ECE171A: Linear Control System Theory
Lecture 10: Nyquist Stability

Nikolay Atanasov

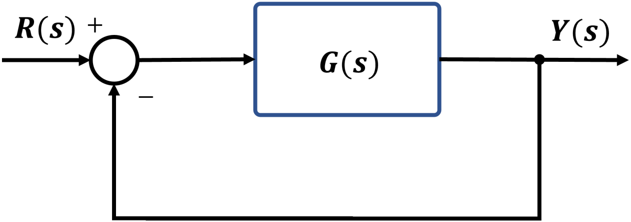
natanasov@ucsd.edu

JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

Outline

Outline

Stability of Feedback Systems



- Consider a feedback control system with open-loop transfer function $G(s)$ (controller and plant) and closed-loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

- Testing BIBO stability using the poles of $T(s)$ requires knowledge of $G(s)$ and gives little guidance for control design, i.e., how should the controller be modified to make an unstable system stable?

- Given a Bode plot of $G(s)$, we aim to understand the stability of $T(s)$

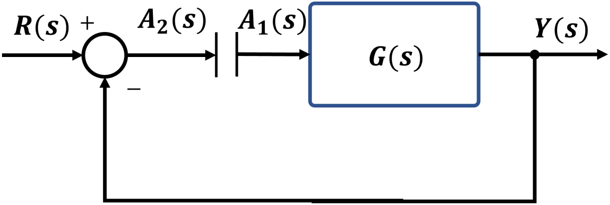
Nyquist's Idea

- **Harry Nyquist** made important contributions to control theory (stability of feedback systems), electronics (thermal noise), and communication theory (telegraph)
- Nyquist proposed an idea to determine the stability of a closed-loop system by investigating how sinusoidal signal propagate around the feedback loop

H. Nyquist

- Similar to return difference, break the feedback loop and ask whether a signal injected at $A_1(s)$ has the same or smaller magnitude when it reaches $A_2(s)$
- Nyquist's idea allows reasoning about **closed-loop stability** based on the **frequency response of the open-loop transfer function**

Nyquist's Idea

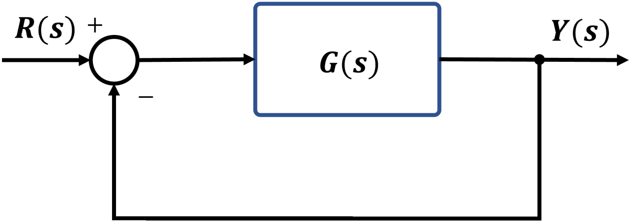


- Suppose that a sinusoid of frequency ω is injected at $A_1(s)$. In steady state, the signal at $A_2(s)$ will be a sinusoid with the same frequency ω, magnitude $|G(j\omega)|$, and phase $180^\circ + \angle G(j\omega)$
- **Critical point**: the signals at $A_1(s)$ and $A_2(s)$ are identical if:

$$|G(j\omega)| = 1 \quad \text{and} \quad \angle G(j\omega) = -180^\circ \quad \Leftrightarrow \quad G(j\omega) = -1$$

- **Nyquist's idea**: Let ω_p be such that $\angle G(j\omega_p) = -180^\circ$. A feedback control system is stable if $|G(j\omega_p)| < 1$ since the signal at $A_2(s)$ will have smaller amplitude than the injected signal at $A_1(s)$.

Open-loop Poles vs Closed-loop Poles



- Open-loop transfer function: $G(s) = \dfrac{b(s)}{a(s)}$
- Closed-loop transfer function: $T(s) = \dfrac{G(s)}{1 + G(s)} = \dfrac{b(s)}{a(s) + b(s)}$
- Let $\Delta(s) = 1 + G(s)$
  - **The closed-loop poles are the zeros of** $\Delta(s)$
  - **The open-loop poles are the poles of** $\Delta(s)$:

$$\Delta(s) = 1 + G(s) = 1 + \frac{b(s)}{a(s)} = \frac{a(s) + b(s)}{a(s)}$$

Outline

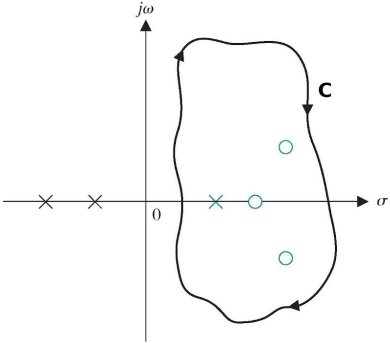
Nyquist's Idea

Principle of the Argument

Nyquist Stability Criterion

Contours in the Complex Plane

- Nyquist's stability criterion utilizes a contour C in the complex plane to relate the locations of the **open-loop poles** and the **closed-loop poles**
- A **contour** C is a piecewise smooth path in the complex plane
- A contour C is **closed** if it starts and ends at the same point
- A contour C is **simple** if it does not cross itself at any point
- A parameterization $z(t) \in \mathbb{C}$ of a contour has direction indicated by increasing the parameter $t \in \mathbb{R}$



