frac{\sqrt{2}}{12}$$

- Residues associated with complex conjugate poles are also complex conjugate!

- The residue associated with $p_2 = p_1^*$ is $r_2 = r_1^* = \frac{1}{3} + j\frac{\sqrt{2}}{12}$

- The partial fraction expansion of $F(s)$ is:

$$F(s) = \frac{r_1}{(s - p_1)} + \frac{r_2}{(s - p_2)}$$

Laplace Transform

- The **Laplace transform** $F(s)$ of a function $f(t)$ is:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^{\infty} f(t)e^{-st}dt$$

where $s = \sigma + j\omega$ is a complex variable

- The **inverse Laplace transform** $f(t)$ of a function $F(s)$ is:

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds$$

$$\underset{\text{residue theorem}}{\overset{\text{Cauchy's}}{=\!=\!=\!=\!=}} \sum_{\text{poles of } F(s)} \text{residues of } F(s)e^{st}$$

where σ is greater than the real part of all singularities of $F(s)$

Laplace Transform Example

- Compute the Laplace transform of $f(t) = e^{at}$:

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt = -\frac{1}{(s-a)} e^{-(s-a)t}\Big|_{t=0}^{t=\infty}$$

$$\xlongequal[\text{Re}(s)>a]{\text{Require}} 0 - \left(-\frac{1}{(s-a)} e^0\right) = \frac{1}{s-a}$$

- Compute the inverse Laplace transform of $F(s) = \frac{1}{s-a}$:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{s-a} e^{st} ds = \frac{e^{at}}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{s-a} e^{(s-a)t} ds$$

$$\xlongequal[\text{residue theorem}]{\text{Cauchy's}} e^{at} \lim_{s \to a} \left\{(s-a)\frac{1}{s-a} e^{(s-a)t}\right\} = e^{at}$$

Initial and Final Value Theorems

Initial Value Theorem

Suppose that $f(t)$ has a Laplace transform $F(s)$. Then:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

Final Value Theorem

Suppose that $f(t)$ has a Laplace transform $F(s)$. Suppose that every pole of $F(s)$ is either in the open left-half plane or at the origin of $\mathbb{C}$. Then:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Laplace Transform Properties

	t domain	s domain
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
convolution	$(f * g)(t)$	$F(s)G(s)$
multiplication	$f(t)g(t)$	$\frac{1}{2\pi j} \int_{Re(\sigma)-j\infty}^{Re(\sigma)+j\infty} F(\sigma)G(s-\sigma)d\sigma$
scaling, $a > 0$	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
s-domain derivative	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
time-domain derivative	$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
s-domain integarion	$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)d\sigma$
time-domain integarion	$\int_0^t f(\tau)d\tau = (H * f)(t)$	$\frac{1}{s}F(s)$
s-domain shift	$e^{at}f(t)$	$F(s-a)$
time-domain shift, $a > 0$	$f(t-a)H(t-a)$	$e^{-as}F(s)$

- Heaviside step function $H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$
- Convolution: $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

$f(t)$	$F(s)$
$\int_{-\infty}^{t} f(t)\, dt$	$\frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} f(t)\, dt$
Impulse function $\delta(t)$	1
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{1}{\omega}[(\alpha-a)^2+\omega^2]^{1/2}e^{-at}\sin(\omega t+\phi)$, $\phi = \tan^{-1}\frac{\omega}{\alpha - a}$	$\frac{s+\alpha}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}t, \zeta < 1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$\frac{1}{a^2+\omega^2}+\frac{1}{\omega\sqrt{a^2+\omega^2}}e^{-at}\sin(\omega t-\phi)$, $\phi = \tan^{-1}\frac{\omega}{-a}$	$\frac{1}{s[(s+a)^2+\omega^2]}$
$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$, $\phi = \cos^{-1}\zeta, \zeta < 1$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
$\frac{\alpha}{a^2+\omega^2}+\frac{1}{\omega}\left[\frac{(\alpha-a)^2+\omega^2}{a^2+\omega^2}\right]^{1/2}e^{-at}\sin(\omega t+\phi)$. $\phi = \tan^{-1}\frac{\omega}{\alpha-a} - \tan^{-1}\frac{\omega}{-a}$	$\frac{s+\alpha}{s[(s+a)^2+\omega^2]}$

