ECE171A: Linear Control System Theory
Lecture 9: PID Control

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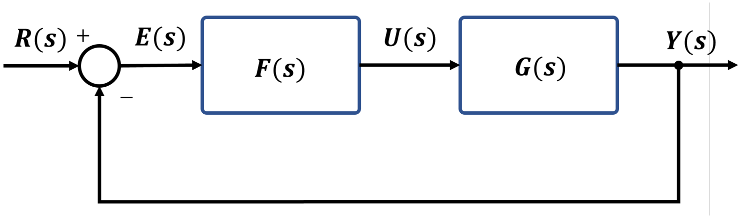
Second-order System Control Design



- Consider a unity-feedback control system with a second-order plant:

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0}$$

- How should the controller $F(s)$ be designed to ensure that the system is **stable** and its **step response has zero steady-state error**?

Proportional (P) Control

- A **proportional (P) controller** uses a constant gain K_p:

$$F(s) = \frac{U(s)}{E(s)} = K_p \qquad\qquad u(t) = K_p e(t)$$

- Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{F(s)G(s)}{1 + F(s)G(s)} = \frac{K_p b_0}{s^2 + a_1 s + (a_0 + K_p b_0)}$$

- P control can accelerate the response of a second-order system by changing the natural frequency $\omega_n^2 = (a_0 + K_p b_0)$

- To ensure stability, we need $a_1 > 0$ and $a_0 + K_p b_0 > 0$. P control can stabilize only some systems because it adjusts one coefficient of the characteristic equation.

- For $a_0 \neq 0$, $F(s)G(s)$ has $q = 0$ poles at the origin (type 0 system). The closed-loop system step response will have a constant finite steady-state error.

Proportional-Integral (PI) Control

- To achieve zero steady-state step error, we need a type 1 system.
- To add a pole at the origin in $F(s)G(s)$, introduce an integrator in $F(s)$
- A **proportional-integral (PI) controller** uses a proportional gain K_p and an integral gain K_i:

$$F(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} \qquad\qquad u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau$$

- Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{F(s)G(s)}{1 + F(s)G(s)} = \frac{b_0(K_p s + K_i)}{s^3 + a_1 s^2 + (a_0 + K_p b_0)s + K_i b_0}$$

- We achieved the steady-state error specification but the closed-loop system might still be unstable if $a_1 < 0$

Proportional-Integral-Derivative (PID) Control

- A **proportional-integral-derivative (PID) controller** uses a proportional gain K_p, an integral gain K_i, and a derivative gain K_d:

$$F(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d s \qquad u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \dot{e}(t)$$

- Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{F(s)G(s)}{1 + F(s)G(s)} = \frac{b_0(K_p s + K_i + K_d s^2)}{s^3 + (a_1 + K_d b_0)s^2 + (a_0 + K_p b_0)s + K_i b_0}$$

- The coefficients of the characteristic polynomial can be set **arbitrarily** via an appropriate choice of K_p, K_i, K_d

- PID control can guarantee stability, good transient behavior, and zero steady-state step error for a second-order plant

PID Control Example

- Consider the plant $G(s) = \frac{1}{s^2 - 3s - 1}$
- Design a controller $F(s)$ to achieve step response with zero steady-state error and place the closed-loop system poles at -5, -6, -7
- PID controller: $F(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d s$
- Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{F(s)G(s)}{1 + F(s)G(s)} = \frac{K_d s^2 + K_p s + K_i}{s^3 + (K_d - 3)s^2 + (K_p - 1)s + K_i}$$

- Vieta's formulas:

$$\begin{aligned}
(-5) + (-6) + (-7) &= -(K_d - 3) \qquad &\Rightarrow \qquad K_d &= 21 \\
(-5)(-6) + (-5)(-7) + (-6)(-7) &= (K_p - 1) \qquad &\Rightarrow \qquad K_p &= 108 \\
(-5)(-6)(-7) &= (-1)^3 K_i \qquad &\Rightarrow \qquad K_i &= 210
\end{aligned}$$

