ECE171A: Linear Control System Theory
Lecture 5: Test Signals and Steady-state Error

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Feedback Control System Performance Measures

- Advantage of feedback control systems: the ability to adjust the transient and steady-state response
- To design and analyze feedback control systems we must define and measure their transient and steady-state performance
- The response of the system to specific test input signals is evaluated according to several performance criteria:
  - Rise time
  - Percent overshoot
  - Settling time
  - Steady-state error
  - Sensitivity to disturbance and noise
  - Sensitivity to parameter variations

Test Signals

- The response of a system to specific input signals, called **test signals**, allows us to study the transient and stead-state response of the system
- Test signals:

Test Signal	$r(t)$	$R(s)$
Impulse	$r(t) = \delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0 \end{cases}$	$R(s) = 1$
Step	$r(t) = H(t) = \int_{-\infty}^{t} \delta(s) ds = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$	$R(s) = \frac{1}{s}$
Ramp	$r(t) = tH(t) = \begin{cases} t, & t \geq 0, \\ 0, & t < 0 \end{cases}$	$R(s) = \frac{1}{s^2}$
Parabola	$r(t) = \frac{t^2}{2} H(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0, \\ 0, & t < 0 \end{cases}$	$R(s) = \frac{1}{s^3}$

- Other test signals may be used: sine wave, square wave, periodic pulse

Test Signal Example

- Consider an open-loop system with gain $G(s)$:

$$Y(s) = G(s)R(s) \qquad G(s) = \frac{9}{s+10}$$

- The impulse response is obtained with $R(s) = 1$ and reveals the gain:

$$Y(s) = G(s)$$

- Transient impulse response:

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{G(s)\right\} = 9e^{-10t}$$

- Steady-state impulse response:

$$\lim_{t\to\infty} x(t) = 0$$

Test Signal Example

- Consider an open-loop system with gain $G(s)$:

$$Y(s) = G(s)R(s) \qquad G(s) = \frac{9}{s+10}$$

- The step response is obtained with $R(s) = \frac{1}{s}$:

$$Y(s) = \frac{G(s)}{s} = \frac{9}{s(s+10)}$$

- Transient step response:

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = 0.9(1 - e^{-10t})$$

- Steady-state step response:

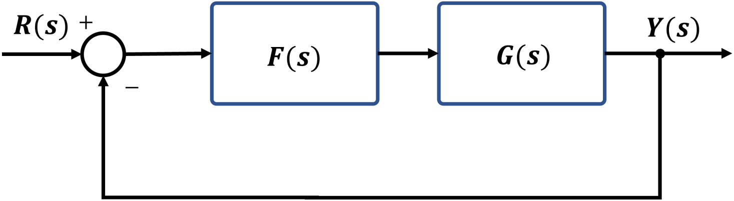
$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 0.9$$

- Error signal: $e(t) = r(t) - y(t)$

- Steady-state step error:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s+1}{s+10} = 0.1$$

Steady-state Error



- Consider a unity-feedback (follow-up) system with control gain $F(s)$ and process gain $G(s)$
- Since the forward-path gain $F(s)G(s)$ is a rational function, it can be expressed as:

$$F(s)G(s) = k\frac{(s - z_1)\cdots(s - z_m)}{s^q(s - p_{q+1})\cdots(s - p_n)}$$

where $0 \leq q \leq n$ explicitly denotes the number of poles equal to zero:

$$p_1 = p_2 = \cdots = p_q = 0$$

Steady-state Error

- We will examine the steady-state error of the unity-feedback system to test signals of the form $r(t) = \frac{t^d}{d!}$ for $t \geq 0$, such as step ($d = 0$), ramp ($d = 1$), parabola ($d = 2$), etc.