Transfer Function

- Consider the LTI ODE with zero initial conditions:

$$a_0 y(t) + \sum_{i=1}^{n} a_i \frac{d^i}{dt^i} y(t) = b_0 r(t) + \sum_{i=1}^{n-1} b_i \frac{d^i}{dt^i} r(t)$$

- Laplace transform:

$$a_0 Y(s) + \sum_{i=1}^{n} a_i s^i Y(s) = b_0 R(s) + \sum_{i=1}^{n-1} b_i s^i R(s)$$

- **Transfer function**: ratio of the Laplace transform of the state variable to the Laplace transform of the input variable with zero initial conditions:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b(s)}{a(s)}$$

  where $a(s) = \sum_{i=0}^{n} a_i s^i$ and $b(s) = \sum_{i=0}^{n-1} b_i s^i$

- The transfer function of this LTI ODE is a **strictly proper rational function**

System Total Response

- Superposition: the general solution $y(t)$ of a nonhomogeneous linear ODE can be obtained as the sum of one particular solution $y_p(t)$ and the general solution $y_h(t)$ to the associated homogeneous ODE:

$$y(t) = y_h(t) + y_p(t)$$

- The complete response of an LTI ODE system consist of a natural response (determined by the initial conditions) plus a forced response (determined by the input):

$$Y(s) = \underbrace{\frac{c(s)}{a(s)}}_{\text{natural response}} + \underbrace{\frac{b(s)}{a(s)}R(s)}_{\text{forced response}}$$

- If the reference input $R(s)$ is a rational function, then the output $Y(s)$ is also a rational function

Spring-Mass-Damper Example

- Consider the spring-mass-damper system:

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

- Laplace transform:

$$M(s^2Y(s) - sy(0) - \dot{y}(0)) + b(sY(s) - y(0)) + kY(s) = R(s)$$

- Natural response (set $r(t) \equiv 0$):

$$Y(s) = \frac{My(0)s + by(0) + M\dot{y}(0)}{Ms^2 + bs + k}$$

- Transfer function (set $y(0) = \dot{y}(0) = 0$):

$$T(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}$$

Spring-Mass-Damper Example

- Consider the natural response with $k/M = 2$ and $b/M = 3$:

$$Y(s) = \frac{(s+3)y(0) + \dot{y}(0)}{s^2 + 3s + 2} = \frac{(s+3)y(0) + \dot{y}(0)}{(s+1)(s+2)}$$
$$= \frac{2y(0) + \dot{y}(0)}{s+1} - \frac{y(0) + \dot{y}(0)}{s+2}$$

- Poles: $p_1 = -1$ and $p_2 = -2$
- Zeros: $z_1 = -\frac{\dot{y}(0)}{y(0)} - 3$
- Residues:

$$r_1 = \left.\frac{(s+3)y(0) + \dot{y}(0)}{(s+2)}\right|_{s=-1} = 2y(0) + \dot{y}(0)$$

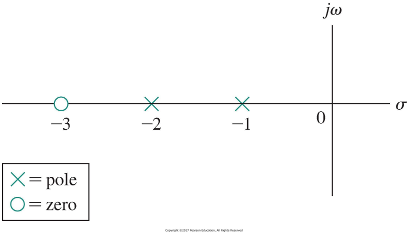
$$r_2 = \left.\frac{(s+3)y(0) + \dot{y}(0)}{(s+1)}\right|_{s=-2} = -y(0) - \dot{y}(0)$$

Spring-Mass-Damper Pole-Zero Map

- Let the initial conditions of the spring-mass-damper system be $y(0) = 1$ and $\dot{y}(0) = 0$

- The poles and zeros are:

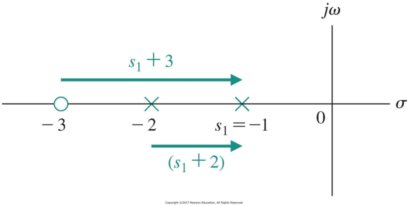
$$p_1 = -1, \quad p_2 = -2, \quad z_1 = -3$$



