ECE171A: Linear Control System Theory
Lecture 4: Sensitivity

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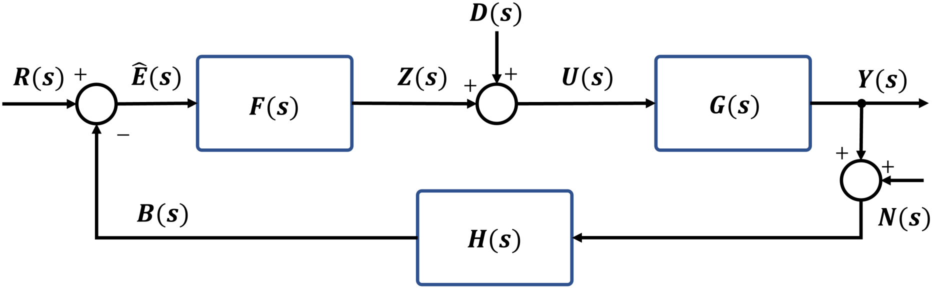
Feedback Control System with Disturbance and Noise



- Reference input: $R(s)$
- Output signal: $Y(s)$
- Feedback signal: $B(s)$
- Measured error: $\hat{E}(s)$
- Controller gain: $F(s)$
- Process gain: $G(s)$
- Process disturbance: $D(s)$
- Sensor noise: $N(s)$
- Feedback gain: $H(s)$
- Loop gain: $L(s) = -F(s)G(s)H(s)$

Output and Error Signals

- Measured error: $\hat{E}(s) = R(s) - B(s) = R(s) - H(s)Y(s) - H(s)N(s)$
- Output signal: $Y(s) = G(s)F(s)\hat{E}(s) + G(s)D(s)$
- Total response:

$$Y(s) = \underbrace{\frac{G(s)F(s)}{1 + G(s)F(s)H(s)}}_{\text{input effect}} R(s) + \underbrace{\frac{G(s)}{1 + G(s)F(s)H(s)}}_{\text{disturbance effect}} D(s) - \underbrace{\frac{G(s)F(s)H(s)}{1 + G(s)F(s)H(s)}}_{\text{noise effect}} N(s)$$

- True error: $E(s) = R(s) - H(s)Y(s)$

$$E(s) = \frac{1}{1 - L(s)} R(s) - \frac{H(s)G(s)}{1 - L(s)} D(s) - \frac{H(s)L(s)}{1 - L(s)} N(s)$$

Noise Sensitivity

- **Noise sensitivity**: $S(s) = \frac{1}{1-L(s)}$
- Complementary sensitivity: $C(s) = 1 - S(s) = \frac{-L(s)}{1-L(s)}$
- Error: $E(s) = S(s)R(s) - S(s)H(s)G(s)D(s) + C(s)H(s)N(s)$
- To minimize the error, we should design the control gain $F(s)$ so that both $S(s)$ and $C(s)$ are small
- Since $S(s) + C(s) = 1$, we cannot make both simultaneously small
- In practice:
  - the measurement noise $N(\sigma + j\omega)$ is associated with high frequencies ω
  - the disturbances $D(\sigma + j\omega)$ are associated with low frequencies ω
  - design $1 - L(\sigma + j\omega)$ to be **large at low frequencies** and **small at high frequencies** ω

Parameter Sensitivity

- Feedback control is useful for disturbance rejection and noise attenuation
- Feedback control is also useful for reducing the sensitivity to parameter variations in the process $G(s)$
- To consider parameter sensitivity, let $D(s) = N(s) = 0$ in the transfer function:
$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)F(s)}{1 + G(s)F(s)H(s)}$$
- Suppose that the process $G(s)$ undergoes a change $\Delta G(s)$ so that the true model is $G(s) + \Delta G(s)$
- What is the change $\Delta T(s)$ in the overall transfer function $T(s)$?

