ECE171A: Linear Control System Theory
Lecture 6: Transient Response Performance Measures

Instructor:

Nikolay Atanasov: natanasov@ucsd.edu

Teaching Assistant:

Chenfeng Wu: chw357@ucsd.edu

UC San Diego

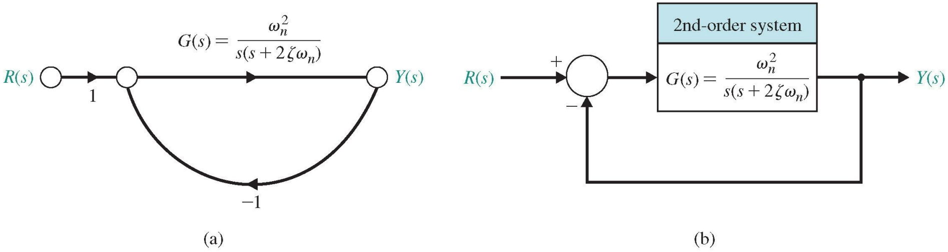
JACOBS SCHOOL OF ENGINEERING

Electrical and Computer Engineering

Feedback Control System Performance Measures

- Advantage of feedback control systems: the ability to adjust the transient and steady-state response
- To design and analyze feedback control systems we must define and measure their transient and steady-state performance
- The response of the system to specific test input signals is evaluated according to several performance criteria:
  - Rise time
  - Percent overshoot
  - Settling time
  - Steady-state error
  - Sensitivity to disturbance and noise
  - Sensitivity to parameter variations

Follow-up Second-order System



- Consider a unity feedback (**follow-up**) second-order system
- Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with **natural frequency** ω_n and **damping ratio** ζ

Second-order System Poles

- Transfer function: $T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
- Transfer function poles:

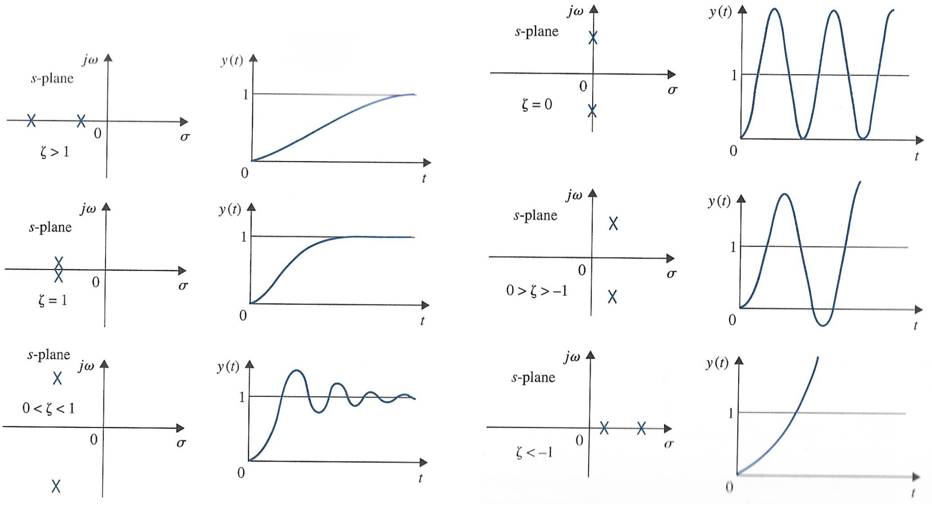
$$p = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Response	Damping ratio	Poles
Underdamped	$\zeta < 1$	$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
Critically damped	$\zeta = 1$	$-\omega_n, -\omega_n$
Overdamped	$\zeta > 1$	$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

- The natural frequency ω_n and damping ratio ζ of a pole p can be obtained as:

$$\omega_n = |p| \qquad \zeta = -\cos(\angle p)$$

Second-order System Step Response



Second-order System Step Response

- If the poles are complex, the step response has oscillations and overshoot
- As the poles move toward the real axis, maintaining a fixed distance from the origin (ζ increasing for fixed ω_n), the oscillations and overshoot decrease
- If ω_n increases, the poles move further left in the left half plane and the oscillations reduce faster
- If all poles are on the negative real axis, there are no oscillations or overshoot
- If there is a pole in the open right half plane, then the step response contains a term that goes to ∞

