ECE171A: Linear Control System Theory
Lecture 7: Stability

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Observations about Impulse Response

- Let $T(s) = \frac{Y(s)}{R(s)}$ be the transfer function of a control system with input $R(s)$ and output $Y(s)$
- The system response to an impulse input $R(s) = 1$ is $Y(s) = T(s)$
- In the time domain, the response $y(t)$ may be obtained by computing a partial fraction expansion of $T(s)$ and applying an inverse Laplace transform
- A pole p of $T(s)$ with residue r appears as an exponential term re^{pt} in the time-domain response $y(t)$
- If the transfer function contains poles in the open right half-plane (ORHP), then the response $y(t)$ will go to infinity
- If all poles of the transfer function are in the open left half-plane (OLHP), then the response $y(t)$ will settle to a steady-state value

Bounded Input Bounded Output

- A signal $f(t)$ is **bounded** if there exists some constant M such that $|f(t)| \leq M$ for all t
- We would like to ensure that the response $y(t)$ of a control system is bounded, if we provide a bounded reference input $r(t)$
- A reference input $r(t)$ with **rational function** Laplace transform $R(s)$ is bounded if and only if:
  - the poles p_i of $R(s)$ satisfy $\text{Re}(p_i) \leq 0$ for all i,
  - all poles with $\text{Re}(p_i) = 0$ are of multiplicity 1 (simple roots).
- Similarly, the response $y(t)$ with rational function Laplace transform $Y(s)$ is bounded if and only if $Y(s)$ has no right half-plane poles or repeated poles on the imaginary axis.

BIBO Stability

- A system is **bounded-input bounded-output (BIBO) stable** if every bounded input leads to a bounded output
- A system is **unstable** if there is at least one bounded input that produces an unbounded output

BIBO Stability of LTI Systems

A linear time-invariant system with transfer function $T(s)$ is:

- **stable**, if all poles of $T(s)$ are in the open left half-plane in the s domain,
- **marginally stable**, if all poles of $T(s)$ are in the closed left half-plane in the s domain and all poles with zero real part are simple roots (of multiplicity 1),
- **unstable**, otherwise.

No Pole-Zero Cancellations

- Consider a linear time-invariant system with transfer function:

$$T(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

- The system is BIBO stable if all poles of $T(s)$ are in the OLHP
- **No pole-zero cancellations**: common poles and zeros in $T(s)$ should not be cancelled before checking stability!
- A cancelled pole will not show up in the input response but will appear in the natural response (when the initial conditions are non-zero) or due to additional inputs (e.g., disturbances)

Pole-Zero Cancellation Example

- Consider the transfer function:

$$T(s) = \frac{Y(s)}{U(s)} = \frac{s-1}{s^2+2s-3} = \frac{s-1}{(s+3)(s-1)}$$

- If we cancel the common pole and zero, $T(s) = 1/(s+3)$ and we might erroneously conclude that the system is BIBO stable since the pole $p_1 = -3$ is in the OLHP

- The ODE description of the system is:

$$\ddot{y}(t) + 2\dot{y}(t) - 3y(t) = \dot{u}(t) - u(t)$$

- If the initial conditions $y(0)$, $\dot{y}(0)$ are not zero, then the Laplace transform will be:

$$s^2 Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) - 3Y(s) = sU(s) - U(s)$$

Pole-Zero Cancellation Example

- Total response with non-zero initial conditions:

$$Y(s) = \underbrace{\frac{s-1}{s^2+2s-3}}_{T(s)} U(s) + \frac{s+2}{s^2+2s-3} y(0) + \frac{1}{s^2+2s-3} \dot{y}(0)$$

- Step response for $U(s) = 1/s$ with non-zero initial conditions:

$$y(t) = \underbrace{\frac{1}{3}(1 - e^{-3t})}_{\text{forced response}} + \underbrace{\frac{y(0)}{4}(3e^{t} + e^{-3t})}_{\text{natural response}} + \underbrace{\frac{\dot{y}(0)}{4}(e^{t} - e^{-3t})}_{\text{natural response}}$$

- Even if the input is bounded, when $y(0)$ or $\dot{y}(0)$ are non-zero, the terms $\frac{3y(0)}{4}e^{t}$ and $\frac{\dot{y}(0)}{4}e^{t}$ are unbounded and the system is **not BIBO stable**

Determining BIBO Stability

- A control system with transfer function

$$T(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

  is BIBO stable if all poles are in the OLHP

- Computing the poles might not always be easy or necessary, e.g., high-order or symbolic characteristic polynomial $a(s)$