- The residues are:

$$r_1 = \left.\frac{(s+3)}{(s+2)}\right|_{s=-1} = 2$$

$$r_2 = \left.\frac{(s+3)}{(s+1)}\right|_{s=-2} = -1$$



Spring-Mass-Damper Response

- The time-domain response of the spring-mass-damper system can be obtained using an inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{2y(0)+\dot{y}(0)}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{y(0)+\dot{y}(0)}{s+2}\right\}$$
$$= \left(2y(0)+\dot{y}(0)\right)e^{-t} - \left(y(0)+\dot{y}(0)\right)e^{-2t}$$

- The **steady-state** response can be obtained via the Final Value Thm:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 0$$

Second-order ODE System

- The spring-mass-damper system is an example of a second-order ODE:

$$\frac{1}{\omega_n^2}\frac{d^2y(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy(t)}{dt} + y(t) = 0$$

  with **natural frequency** $\omega_n = \sqrt{k/M}$ and **damping ratio** $\zeta = b/(2\sqrt{kM})$

- The s-domain response is:

$$Y(s) = \frac{(s + 2\zeta\omega_n)y(0) + \dot{y}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Characteristic equation $a(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

Second-order System Poles

- The system response is determined by the poles:
  - **Overdamped** ($\zeta > 1$): the poles are real:

$$p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \qquad p_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

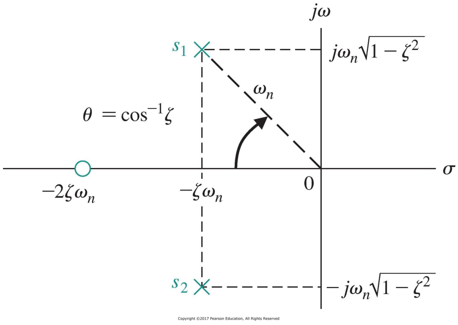
  - **Critically damped** ($\zeta = 1$): the poles are repeated and real:

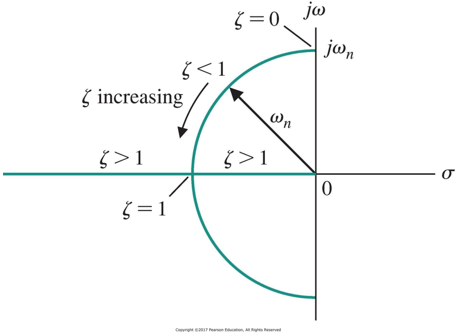
$$p_1 = p_2 = -\omega_n$$

  - **Underdamped** ($\zeta < 1$): the poles are complex:

$$p_1 = -\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2} \qquad p_2 = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$$

Spring-Mass-Damper Locus of Roots





- s-domain plot of the poles ($\times$) and zeros ($\circ$) of $Y(s)$ with $\dot{y}(0) = 0$
- For constant ω_n, as ζ varies, the complex conjugate roots follow a circular locus
- The poles and zeros can be expressed either in Euclidean coordinates or Polar coordinates (e.g., magnitude ω_n and angle $\theta = \cos^{-1}(\zeta)$)

Spring-Mass-Damper Response

- The time domain response can be obtained by determining the residues and applying an inverse Laplace transform:
  - **Overdamped** ($\zeta > 1$):

$$y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t}$$

    where $p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$, $p_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$, $r_1 = \frac{p_2 y(0) + \dot{y}(0)}{p_2 - p_1}$, and $r_2 = -\frac{p_1 y(0) + \dot{y}(0)}{p_2 - p_1}$
  - **Critically damped** ($\zeta = 1$):

$$y(t) = y(0)e^{-\omega_n t} + (\dot{y}(0) + \omega_n y(0))te^{-\omega_n t}$$

  - **Underdamped** ($\zeta < 1$):

$$y(t) = e^{-\zeta\omega_n t}\left(c_1 \cos(\omega_n\sqrt{1-\zeta^2}t) + c_2 \sin(\omega_n\sqrt{1-\zeta^2}t)\right)$$

    where $c_1 = y(0)$ and $c_2 = \frac{\dot{y}(0) + \zeta\omega_n y(0)}{\omega_n\sqrt{1-\zeta^2}}$

Spring-Mass-Damper Response with $\dot{y}(0) = 0$