Underdamped Second-order System Impulse Response

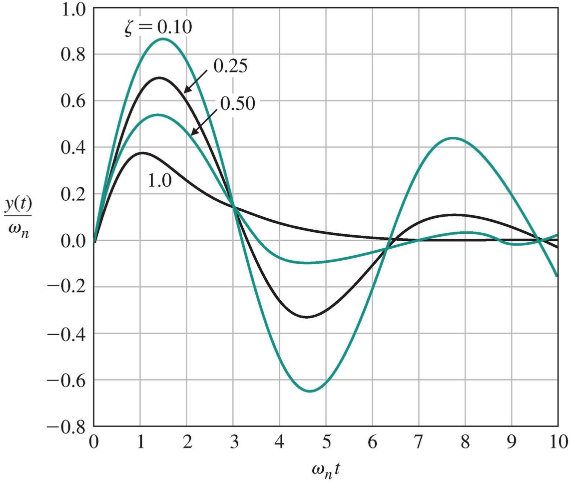
- Consider the underdamped and critically damped cases ($0 \leq \zeta \leq 1$)
- The impulse response is obtained with $R(s) = 1$ and reveals the transfer function:

$$Y(s) = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \alpha)^2 + \omega_d^2}$$

  where we introduced the terms:
  - **damping constant**: $\alpha = \zeta\omega_n$
  - **damped frequency**: $\omega_d = \omega_n\sqrt{1 - \zeta^2}$
- Transient impulse response:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} &= \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1 - \zeta^2}\, t) \\ &= \left(\frac{\alpha^2}{\omega_d} + \omega_d\right) e^{-\alpha t} \sin(\omega_d t) \end{aligned}$$

Underdamped Second-order System Impulse Response



- As the damping ζ decreases, the poles approach the imaginary axis and the response becomes increasingly oscillatory

Underdamped Second-order System Step Response

- The step response is obtained with $R(s) = \frac{1}{s}$:

$$Y(s) = \frac{G(s)}{s(1 + G(s))} = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

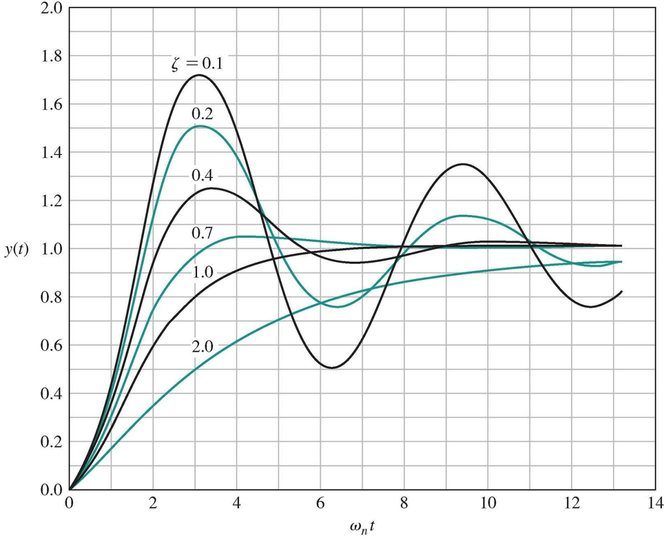
- Transient step response with $\theta = \cos^{-1}(\zeta)$:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \theta) \\ &= 1 - e^{-\alpha t}\left(\cos(\omega_d t) + \frac{\alpha}{\omega_d}\sin(\omega_d t)\right) \end{aligned}$$

- The derivative of the step response is equal to the impulse response:

$$\frac{d}{dt} y(t) = \left(\frac{\alpha^2}{\omega_d} + \omega_d\right) e^{-\alpha t} \sin(\omega_d t)$$

Underdamped Second-order System Step Response



- As the damping ζ decreases, the poles approach the imaginary axis and the response becomes increasingly oscillatory