- Whether the poles are in the OLHP can be verified from the polynomial coefficients rather than from the actual pole values

- **Vieta's formulas** relate the coefficients of a polynomial to its roots

Vieta's Formulas

- Consider the characteristic polynomial with roots p_1, ..., p_n:

$$a(s) = a_n s^n + \ldots + a_1 s + a_0 = a_n(s - p_1)\cdots(s - p_n)$$

- **Vieta's formulas** relate the coefficients a_i to the roots p_i:

$$\sum_{i=1}^{n} p_i = -\frac{a_{n-1}}{a_n} \qquad \prod_{i=1}^{n} p_i = (-1)^n \frac{a_0}{a_n} \qquad \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \prod_{j=1}^{k} p_{i_j} = (-1)^k \frac{a_{n-k}}{a_n}$$

- Examples for $n = 2$ and $n = 3$:

$$a_2(s - p_1)(s - p_2) = a_2 s^2 \underbrace{- a_2(p_1 + p_2)}_{a_1} s + \underbrace{a_2 p_1 p_2}_{a_0}$$

$$\begin{aligned} a_3(s - p_1)(s - p_2)(s - p_3) = a_3 s^3 &\underbrace{- a_3(p_1 + p_2 + p_3)}_{a_2} s^2 \\ &+ \underbrace{a_3(p_1 p_2 + p_1 p_3 + p_2 p_3)}_{a_1} s \underbrace{- a_3 p_1 p_2 p_3}_{a_0} \end{aligned}$$

Necessary Condition for BIBO Stability

- If all poles $p_1, \ldots, p_n$ are in the OLHP, then all characteristic polynomial coefficients $a_0, \ldots, a_n$ have the same sign and are non-zero
- This requirement is necessary but not sufficient
- If the necessary condition is not satisfied, then the system is BIBO unstable
- If the necessary condition is satisfied, additional information is needed to decide if the system is BIBO stable

Necessary Condition for BIBO Stability of LTI Systems

If all poles of the transfer function $T(s) = b(s)/a(s)$ of an LTI system are in the open left half-plane in the s domain, then all coefficients of the characteristic polynomial $a(s)$ will be non-zero and have the same sign.

Necessary Condition for BIBO Stability Example

- Consider an LTI system with transfer function $T(s) = b(s)/a(s)$ and characteristic polynomial $a(s)$ shown below. Is this system BIBO stable?
  - $a(s) = s^3 - 2s^2 + s + 1$
  - $a(s) = s^4 + s^2 + s + 1$
  - $a(s) = s^3 + 2s^2 + 2s + 1$
  - $a(s) = s^3 + 2s^2 + s + 12$

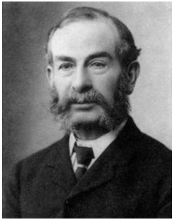
Necessary and Sufficient Condition for BIBO Stability

- In the 1870s-1890s, Edward Routh and Adolf Hurwitz independently developed a method for determining the locations in the s plane but not the actual values of the roots of a polynomial with constant real coefficients

- Characteristic polynomial:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$$

- The Routh-Hurwitz method constructs a table with $n+1$ rows from the coefficients a_i of an n-th degree polynomial $a(s)$ and relates the number of sign changes in the first column of the table to the number of roots in the closed right half-plane



E. Routh

A. Hurwitz

Routh Table

- $a(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	a_0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	$b_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$	$b_{n-3} = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$	b_{n-5}	$\cdots$	0
s^{n-3}	$c_{n-1} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$	$c_{n-3} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}}$	c_{n-5}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
s^0	a_0	0	0	$\cdots$	0

- Any row can be multiplied by a positive constant without changing the result

Routh Table ($n = 6$)

- $a(s) = a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$

s^6	a_6	a_4	a_2	a_0
s^5	a_5	a_3	a_1	0
s^4	$b_5 = -\frac{1}{a_5}\begin{vmatrix} a_6 & a_4 \\ a_5 & a_3 \end{vmatrix}$	$b_3 = -\frac{1}{a_5}\begin{vmatrix} a_6 & a_2 \\ a_5 & a_1 \end{vmatrix}$	$b_1 = a_0$	0
s^3	$c_5 = -\frac{1}{b_5}\begin{vmatrix} a_5 & a_3 \\ b_5 & b_3 \end{vmatrix}$	$c_3 = -\frac{1}{b_5}\begin{vmatrix} a_5 & a_1 \\ b_5 & b_1 \end{vmatrix}$	0	0
s^2	$d_5 = -\frac{1}{c_5}\begin{vmatrix} b_5 & b_3 \\ c_5 & c_3 \end{vmatrix}$	$d_3 = a_0$	0	0
s^1	$e_5 = -\frac{1}{d_5}\begin{vmatrix} c_5 & c_3 \\ d_5 & d_3 \end{vmatrix}$	0	0	0
s^0	a_0	0	0	0