- Consider the error signal $e(t) = r(t) - y(t)$ with Laplace transform:

$$E(s) = R(s) - Y(s) = R(s) - F(s)G(s)E(s)$$

- The reference-to-error transfer function is:

$$E(s) = \frac{1}{1 + F(s)G(s)} R(s)$$

- The steady-state error with reference input $R(s) = \frac{1}{s^{d+1}}$ can be obtained by the final value theorem:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{1}{(1 + F(s)G(s))s^d}$$

Position Error Coefficient

- **Unit step response**: when $r(t)$ is a unit step such that $d = 0$ and $R(s) = 1/s$, the steady state error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{1 + F(s)G(s)} = \frac{1}{1 + K_p}$$

- **Position Error Coefficient**: $K_p = \lim_{s\to 0} F(s)G(s)$

- **Example**: if a steady-state error to a unit step of at most 10% is desired, then we need to choose the control gain $F(s)$ such that $K_p \geq 9$

Velocity Error Coefficient

- **Ramp response**: when $r(t)$ is a ramp such that $d = 1$ and $R(s) = 1/s^2$, the steady-state error is:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{(1 + F(s)G(s))s} = \frac{1}{K_v}$$

- **Velocity Error Coefficient**: $K_v = \lim_{s \to 0} sF(s)G(s)$

- **Example**: if a steady-state error to a ramp input of at most 1% is desired, then we need to choose the control gain $F(s)$ such that $K_v \geq 100$

Acceleration Error Coefficient

- **Parabolic response**: when $r(t)$ is a parabola such that $d = 2$ and $R(s) = 1/s^3$, the steady-state error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{(1 + F(s)G(s))s^2} = \frac{1}{K_a}$$

- **Acceleration Error Coefficient**: $K_a = \lim_{s\to 0} s^2 F(s)G(s)$

- **Example**: if a steady-state error to a parabola input of at most 5% is desired, then we need to choose the control gain $F(s)$ such that $K_a \geq 20$

Steady-state Error

- When $r(t) = t^d/d!$ and $R(s) = 1/s^{d+1}$, the steady-state error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{(1+F(s)G(s))s^d}$$

- The error is determined by the term:

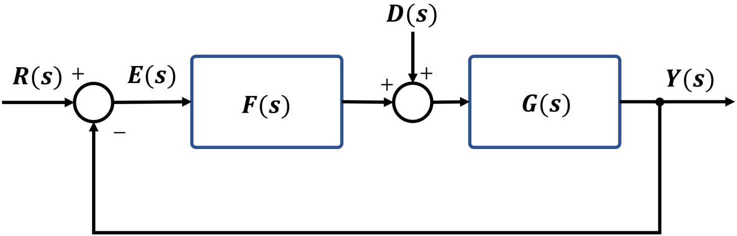
$$s^d F(s)G(s) = k\frac{s^d(s-z_1)\cdots(s-z_m)}{s^q(s-p_{q+1})\cdots(s-p_n)}$$

- Three cases are possible, assuming that the control system is **stable** (all poles of $sE(s)$ are in the open left-half plane):
  - If $d < q$, then $s^d F(s)G(s)$ will contain a term s^{q-d} in the denominator and $sE(s)$ will contain $q-d$ zeros at the origin. Hence, $\lim_{s\to 0} sE(s) = 0$ and **zero steady-state error** will be achieved.
  - If $d = q$, then $sE(s)$ will contain no zeros at the origin and a **constant finite steady-state error** will be achieved.
  - If $d > q$, then $sE(s)$ will have $d-q$ poles at the origin. Hence, $\lim_{s\to 0} sE(s) = \infty$ and an **infinite steady-state error** will be achieved. In other words, the system output will not track the reference input at all.

Control System Type

- The results on the previous slide indicate that the number q of poles at the origin in $F(s)G(s)$ determines the type of reference inputs that the closed-loop system is able to track
- The number q of poles at the origin in $F(s)G(s)$ is called **system type**
- A system of type q can track polynomial reference signals of degree q or less to within a constant finite steady-state error
- During control design, the controller gain $F(s)$ can be chosen to achieve a certain number of poles at the origin if the process $G(s)$ does not have the required number of poles to track a desired reference signal
- It appears that having more integrators $(1/s)$ in $F(s)G(s)$ is better since it allow tracking higher-order reference signals. However, the larger q is, the harder it is to stabilize the system since integrators slow the response down.