Step Response Performance Measures

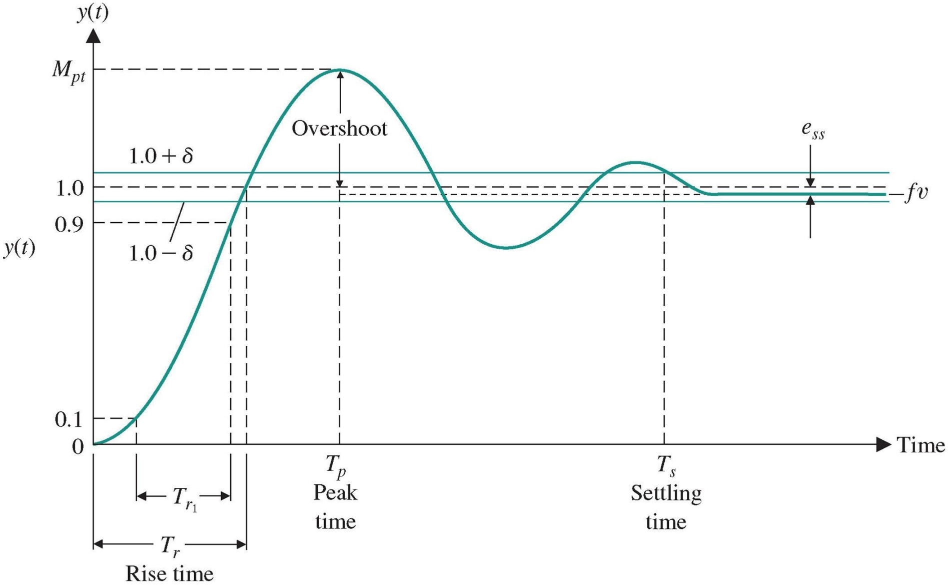
- Standard performance measures are defined in terms of the step response of the closed-loop system
- **Rise time** t_r: time for the system step response $y(t)$ to go from $\delta\%$ to $1 - \delta\%$ of the steady-state value
- **Peak time** t_p: time at which the system step response $y(t)$ achieves its maximum value (**defined only for underdamped systems**)
- **Percent overshoot**: the max value of the system step response, $y(t_p)$, expressed as a percentage of the steady-state value, $y(\infty) = \lim_{t \to \infty} y(t)$:

$$\text{percent overshoot} = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$

- **Settling time** t_s: the time required for the step response to settle within $\delta\%$ of the steady-state value, i.e., for all $t \geq t_s$:

$$|y(t) - y(\infty)| \leq \frac{\delta}{100}$$

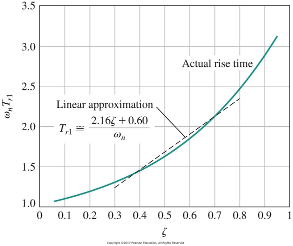
Underdamped Second-order System Step Response



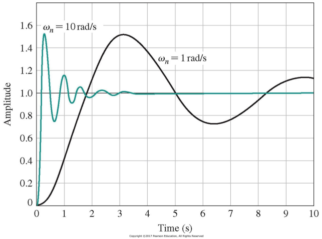
Underdamped Second-order System Rise Time

- **Rise time**: an exact expression for t_r is challenging to obtain.
- The best linear fit to the 10%-to-90% rise time is accurate for $0.3 < \zeta < 0.8$:

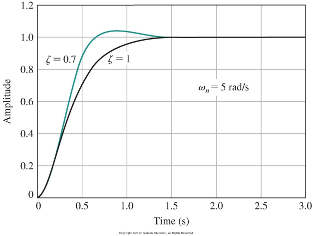
$$t_r \approx \frac{2.16\zeta + 0.6}{\omega_n}$$



(a) Rise time



(b) Effect of ω_n for $\zeta = 0.2$



(c) Effect of ζ for $\omega_n = 5$

Underdamped Second-order System Peak Time

- **Peak time**: obtained by setting the derivative of the response to zero and solving for t:

$$0 = \left(\frac{\alpha^2}{\omega_d} + \omega_d\right) e^{-\alpha t} \sin(\omega_d t) \quad \Rightarrow \quad t = \frac{k\pi}{\omega_d},\ k = 0, 1, 2, \ldots$$

- The maximum overshoot occurs at the first peak:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

- The max value of the system step response is:

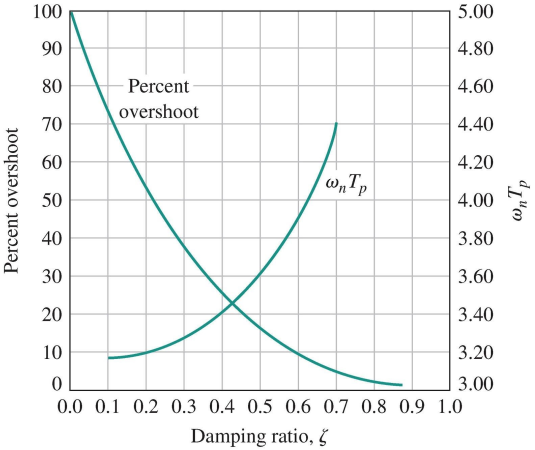
$$y(t_p) = 1 + e^{-\alpha \frac{\pi}{\omega_d}}$$