- **Cauchy's Principle of the Argument**: relates the arguments (phases) of the zeros and poles of $G(s)$ inside a contour C to the shape of $G(C)$
- $G(C)$ is a new closed contour obtained by evaluating $G(s)$ at all s on C

Open-loop Transfer Function

- Consider a control system with open-loop transfer function:

$$G(s) = \kappa \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- At each s, $G(s)$ is a complex number with magnitude and phase:

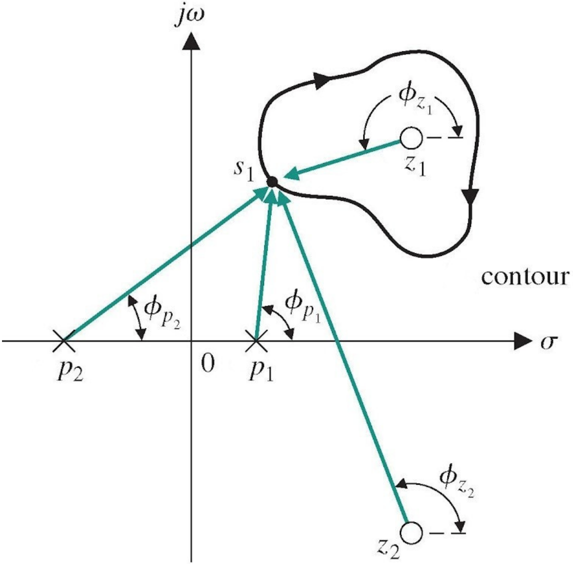
$$|G(s)| = |\kappa| \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{i=1}^{n} |s - p_i|} \qquad \angle G(s) = \angle \kappa + \sum_{i=1}^{m} \angle (s - z_i) - \sum_{i=1}^{n} \angle (s - p_i)$$

- Graphical evaluation of the magnitude and phase:
  - $|s - z_i|$ is the length of the vector from z_i to s
  - $|s - p_i|$ is the length of the vector from p_i to s
  - $\angle (s - z_i)$ is the angle from the real axis to the vector from z_i to s
  - $\angle (s - p_i)$ is the angle from the real axis to the vector from p_i to s

Evaluating $G(s)$ along a Contour

- Let C be a simple closed clockwise contour C in the complex plane
- Evaluating $G(s)$ at all points on C produces a new closed contour $G(C)$
- **Assumption**: C does not pass through the origin or any of the poles or zeros of $G(s)$ (otherwise $\angle G(s)$ is undefined)
- A zero z_i outside the contour C:
  - As s moves around the contour C, the vector $s - z_i$ swings up and down but not all the way around
  - The net change in $\angle(s - z_i)$ is 0
- A zero z_i inside the contour C:
  - As s moves around the contour C, the vector $s - z_i$ turns all the way around
  - The net change in $\angle(s - z_i)$ is -360°
- A pole p_i outside the contour C: the net change in $\angle(s - p_i)$ is 0
- A pole p_i inside the contour C: the net change in $\angle(s - p_i)$ is -360°

Evaluating $G(s)$ along a Contour



Principle of the Argument

- Let Z and P be the number of zeros and poles of $G(s)$ inside C
- As s moves around C, $\angle G(s)$ undergoes a net change of $-(Z-P)360^\circ$
- A net change of -360° means that the vector from 0 to $G(s)$ swings clockwise around the origin one full rotation
- A net change of $-(Z-P)360^\circ$ means that the vector from 0 to $G(s)$ must encircle the origin in clockwise direction $(Z-P)$ times

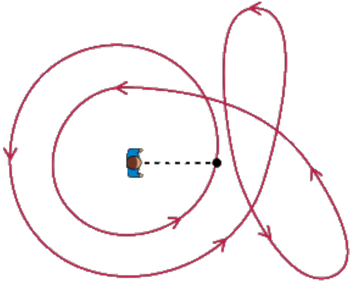
Cauchy's Principle of the Argument

Consider a transfer function $G(s)$ and a simple closed clockwise contour C. Let Z and P be the number of zeros and poles of $G(s)$ inside C. Then, the contour generated by evaluating $G(s)$ along C will encircle the origin in a clockwise direction $Z-P$ times.