PID Control Gain Tuning

- **PID tuning**: the process of determining satisfactory PID control gains
  - Manual PID tuning
  - Ziegler-Nichols method (see Dorf-Bishop Ch. 7.6)
- **Manual PID tuning**:
  - Set $K_i = K_d = 0$
  - Increase K_p slowly until the output of the closed-loop system oscillates on the verge of instability
  - Reduce K_p to achieve **quarter amplitude decay** of the closed-loop response, i.e., the amplitude should be one-fourth of the maximum value during the oscillatory period
  - Increase K_i and K_d to achieve the desired response

Table 7.4 Effect of Increasing the PID Gains K_p, K_D, and K_I on the Step Response

PID Gain	Percent Overshoot	Settling Time	Steady-State Error
Increasing K_P	Increases	Minimal impact	Decreases
Increasing K_I	Increases	Increases	Zero steady-state error
Increasing K_D	Decreases	Decreases	No impact

PID Control: Implementation Issues

- PID control is easy to implement by tuning the knobs K_p, K_i, K_d
- Derivative control requires differentiation of the error signal:

$$\dot{e}(t) \approx \frac{e(t) - e(t-\tau)}{\tau}$$

- In practice, the error signal is measured and contains high-frequency noise, which should not be differentiated
- The derivative term $K_d s$ is implemented in conjunction with a low-pass filter $H(s) = \frac{1}{\tau_f s + 1}$ for small τ_f
- PID control with high-frequency noise attenuation:

$$F(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + \frac{K_d s}{\tau_f s + 1}$$

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \dot{e}_f(t)$$

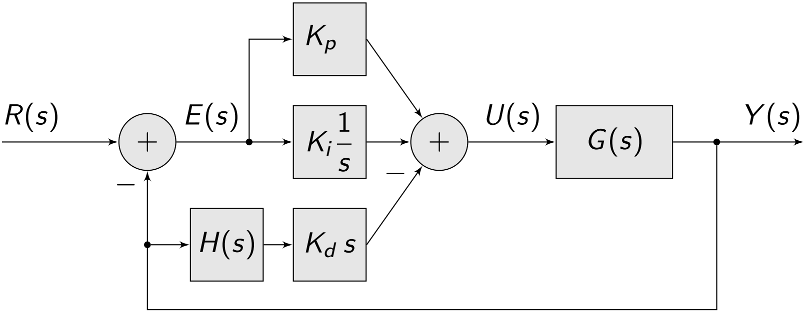
$$\tau_f \dot{e}_f(t) = -e_f(t) + e(t)$$

PID Control: Implementation Issues

- **Discrete-time PID control**:
  - sampling interval: τ_s
  - filter time constant: τ_f
  - sampled error: $e[k] = e(k\tau_s)$
  - filtered error: $e_f[k] = \frac{\tau_s}{\tau_f} e[k] + \left(1 - \frac{\tau_s}{\tau_f}\right) e_f[k-1]$
  - derivative error: $e_d[k] = \frac{e_f[k] - e_f[k-1]}{\tau_s}$
  - integral error: $e_i[k] = e_i[k-1] + \tau_s e[k-1]$
  - control: $u[k] = K_p e[k] + K_i e_i[k] + K_d e_d[k]$

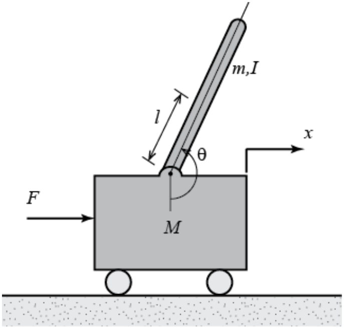
PID Control: Implementation Issues

- **Derivative kick**: if the reference $r(t)$ changes suddenly, the derivative component may become very large
- Note that for constant $r(t)$, $\dot{e}(t) = -\dot{y}(t)$
- **Derivative on measurement**: use $-\dot{y}(t)$ instead of $\dot{e}(t)$ to avoid derivative kick