Underdamped Second-order System Percent Overshoot

- **Percent overshoot**: since $y(\infty) = \lim_{t\to\infty} y(t) = 1$:

$$\text{percent overshoot} = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\% = e^{-\alpha\frac{\pi}{\omega_d}} \times 100\%$$

- There is a trade-off between swiftness of response and percent overshoot



Underdamped Second-order System Settling Time

- Underdamped second-order system step response:

$$y(t) = 1 - e^{-\alpha t} \left(\cos(\omega_d t) + \frac{\alpha}{\omega_d} \sin(\omega_d t) \right)$$

- **Settling time**: since the cosine and sine terms oscillate, approximate the time required for the step response to settle within $\delta\%$ of the steady-state value by calculating the time at which the exponential term $e^{-\alpha t}$ becomes equal to $\delta/100$:

$$e^{-\alpha t_s} \approx \frac{\delta}{100} \qquad \Rightarrow \qquad t_s \approx -\frac{1}{\alpha} \ln \frac{\delta}{100}$$

- For $\delta = 2\%$, the settling time is: $t_s \approx \dfrac{4}{\alpha} = \dfrac{4}{\zeta \omega_n}$

Underdamped Second-order System Performance Measures

- It is desirable to achieve small t_r, small percent overshoot, and small t_s
- As ω_n increases with fixed ζ, t_r decreases, t_p decreases, the percent overshoot stays the same, and t_s decreases
- As ζ increases with fixed ω_n, t_r stays the same, t_p increases, the percent overshoot decreases, and t_s decreases
- If desired upper bounds are given:

$$t_r \leq \bar{t}_r \qquad t_p \leq \bar{t}_p \qquad \text{p.o.} \leq \bar{\text{p.o.}} \qquad t_s \leq \bar{t}_s$$

  we can obtain constraints for ζ and ω_n, which determine valid regions for the transfer function poles $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ in the complex plane:

$$\frac{2.16\zeta + 0.6}{\omega_n} \leq \bar{t}_r \qquad \frac{\zeta}{\sqrt{1-\zeta^2}}\pi \geq -\ln\frac{\bar{\text{p.o.}}}{100}$$

$$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \leq \bar{t}_p \qquad \frac{4}{\zeta\omega_n} \leq \bar{t}_s$$

Effect of Additional Poles or Zeros

- So far we analyzed the step response of an underdamped second-order system with transfer function:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- What happens if the transfer function contains zeros or additional poles?

Effect of Poles on the Step Response

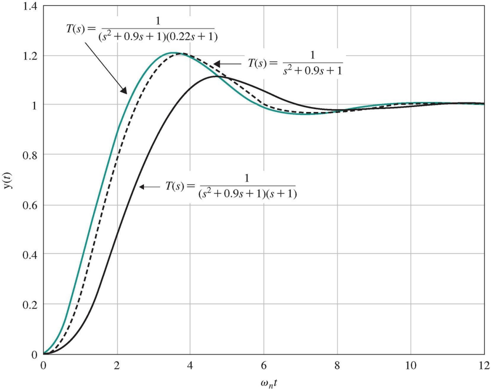
- From the partial fraction expansion of the transfer function, we know that a pole p contributes a term of the form e^{pt}
- If any pole is in the right half-plane ($\text{Re}(p) > 0$), then the step response will go to infinity (unstable system)
- If any pole is far left in the left half-plane ($\text{Re}(p) \ll 0$), then its contribution to the step response dies out quickly
- If the poles can be divided into a set that is close to the origin, and another set that is far away, then the poles that are close to the origin are called **dominant poles**. The exponential terms in the step response of the dominant poles determine the overall system response.
- Adding a left half-plane pole to the transfer function makes the response **slower** because an additional exponential term must die out before the system reaches its final value

Introducing a Pole in a Second-order System

- Introduce a pole $s = -1/\gamma$ in the transfer function:

$$T_\gamma(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(\gamma s + 1)}$$

- If $|1/\gamma| \geq 10|\zeta\omega_n|$, then $T_\gamma(s)$ can be approximated by $T(s)$ since the contribution of the new pole to the step response is dominated by the original two poles



Introducing a Zero in a Second-order System

- Introduce a zero $s = -a$ in the transfer function:

$$T_a(s) = \frac{(\frac{1}{a}s + 1)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- The reason for writing $(\frac{1}{a}s + 1)$ instead of $s + a$ is to maintain a steady-state value of 1