Parameter Sensitivity

- Since $T(s)$ and $G(s)$ might have different units, parameter sensitivity is defined as a percentage change in $T(s)$ over percentage change in $G(s)$
- **Parameter sensitivity** is the ratio of the incremental change in the system transfer function to the incremental change in the process transfer function:

$$\boxed{S_G^T(s) = \frac{dT(s)}{dG(s)}\frac{G(s)}{T(s)}} \approx \frac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)}$$

- Ideally, the parameter sensitivity should be small to allow robustness to changes in $G(s)$
- Conversely, the gain of elements with high sensitivity should be estimated well because minor mismatch might have a significant effect on the overall system transfer function. These are the system elements we should really be careful about.

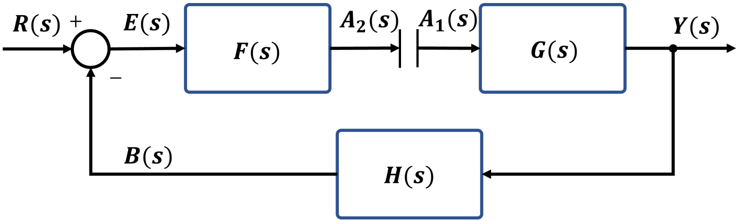
Return Difference

- Hendrik Bode was interested in measuring the effect of feedback on a specific element in a closed-loop control system

- Bode defined **return difference** as an impulse input (1 in the s-domain) at an element minus the loop transfer function $L(s)$ back to the element

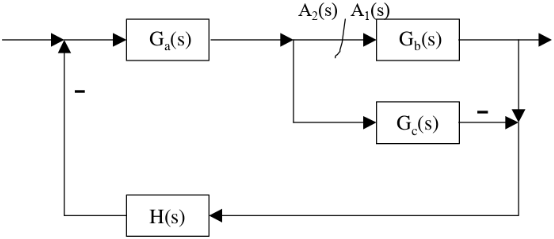
- To find the return difference:
  - open the feedback loop immediately prior to the element of interest
  - compute the transfer function $L(s) = \frac{A_2(s)}{A_1(s)}$ from the element input (A_1) back to the cut connection (A_2)
  - the return difference is $\rho(s) = 1 - L(s)$

Return Difference Example 1



- To find the return difference with respect to $G(s)$, cut the loop immediately prior to $G(s)$
- Compute the loop gain: $L(s) = \frac{A_2(s)}{A_1(s)} = -G(s)H(s)F(s)$
- Return difference: $\rho_G(s) = 1 - L(s) = 1 + G(s)H(s)F(s)$

Return Difference Example 2



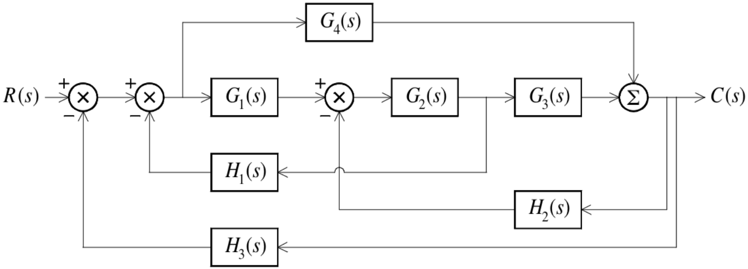
- To find the return difference with respect to $G_b(s)$, cut the loop immediately prior to $G_b(s)$
- Compute the loop gain via Mason's formula:

$$L(s) = \frac{G_1(s)\Delta_1(s)}{\Delta(s)} = \frac{-H(s)G_a(s)G_b(s)}{1 - H(s)G_a(s)G_c(s)}$$

- Return difference:

$$\rho_{G_b}(s) = 1 - L(s) = 1 + \frac{H(s)G_a(s)G_b(s)}{1 - H(s)G_a(s)G_c(s)} = \frac{1 + H(s)G_a(s)(G_b(s) - G_c(s))}{1 - H(s)G_a(s)G_c(s)}$$