Winding Number

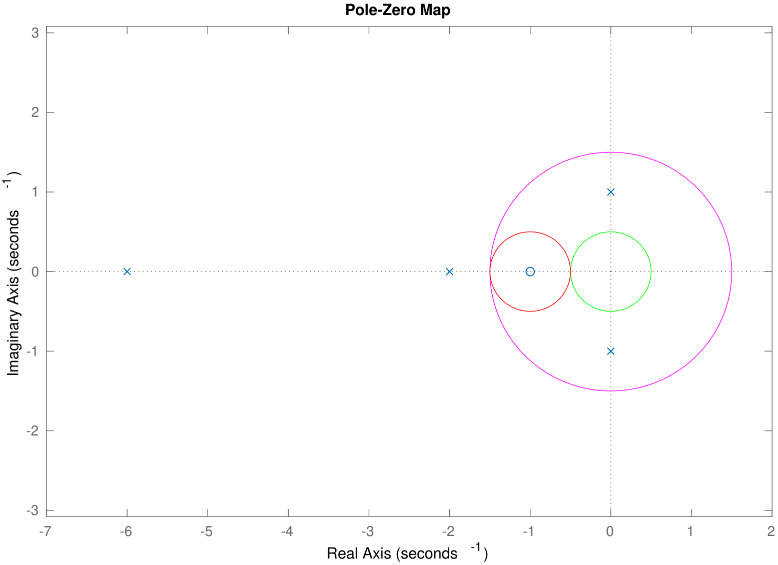
- To determine the number of encirclements of a point s by a contour Γ:
  1. Fix a pin at s pointing out of the page
  2. Attach a string from the pin to the contour Γ
  3. Let the end of the string attached to Γ traverse the contour
- The **winding number** $n(\Gamma, s)$ of Γ about s is equal to the number of times the string winds up on the pin when Γ is traversed:

$$n(\Gamma, s) = \frac{1}{2\pi j} \oint_{\Gamma} \frac{1}{s-z} dz$$



Principle of the Argument: Example

- Pole-zero map for $G(s) = \frac{10(s+1)}{(s+2)(s^2+1)(s+6)}$



Principle of the Argument: Example

- A circle contour C centered at the origin with radius 0.5 (green)
- The contour may be parameterized by $z(t) = 0.5e^{-jt}$ for $t \in [0, 2\pi]$
- The contour C is mapped by $G(s)$ to a new contour (from blue to red), e.g., parameterized by $G(z(t))$ for $t \in [0, 2\pi]$

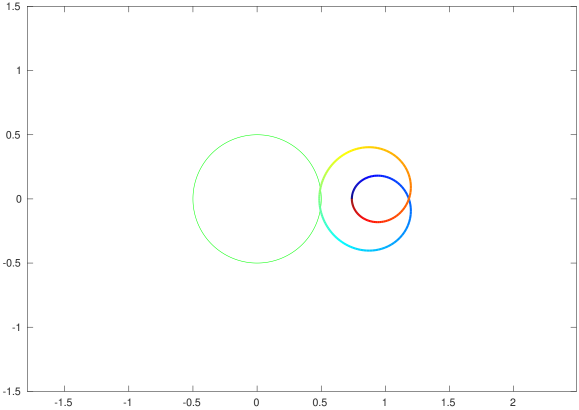


Figure: The origin is encircled 0 times clockwise

Principle of the Argument: Example

- A circle contour C centered at $(-1, 0)$ with radius 1 (red)
- The contour C is mapped by $G(s)$ to a new contour (from blue to red)

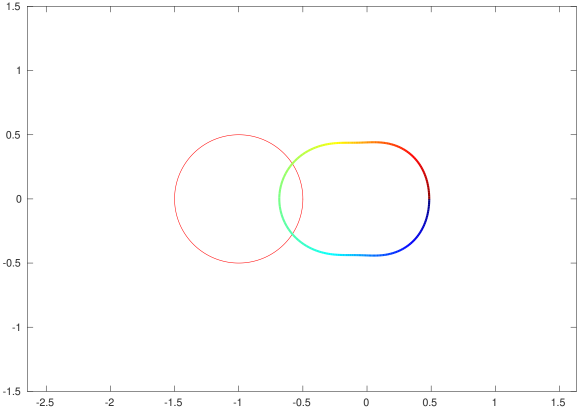


Figure: The origin is encircled 1 time clockwise

Principle of the Argument: Example

- A circle contour C centered at the origin with radius 1.5 (magenta)
- The contour C is mapped by $G(s)$ to a new contour (from blue to red)