- The new transfer function can be decomposed as:

$$T_a(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{s}{a}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = T(s) + \frac{s}{a}T(s)$$

- The response of the third order system to a step $R(s) = 1/s$ is:

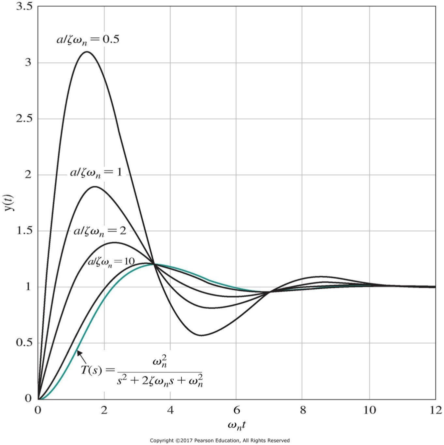
$$Y_a(s) = \left(T(s) + \frac{s}{a}T(s)\right)\frac{1}{s} = Y(s) + \frac{s}{a}Y(s)$$

$$y_a(t) = y(t) + \frac{1}{a}\dot{y}(t)$$

where $Y(s)$ and $y(t)$ are the s- and t-domain step response of the original second-order system

Introducing a Zero in a Second-order System

- Step response of a system with transfer function $T_a(s) = \frac{(\frac{1}{a}s+1)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ and $\zeta = 0.45$

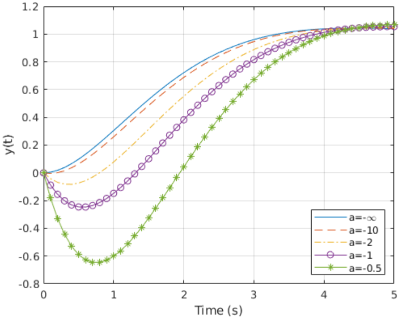


- As a increases, the zero moves farther into the left half-plane and the step response of $T_a(s)$ approaches that of the second-order system $T(s)$

Introducing a Zero in a Second-order System

- We can see from the step-response of $T_a(s)$ that adding a zero in the left half-plane makes the step response **faster**:
  - the rise time decreases
  - the peak time decreases
  - the overshoot increases
  - the settling time does not change

- If the zero is added in the right half-plane (i.e., $a < 0$), then $\dot{y}(t)$ is subtracted from $y(t)$ to produce $y_a(t)$. The response is **slower** and can go decrease before before rising to its steady state value (**undershoot**).



Dominant Pole-Zero Approximation

- If a high-order system has a cluster of poles and zeros that are much closer (e.g., 5 times or more) to the origin than the remaining poles and zeros, then the system can be approximated by a lower order system with only those dominant poles and zeros
- **Example**: if $a \gg \zeta\omega_n > 0$ and $1/\gamma \gg \zeta\omega_n > 0$, then:

$$T_{a,\gamma}(s) = \frac{\omega_n^2(\frac{1}{a}s + 1)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(\gamma s + 1)} \approx T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Example: Dorf-Bishop Problem AP5.1

- Consider a control system with transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}$$

(a) Determine the steady-state error for a unit step input.

(b) Assume that the complex poles are dominant. Determine the percent overshoot and the settling time to within 2% of the steady-state value.

(c) Plot the actual system response and compare it with the estimates of part (b).

Example AP5.1: Part (a)

- The error is:

$$E(s) = R(s) - Y(s) = R(s) - T(s)R(s) = (1 - T(s))R(s)$$

- The steady-state error for input $R(s) = 1/s$ is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}(1 - T(s))$$
$$= \lim_{s\to 0}\left(1 - \frac{108(s+3)}{(s+9)(s^2+8s+36)}\right) = 1 - \frac{108(3)}{9(36)} = 0$$

Example AP5.1: Part (b)

- Assuming that the complex poles are dominant:

$$T(s) = \frac{36(\frac{s}{3}+1)}{(s+9)(s^2+8s+36)} \approx \frac{36}{s^2+8s+36}$$

- The second-order system approximation has natural frequency $\omega_n = 6$ and damping ratio $\zeta = \frac{8}{2\omega_n} = \frac{2}{3}$.