Return Difference Example 3



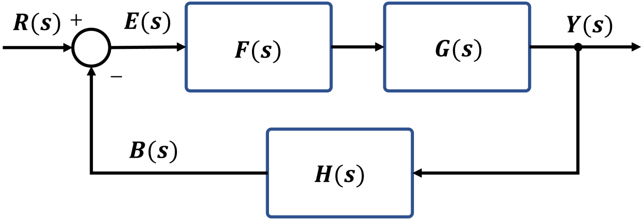
- To find the return difference with respect to $G_2(s)$, cut the loop immediately prior to $G_2(s)$
- Compute the loop gain via Mason's formula:

$$L(s) = \frac{-G_2(s)H_1(s)G_1(s) - G_2(s)G_3(s)H_2(s) - G_2(s)G_3(s)H_3(s)G_1(s) + G_2(s)H_1(s)G_4(s)H_2(s)}{1 + G_4(s)H_3(s)}$$

- Return difference: $\rho_{G_2}(s) = 1 - L(s)$

Parameter Sensitivity

- How is parameter sensitivity related to return difference?



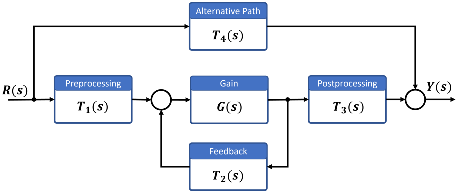
- For a control system with a single feedback loop, parameter sensitivity $S_G(s)$ is equal to the inverse of the return difference $\rho_G(s)$:

$$
\begin{aligned}
S_G(s) &= \frac{dT(s)}{dG(s)}\frac{G(s)}{T(s)} = \frac{d}{dG(s)}\left(\frac{G(s)F(s)}{1+G(s)F(s)H(s)}\right)\frac{G(s)}{T(s)} \\
&= \frac{F(s)}{(1+G(s)F(s)H(s))^2}\frac{G(s)}{T(s)} = \frac{1}{1+G(s)F(s)H(s)} \\
&= \frac{1}{1-L(s)} = \frac{1}{\rho_G(s)}
\end{aligned}
$$

Canonical Feedback Control Architecture

- Transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = T_4(s) + \frac{T_1(s)G(s)T_3(s)}{1 - G(s)T_2(s)}$$



- Sensitivity of $T(s)$ with respect to $G(s)$:

$$\frac{dT}{dG} = T_1 T_3 \left(\frac{1}{1 - GT_2} + \frac{GT_2}{(1 - GT_2)^2} \right) = \frac{T_1 T_3}{(1 - GT_2)^2}$$

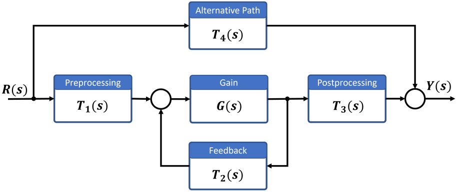
$$S_G^T = \frac{G}{T}\frac{dT}{dG} = \frac{G(1 - GT_2)}{T_4(1 - GT_2) + T_1 T_3 G} \frac{T_1 T_3}{(1 - GT_2)^2}$$

$$= \frac{GT_1 T_3}{T_4(1 - GT_2)^2 + T_1 T_3 G(1 - GT_2)}$$

Canonical Feedback Control Architecture

- Transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = T_4(s) + \frac{T_1(s)G(s)T_3(s)}{1 - G(s)T_2(s)}$$



- Sensitivity of $T(s)$ with respect to $G(s)$:

$$S_G^T(s) = \frac{G(s)T_1(s)T_3(s)}{T_4(s)(1 - G(s)T_2(s))^2 + T_1(s)T_3(s)G(s)(1 - G(s)T_2(s))}$$

- Note that $G(s)$ does not affect $T_4(s)$ in the transfer function. Consider only the portion that $G(s)$ affects:

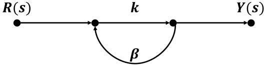
$$T'(s) = \frac{T_1(s)G(s)T_3(s)}{1 - G(s)T_2(s)}$$