Steady-state Error with Disturbance



- Consider the unity-feedback system with distance
- The disturbance-to-error transfer function is:

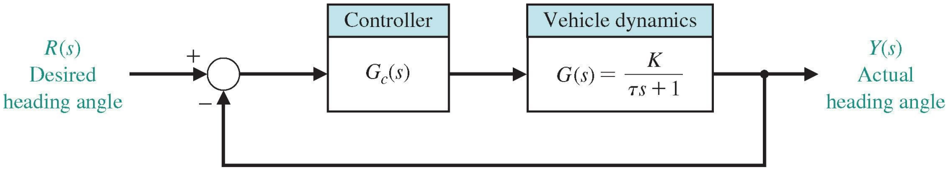
$$\frac{E(s)}{D(s)} = \frac{-G(s)}{1 + F(s)G(s)}$$

- The steady-state error by the final value theorem is:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{-sG(s)}{1 + F(s)G(s)}$$

- The control gain $F(s)$ should be designed as large as possible to minimize the effect of the disturbance

Example: Mobile Robot Heading Angle Control



- Consider a heading-angle steering control system for a mobile robot:

  Heading dynamics: $G(s) = \dfrac{K}{\tau s + 1}$ Control gain: $G_c(s) = K_1 + \dfrac{K_2}{s}$

- What is the steady-state error of the closed-loop system for a step input and a ramp input?

Example: Mobile Robot Heading Angle Control

- If $K_2 = 0$:
  - the forward path gain is: $G_c(s)G(s) = \frac{KK_1}{\tau(s+1/\tau)}$
  - the system is type 0 with position error coefficient:

$$K_p = \lim_{s \to 0} G_c(s)G(s) = KK_1$$

  - the steady-state error for a step input is:

$$\lim_{t \to \infty} e(t) = \frac{1}{1 + K_p} = \frac{1}{1 + KK_1}$$

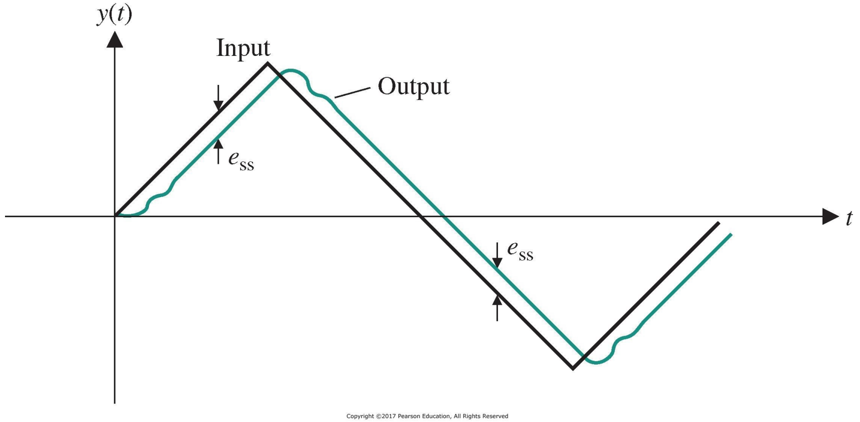
- If $K_2 > 0$:
  - the forward path gain is: $G_c(s)G(s) = \frac{KK_1(s+K_2/K_1)}{\tau s(s+1/\tau)}$
  - the system is type 1 with velocity error coefficient:

$$K_v = \lim_{s \to 0} sG_c(s)G(s) = KK_2$$

  - the steady-state error for a ramp input is:

$$\lim_{t \to \infty} e(t) = \frac{1}{K_v} = \frac{1}{KK_2}$$

Example: Mobile Robot Heading Angle Control



- Transient response of the heading-angle steering control system to a triangular wave reference input
- The response shows the effect of the non-zero steady-state error $e_{ss} = 1/(KK_2)$