- The percent overshoot is:

$$\text{p.o.} = 100 \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi\right) = 100 \exp\left(-\frac{2\pi}{\sqrt{5}}\right) \approx 6\%$$

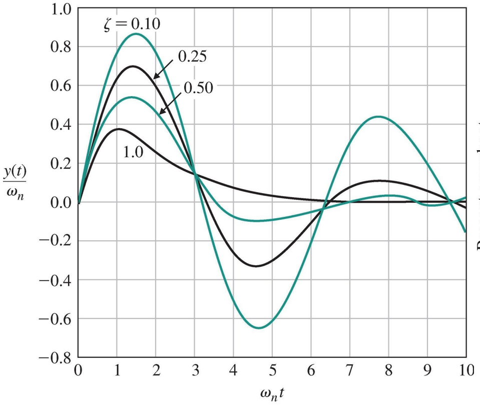
- The settling time to within 2% of the steady-state value is:

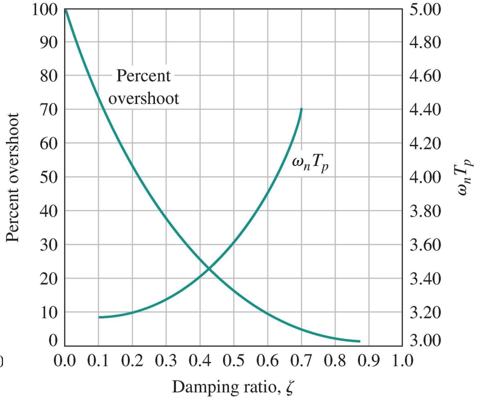
$$t_s \approx \frac{4}{\zeta\omega_n} = 1 \text{ second.}$$

- The percent overshoot can also be determined approximately from

Example AP5.1: Part (b)

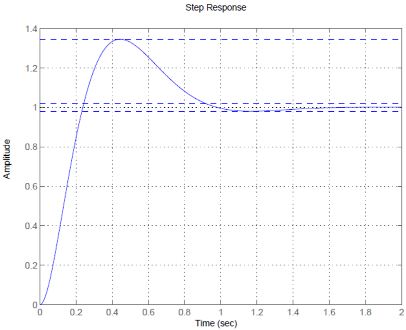
- The percent overshoot can also be determined approximately from the second-order system plots on Slide 10 and 15.





Example AP5.1: Part (c)

- The step response of the original system is:




```
sys = tf([108 324],[1 17 108 324]);
characteristics = stepinfo(sys,
    'RiseTimeLimits',[0.05,0.95],
    'SettlingTimeThreshold', 0.02);
stepplot(sys);
hold on;
plot([0,2],[characteristics.Peak,
    characteristics.Peak],'b--');
```

- The actual percent overshoot and settling time are:

$$\text{p.o} = 34.4\% \qquad \text{and} \qquad t_s = 1.18 \text{ second.}$$

- The difference in the actual and estimated percent overshoot is due to the term $(\frac{s}{a} + 1)$ in the numerator, which does not satisfy the requirement for an accurate dominant pole-zero approximation:

$$3 = a \not\gg \zeta\omega_n = 4$$

s-Plane Root Location

- To understand the effect of the transfer function poles and zeros on the system response in general, consider an abstract control system without repeated poles

- Suppose that the step response achieves a unit steady-state value and has partial fraction expansion:

$$Y(s) = \frac{1}{s} + \sum_i \frac{A_i}{s+\sigma_i} + \sum_k \frac{B_k s + C_k}{s^2 + 2\alpha_k s + (\alpha_k^2 + \omega_k^2)}$$

  where A_i, B_k, and C_k are some constants.

- The poles are real ($s = -\sigma_i$) or complex conjugate pairs ($s = -\alpha_k \pm j\omega_k$)

- The inverse Laplace transform of the step response is:

$$y(t) = 1 + \sum_i A_i e^{-\sigma_i t} + \sum_k D_k e^{-\alpha_k t} \sin(\omega_k t + \theta_k)$$

  where D_k, θ_k depend on B_k, C_k, α_k, and ω_k

s-Plane Root Location

- The response is composed of the steady-state value, exponential terms, and damped sinusoidal terms
- The response achieves its steady-state value only if the real part of the poles is in the left half of the s-plane, which ensures that the exponential terms decay
- It is important to understand the effect of adding, deleting, or moving poles and zeros of $T(s)$ in the s-plane on the step and impulse response
  - The poles of $T(s)$ determine the particular response modes (exponential terms) that will be present
  - The zeros of $T(s)$ establish the relative weights (A_i and D_k) of the response modes. Moving a zero closer to a pole will reduce the weights of the corresponding exponential term.

s-Plane Root Location

- Impulse response of an abstract control system for various transfer function pole locations in the *s*-plane