- Letting $T_4(s) = 0$ in $S_G^T(s)$ shows that $S_G^{T'}(s)$ is the inverse of the return difference:

$$S_G^{T'}(s) = \frac{1}{1 - G(s)T_2(s)} = \frac{1}{\rho_G^{T'}(s)}$$

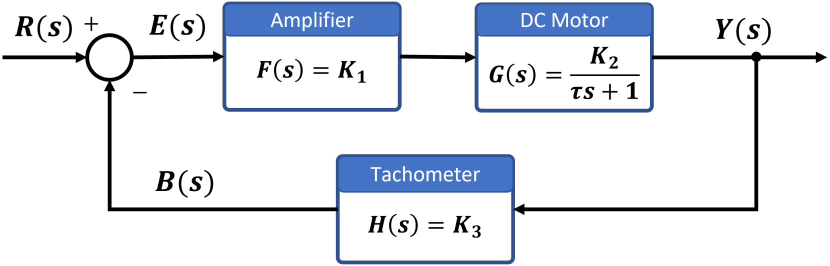
Example: Feedback OpAmp Sensitivity

- Consider a feedback amplifier with input voltage $R(s)$, feedforward gain k, feedback gain β, and output voltage $Y(s)$



- Transfer function: $T(s) = \frac{Y(s)}{R(s)} = \frac{k}{1-k\beta}$
- Return difference: $\rho_k = 1 - k\beta$
- Sensitivity wrt k: $S_k^T = \frac{1}{1-k\beta}$
- Sensitivity wrt β: $S_\beta^T = \frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta(1-k\beta)}{k}\frac{k^2}{(1-\beta k)^2} = \frac{k\beta}{1-k\beta}$
- Example design specification: the sensitivity with respect to k at $s = 0$ has to be less than 1%.
- When $k \approx 10^3$ and $\beta \approx -0.1$, then $S_k^T \approx 0$ and $S_\beta^T \approx -1$.
- When designing an OpAmp, the forward gain k can be arbitrary but we need to be careful with the design of β because it affects the response almost one-to-one.

Open-loop vs Closed-loop Response



- Open-loop response:

$$Y(s) = F(s)G(s)R(s) = \frac{K_1 K_2}{\tau s + 1} R(s)$$

- Closed-loop response:

$$Y(s) = \frac{F(s)G(s)}{1 + H(s)F(s)G(s)} R(s) = \frac{K_1 K_2}{\tau s + 1 + K_1 K_2 K_3} R(s)$$

Transient Response

- The **transient response** of a system is the response before the output settles at its final value
- The transient response is an important characteristic that must be adjusted until it is satisfactory, e.g., to prevent oscillations
- Open-loop response with $R(s) = 1/s$ using partial fraction expansion and inverse Laplace transform:

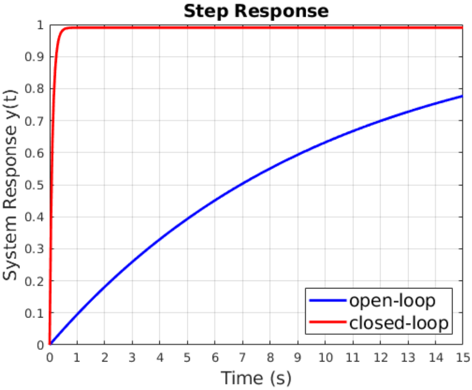
$$y(t) = \mathcal{L}^{-1}\left\{\frac{K_1K_2}{s(\tau s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{K_1K_2}{s} - \frac{K_1K_2}{s+1/\tau}\right\} = K_1K_2(1-e^{-t/\tau})$$

- Closed-loop response with $R(s) = 1/s$:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{K_1K_2}{s(\tau s+1+K_1K_2K_3)}\right\} = \frac{K_1K_2}{1+K_1K_2K_3}\left(1 - e^{-(1+K_1K_2K_3)t/\tau}\right)$$