Routh-Hurwitz BIBO Stability Criterion

- The Routh-Hurwitz criterion is a necessary and sufficient criterion for BIBO stability of linear time-invariant systems

Necessary and Sufficient Condition for BIBO Stability of LTI Systems

Consider a Routh table constructed from the characteristic polynomial $a(s)$ of an LTI system with transfer function $T(s) = b(s)/a(s)$. The number of sign changes in the first column of the Routh table is equal to the number of roots of $a(s)$ in the closed right half-plane. The system is BIBO stable if and only if there are no sign changes in the first column of the Routh table.

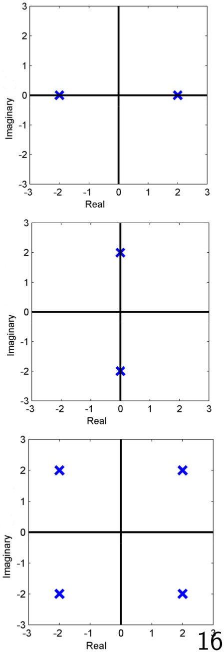
- There are two special cases related to the Routh table:
  1. The first element of a row is 0 but some of the other elements are not
     - **Solution**: replace the 0 with an arbitrary small ϵ
  2. All elements of a row are 0
     - **Solution**: replace the zero row with the coefficients of $\frac{dA(s)}{ds}$, where $A(s)$ is an **auxiliary polynomial** with coefficients from the row just above the zero row

Routh-Hurwitz Auxiliary Polynomial

- The Routh table associated with polynomial $a(s)$ contains an all zero row when $a(s)$ has roots located symmetrically about the origin, e.g.,:

$$(s+\sigma)(s-\sigma) \qquad \text{or} \qquad (s+j\omega)(s-j\omega)$$

- This special case is resolved using an auxiliary polynomial $A(s)$ with coefficients obtained from the row that precedes the zero row in the Routh table
- The roots of the auxiliary polynomial $A(s)$ satisfy the original characteristic equation ($a(s)=0$) and are located symmetrically about the origin
- Since the auxiliary polynomial roots are symmetric about the origin, the system is either **unstable** or **marginally stable** but cannot be stable



Routh-Hurwitz Stability: Second-order System

- Consider the characteristic polynomial of a second-order system:

$$a(s) = as^2 + bs + c$$

- The Routh table is:

s^2	a	c
s^1	b	0
s^0	$-\frac{1}{b}(0 - bc) = c$	0

- A necessary and sufficient condition for BIBO stability of a second-order system is that all coefficients of the characteristic polynomial are non-zero and have the same sign.

Routh-Hurwitz Stability: Third-order System

- Consider the characteristic polynomial of a third-order system:

$$a(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

- The Routh table is:

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$-\frac{1}{a_2}(a_3 a_0 - a_1 a_2)$	0
s^0	a_0	0

- A necessary and sufficient condition for BIBO stability of a third-order system is that all coefficients of the characteristic polynomial are non-zero, have the same sign, and $a_1 a_2 > a_0 a_3$.

- If $a_1 a_2 = a_0 a_3$, one pair of roots lies on the imaginary axis in the s plane and the system is marginally stable. This results in an all zero row in the Routh table.