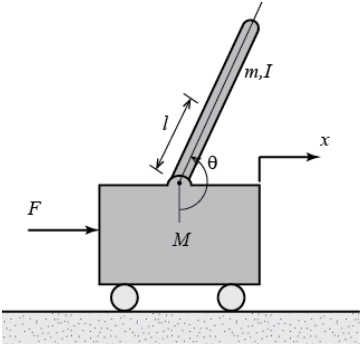
Inverted Pendulum Example

- Consider an inverted pendulum mounted on a motorized cart
- **Objective**: control the cart force to balance the inverted pendulum in an upright position
- Popular example in control theory and reinforcement learning
- Nonlinear system that is unstable without control



Inverted Pendulum: Parameters

- Cart mass: $M = 0.5$ kg
- Pendulum mass: $m = 0.2$ kg
- Cart friction coefficient: $b = 0.1$ N/m/sec
- Length to pendulum center of mass: $\ell = 0.3$ m
- Pendulum moment of inertia: $I = 0.006$ kg m^2
- Cart input force: F
- Cart position: x
- Pendulum angle: θ



Inverted Pendulum: System Model

- Horizontal direction force balance for the cart:

$$M\ddot{x} + b\dot{x} + N = F$$

- Horizontal direction force balance for the pendulum:

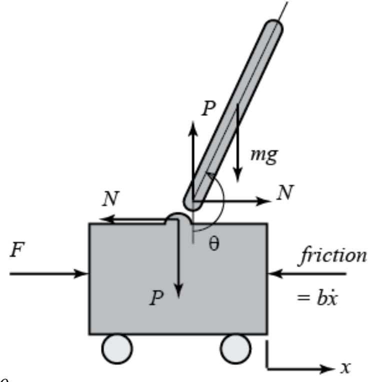
$$N = m\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta$$

- Force balance perpendicular to the pendulum:

$$P\sin\theta + N\cos\theta - mg\sin\theta = m\ell\ddot{\theta} + m\ddot{x}\cos\theta$$

- Torque balance about the pendulum centroid:

$$-P\ell\sin\theta - N\ell\cos\theta = I\ddot{\theta}$$



Inverted Pendulum: System Model

- Eliminating the reaction force N and the normal force P and denoting the input force F by u, we get the cart-pole equations of motion:

$$(M+m)\ddot{x} + b\dot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta = u$$
$$(I+m\ell^2)\ddot{\theta} + mg\ell\sin\theta = -m\ell\ddot{x}\cos\theta$$

- Since our control techniques apply to linear time-invariant systems only, we need to linearize the equations of motion
- Linearize about the upright pendulum position π and assume that the pendulum remains within a small neighborhood: $\theta = \pi + \phi$
- Small angle approximation:

$$\cos\theta = \cos(\pi+\phi) \approx -1 \qquad \sin\theta = \sin(\pi+\phi) \approx -\phi \qquad \dot{\theta}^2 = \dot{\phi}^2 \approx 0$$

- Linearized equations of motion:

$$(M+m)\ddot{x} + b\dot{x} - m\ell\ddot{\phi} = u$$
$$(I+m\ell^2)\ddot{\phi} - mg\ell\phi = m\ell\ddot{x}$$

Inverted Pendulum: Transfer Function

- Laplace transform of the equations of motion with zero initial conditions:

$$(M+m)s^2X(s) + bsX(s) - m\ell s^2\Phi(s) = U(s)$$
$$(I+m\ell^2)s^2\Phi(s) - mg\ell\Phi(s) = m\ell s^2 X(s)$$

- Eliminating $X(s)$ leads to:

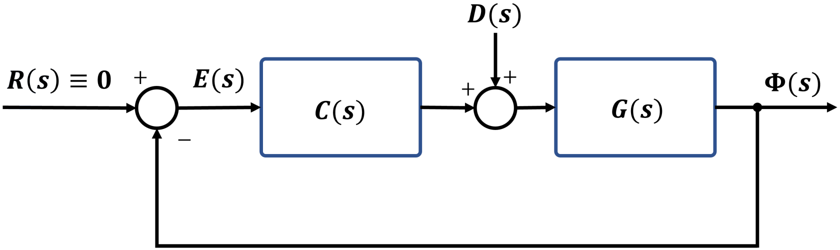
$$(M+m)\left(\frac{I+m\ell^2}{m\ell} - \frac{g}{s^2}\right)s^2\Phi(s) + b\left(\frac{I+m\ell^2}{m\ell} - \frac{g}{s^2}\right)s\Phi(s) - m\ell s^2\Phi(s) = U(s)$$

- Pendulum transfer function with $q = (M+m)(I+m\ell^2) - (m\ell)^2$:

$$G(s) = \frac{\Phi(s)}{U(s)} = \frac{m\ell s^2}{qs^4 + b(I+m\ell^2)s^3 - (M+m)mg\ell s^2 - bmgls}$$

Inverted Pendulum: PID Control

- Design a controller $C(s)$ to maintain the pendulum vertically upward when the cart input F is subjected to a 1-Nsec impulse disturbance $D(s)$
- Design specifications:
  - Settling time of less than 5 seconds
  - Maximum pendulum deviation from the vertical position of 0.05 rad



Inverted Pendulum: PID Control

- Pendulum transfer function with $q = (M+m)(I+m\ell^2) - (m\ell)^2$:

$$G(s) = \frac{\Phi(s)}{U(s)} = \frac{m\ell s^2}{qs^4 + b(I+m\ell^2)s^3 - (M+m)mg\ell s^2 - bmgls}$$

```
M = 0.5; m = 0.2; b = 0.1; I = 0.006;
g = 9.8; l = 0.3; q = (M+m)*(I+m*l^2)-(m*l)^2;
s = tf('s');
G = (m*l*s^2)/(q*s^4 + b*(I + m*l^2)*s^3 -(M + m)*m*g*l*s^2 -b*m*g*l*s);
```

- PID control design: $C(s) = K_p + K_i\frac{1}{s} + K_d s$

```
Kp = 100; Ki = 1; Kd = 1;
C = pid(Kp,Ki,Kd);
```

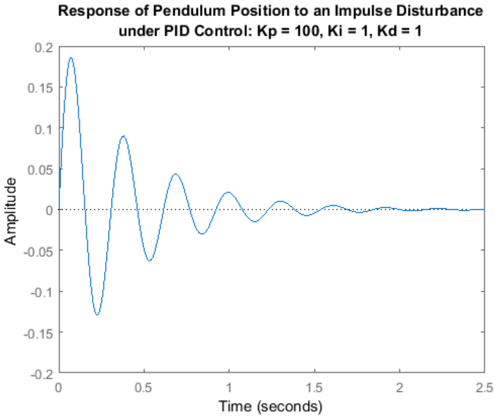
- Closed-loop transfer function from $D(s)$ to $\Phi(s)$:

$$T(s) = \frac{\Phi(s)}{D(s)} = \frac{G(s)}{1 + C(s)G(s)}$$

```
T = feedback(G,C);
```

Inverted Pendulum: PID Control

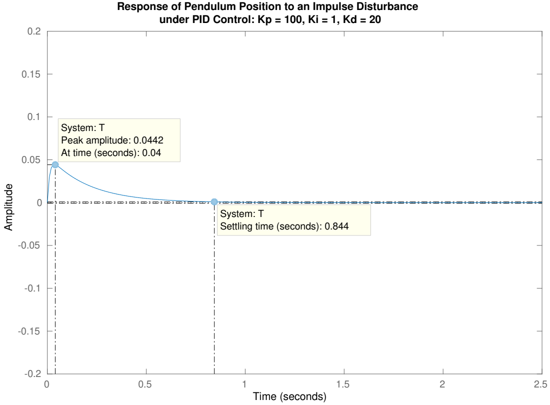
```
t=0:0.01:10;
impulse(T,t)
axis([0, 2.5, -0.2, 0.2]);
title({'Response of Pendulum Position to an Impulse Disturbance';'under
    PID Control: Kp = 100, Ki = 1, Kd = 1'});
```