Transient Response

- System parameters: $\tau = 10$, $K_3 = 1$
- Open-loop design: $K_1 K_2 = 1$
- Closed-loop design: $K_1 K_2 = 100$



Transient Response

- The closed-loop system responds 100 times faster than the open-loop system (open-loop pole: $1/\tau = 0.1$ vs closed-loop pole: $K_1 K_2 K_3/\tau = 10$)

- The closed-loop system requires a large control gain K_1 (e.g., powerful motor)

- The open-loop sensitivity to a variation in K_2 is: 1

- The closed-loop sensitivity to a variation in K_2 is:

$$S_{K_2}(s) = \frac{dT(s)}{dK_2}\frac{K_2}{T(s)} = \frac{s + 1/\tau}{s + (1 + K_1 K_2 K_3)/\tau}$$

- Using the values $\tau = 10$, $K_3 = 1$, $K_1 K_2 = 100$, we have $S_{K_2}(s) = \frac{s+0.1}{s+10.1}$. At low frequency, e.g., $s = 0 + j1$, $|S_{K_2}(s)| \approx 0.1$.

Steady-state Response

- The **steady-state response** of a system is the response after the output settles at its final value
- Error: $E(s) = R(s) - H(s)Y(s)$
- Open-loop error: $E_o(s) = (1 - H(s)F_o(s)G(s))R(s)$
- Closed-loop error: $E_c(s) = \frac{1}{1+H(s)F_c(s)G(s)}R(s)$
- Steady-state time-domain error via the Final Value Theorem:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

- Open-loop steady-state error with $R(s) = 1/s$:

$$\lim_{t\to\infty} e_o(t) = (1 - H(0)F_o(0)G(0))$$

- Closed-loop steady-state error with $R(s) = 1/s$:

$$\lim_{t\to\infty} e_c(t) = \frac{1}{1 + H(0)F_c(0)G(0)}$$

Advantages of Closed-loop Control

- By calibrating the control gain $F_o(0)$, we can make the open-loop steady-state error zero.
- What is the advantage of closed-loop control?
- During system operation it is inevitable that the parameters of $G(0)$ will change, making the open-loop steady-state error non-zero.
- In contrast, the closed-loop system monitors the system response and actively reduces the steady-state error even when $G(0)$ is changing.
- The **advantage of closed-loop control** is that it reduces the steady-state error resulting from parameter changes and calibration errors.

Advantages of Closed-loop Control

- For example, consider the same parameters as before: $H(0) = K_3 = 1$, $F_o(0)G(0) = 1$, $F_c(0)G(0) = 100$

- If the parameters are accurate:

$$\lim_{t\to\infty} e_o(t) = 0 \qquad \lim_{t\to\infty} e_c(t) = \frac{1}{101}$$

- If there is a 10% error in the parameters, e.g., $F_o'(0)G'(0) = 0.9$, $F_c'(0)G'(0) = 90$:

$$\lim_{t\to\infty} e_o'(t) = 0.1 \qquad \lim_{t\to\infty} e_c'(t) = \frac{1}{91}$$

- The precent change from the calibrated setting is:

$$\frac{|e_o(\infty) - e_o'(\infty)|}{|r(\infty)|}\times 100\% = 10\% \qquad \frac{|e_c(\infty) - e_c'(\infty)|}{|r(\infty)|}\times 100\% = 0.11\%$$

where $r(\infty) = 1$ is the steady-state value of the step reference input

Disadvantages of Closed-loop Control

- **Increased system complexity**: a sensing component is necessary, which also introduced sensor noise
- **Loss of gain**: the forward gain in a closed-loop system is smaller by a factor of $\frac{1}{1+H(s)F(s)G(s)}$ than the forward gain of an open-loop system. This is exactly the factor that reduces the sensitivity of the system to parameter variations and distrubances. Usually the gain in robustness from closed-loop control significantly outweights the loss of gain.
- **Potential for instability**: the introduction of closed-loop control may lead to system instability, even if the open-loop system is stable