Routh-Hurwitz Stability: Higher-order System

- Consider the characteristic polynomial of a fifth-order system:

$$a(s) = s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240$$

- The Routh table is:

s^5	1	10	152
s^4	1	72	240
s^3	-62	-88	0
s^2	70.6	240	0
s^1	122.6	0	0
s^0	240	0	0

- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**

- The roots of $a(s)$ are:

$$a(s) = (s + 3)(s + 1 \pm j\sqrt{3})(s - 2 \pm j4)$$

Routh-Hurwitz Stability: Special Case 1

- Consider the polynomial:

$$a(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

- The Routh table is:

s^5	1	2	11
s^4	2	4	10
s^3	$\not{0} \nearrow \epsilon$	6	0
s^2	$c_4 = \frac{1}{\epsilon}(4\epsilon - 12)$	10	0
s^1	$d_4 = \frac{1}{c_4}(6c_4 - 10\epsilon)$	0	0
s^0	10	0	0

- For $0 < \epsilon \ll 1$, we see that $c_4 < 0$ and $d_4 > 0$

- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**

Routh-Hurwitz Stability: Special Case 1

- Consider the polynomial:

$$a(s) = s^4 + s^3 + 2s^2 + 2s + 3$$

- The Routh table is:

s^4	1	2	3
s^3	1	2	0
s^2	$\not{0} \nearrow \epsilon$	3	0
s^1	$2 - \frac{3}{\epsilon}$	0	0
s^0	3	0	0

- For $0 < \epsilon \ll 1$, we see that $2 - \frac{3}{\epsilon} < 0$

- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**

Routh-Hurwitz Stability: Special Case 2

- Consider the polynomial:

$$a(s) = s^3 + 2s^2 + 4s + 8$$

- The Routh table is:

s^3	1	4
s^2	2	8
s^1	0	0
s^0	8	0

- There is an all-zero row at s^1
- The auxiliary polynomial is: $A(s) = 2s^2 + 8 = 2(s + j2)(s - j2)$
- There are two roots on the $j\omega$-axis and the system is **marginally stable**

Routh-Hurwitz Stability: Special Case 2

- Consider the polynomial:

$$a(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$$

- The Routh table is:

s^5	1	2	1
s^4	1	2	1
s^3	0	0	0
s^2	1	1	0
s^1	0	0	0
s^0	1	0	0

- There is an all-zero row at s^3 and s^1

- The auxiliary polynomial at the s^3 row is:

$$A(s) = s^4 + 2s^2 + 1 = (s^2 + 1)^2 = (s + j)(s - j)(s + j)(s - j)$$

- There are repeated roots on the $j\omega$-axis and the system is **unstable**

Routh-Hurwitz Stability: Special Case 2

- Consider the polynomial:

$$a(s) = s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4$$

- The Routh table is:

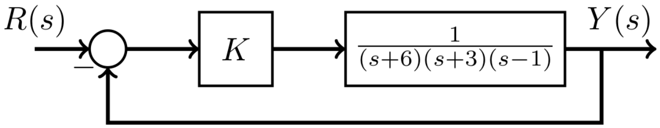
s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	0
s^1	~~0~~ → 8	0	0
s^0	4	0	0

- There is an all-zero row at s^1 with auxiliary polynomial

$$A(s) = 4s^2 + 4 = 4(s^2 + 1) = 4(s + j)(s - j)$$

- There are two roots on the $j\omega$-axis and the system is **marginally stable**

Routh-Hurwitz Stability: Parametric System



- The Routh-Hurwitz stability criterion can be used to determine the range of system parameters for which the system is stable
- Transfer function: $T(s) = \frac{K}{s^3+8s^2+9s+(K-18)}$
- Characteristic polynomial: $a(s) = s^3 + 8s^2 + 9s + (K - 18)$

Routh-Hurwitz Stability: Parametric System

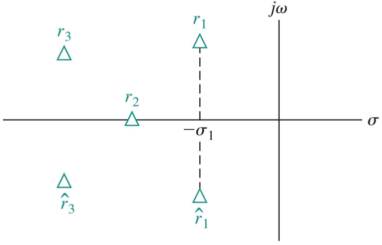
- Characteristic polynomial: $a(s) = s^3 + 8s^2 + 9s + (K - 18)$
- The Routh table is:

s^3	1	9
s^2	8	$(K - 18)$
s^1	$\frac{90-K}{8}$	0
s^0	$(K - 18)$	0

- There will be no sign changes in the first column of the Routh table if $(90 - K) > 0$ and $(K - 18) > 0$
- The system is BIBO stable if and only if $18 < K < 90$

Relative Stability

- Even if all poles of a transfer function have negative real parts, it might be necessary to check their relative distances to the imaginary axis
- For example, r_2 is relatively more stable than r_1 and $\hat{r}_1$



- All roots of $a(s)$ have real parts less than σ if and only if all roots of $\bar{a}(s) = a(s + \sigma)$ are in the open left half-plane
- Use the Routh-Hurwitz criterion on $\bar{a}(s)$ to check whether all roots of $a(s)$ lie to the left of σ