- **Settling time**: 1.64 sec meets the specifications (no additional integral control is needed)
- **Peak response**: 0.2 rad exceeds the requirement of 0.05 rad (the overshoot can be reduced by increasing the derivative control gain)

Inverted Pendulum: PID Control

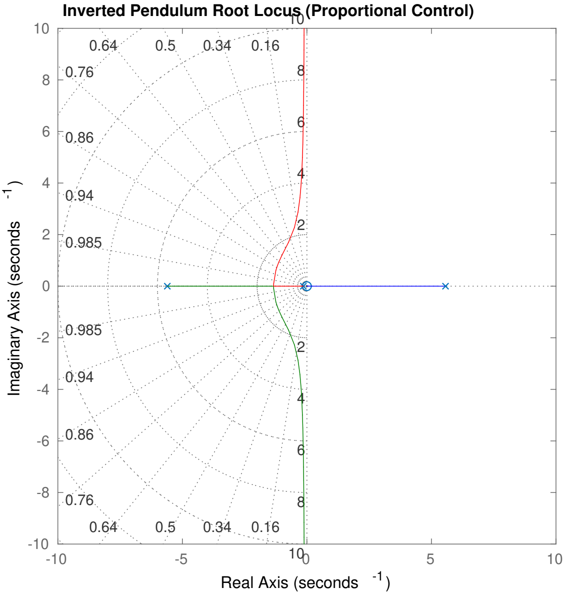
```
t=0:0.01:10;
impulse(T,t)
axis([0, 2.5, -0.2, 0.2]);
title({'Response of Pendulum Position to an Impulse Disturbance';'under
    PID Control: Kp = 100, Ki = 1, Kd = 20'});
```



- **Settling time**: 0.844 sec meets the specifications
- **Peak response**: 0.044 rad meets the specifications

Inverted Pendulum: Root Locus with Proportional Control

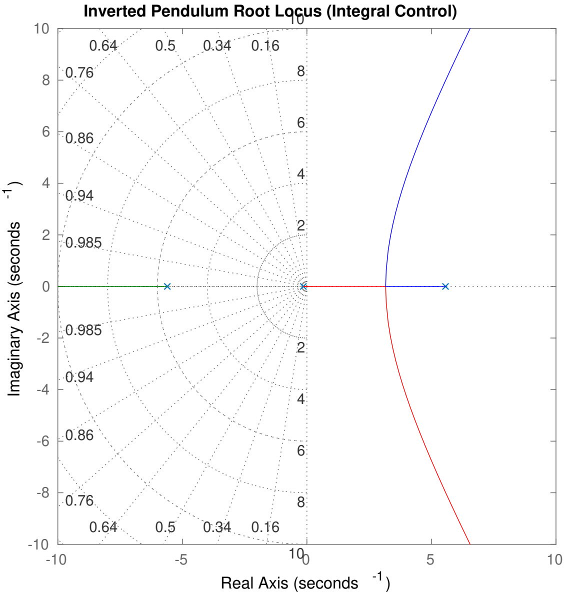
- Positive root locus for the inverted pendulum plant $G(s)$



- One branch entirely in the right half-plane
- Need to add a pole at the origin (integrator) to cancel the plant zero at the origin
- This will produce two closed-loop poles in the right half-plane that we can then draw to the left-half plane to stabilize the closed-loop system

Inverted Pendulum: Root Locus with Integral Control

- Positive root locus for integral control of the inverted pendulum $\frac{1}{s}G(s)$



- We need to draw the two branches to the left-half plane to stabilize the closed-loop system
- Adding a zeros to the controller will pull the branches to the left

Inverted Pendulum: Root Locus Manipulation

- Poles and zeros of $\frac{1}{s}G(s) = \frac{m\ell s^2}{qs^5+b(I+m\ell^2)s^4-(M+m)mg\ell s^3-bmgls^2}$:

$$z_1 = z_2 = 0$$
$$p_1 = p_2 = 0, \quad p_3 = -0.143, \quad p_4 = -5.604 \quad p_5 = 5.565$$

- Suppose we introduce a zero to the controller: $\frac{(s-z_3)}{s}G(s)$
- There will be $5-3=2$ asymptotes with angles $\frac{\pi}{2}$, $\frac{3\pi}{2}$ and centroid:

$$\alpha = \frac{1}{2}(-5.604+5.565-0.143-z_3) = -\frac{0.182+z_3}{2}$$

- We cannot have z_3 in the right half-plane so the best we can do to pull the root locus branches is to have $z_3 \approx 0$ so that $\alpha \approx -0.1$.
- The real parts of the two poles $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ will approach $\alpha \approx -0.1$ as $K \to \infty$
- This design is insufficient to meet the settling time specification:

$$t_s \approx \frac{4}{\zeta\omega_n} \approx \frac{4}{0.1} = 40$$

Inverted Pendulum: Root Locus Manipulation

- Adding a single zero to the controller is not sufficient to pull the root locus branches far enough to the left
- Add two zeros between $p_3 = -0.143$ and $p_4 = -5.604$ to pull the root locus branches towards them, leaving a single asymptote at $-\pi$
- Let $z_3 = -3$ and $z_4 = -4$ and consider the controller:

$$C(s) = \frac{(s+3)(s+4)}{s} = 7 + 12\frac{1}{s} + s$$

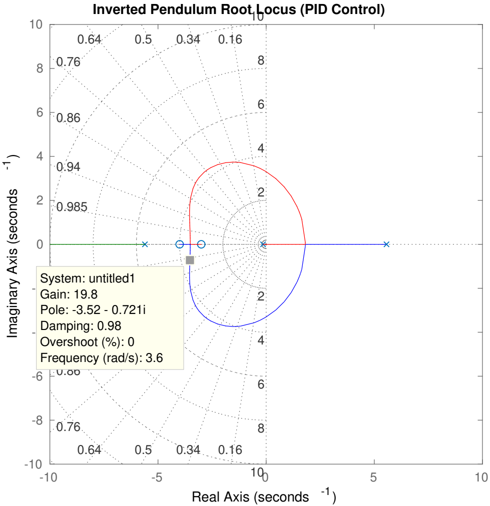
- Note that $KC(s)$ is a PID controller:

$$K_p = 7K \qquad K_i = 12K \qquad K_d = 1K$$

Inverted Pendulum: Root Locus with PID Control

- Positive root locus for PID control of the inverted pendulum:

$$\frac{(s+3)(s+4)}{s}G(s)$$



- To achieve $t_s \leq 5$ sec, we need the real parts of the dominant closed-loop poles to be less than $-4/5 = -0.8$
- To ensure that p.o. $\leq 5\%$, we also need sufficient damping for the dominant closed-loop poles
- Placing the dominant poles near the real axis increases the damping ratio ζ
- Choose $K \approx 20$

Inverted Pendulum: PID Control

```
T = feedback(G,20*(s+3)*(s+4)/s);
t=0:0.01:10;
impulse(T,t);
title({'Impulse Disturbance Response of Pendulum Angle'; 'under PID
    Control: Kp = 140, Ki = 240, Kd = 20'});
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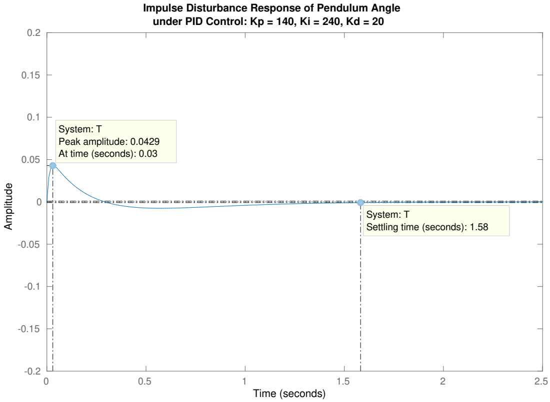

- **Settling time**: 1.580 sec meets the specifications
- **Peak response**: 0.043 rad meets the specifications