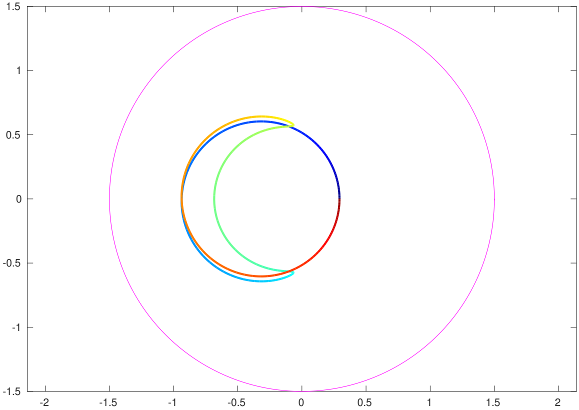


Figure: The origin is encircled 1 time counterclockwise

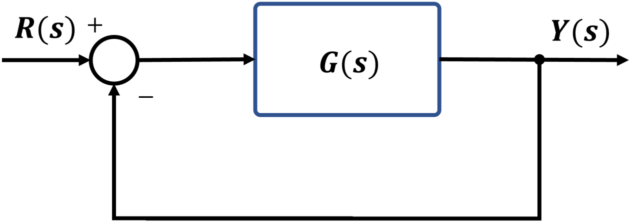
Outline

Nyquist's Idea

Principle of the Argument

Nyquist Stability Criterion

Open-loop Poles vs Closed-loop Poles

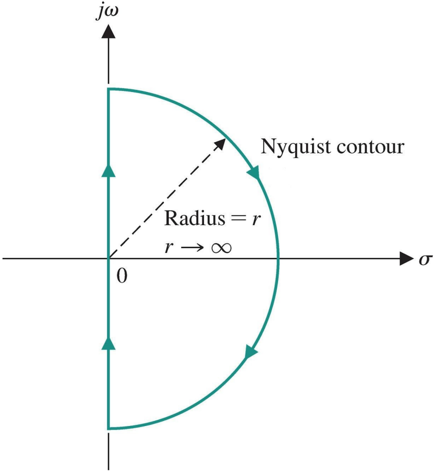


- Open-loop transfer function: $G(s) = \dfrac{b(s)}{a(s)}$
- Closed-loop transfer function: $T(s) = \dfrac{G(s)}{1 + G(s)} = \dfrac{b(s)}{a(s) + b(s)}$
- Let $\Delta(s) = 1 + G(s)$
  - **The closed-loop poles are the zeros of** $\Delta(s)$
  - **The open-loop poles are the poles of** $\Delta(s)$:

$$\Delta(s) = 1 + G(s) = 1 + \frac{b(s)}{a(s)} = \frac{a(s) + b(s)}{a(s)}$$

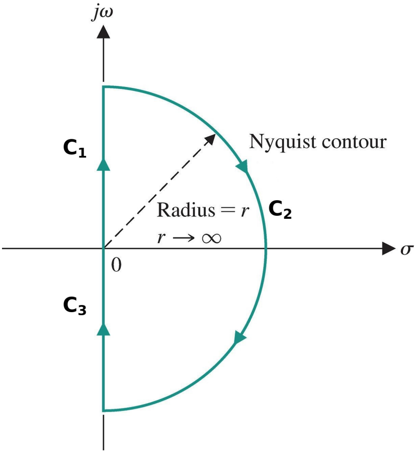
Nyquist Contour

- To determine how many closed-loop poles lie in the closed right half-plane, we apply the Principle of the Argument to $\Delta(s)$
- Define a clockwise contour C that covers the closed right half-plane



Nyquist Contour

- The Nyquist contour is made up of three parts:
  - **Contour** C_1: points $s = j\omega$ on the positive imaginary axis, as ω ranges from 0 to ∞
  - **Contour** C_2: points $s = re^{j\theta}$ on a semi-circle as $r \to \infty$ and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
  - **Contour** C_3: points $s = j\omega$ on the negative imaginary axis, as ω ranges from $-\infty$ to 0



Nyquist Plot

- A **Nyquist plot** evaluates $\Delta(s) = 1 + G(s)$ over the Nyquist contour C
- Contour $\Delta(C)$ is obtained by shifting contour $G(C)$ by one unit to the right

$$\text{Nyquist contour } C \quad \Rightarrow \quad \text{Nyquist plot } G(C)$$

- The contour $G(C)$ is obtained by combining $G(C_1)$, $G(C_2)$, and $G(C_3)$:
  - **Contour** C_1:
    - plot $G(j\omega)$ for $\omega \in (0, \infty)$ in the complex plane
    - equivalent to a **polar plot** for $G(s)$
  - **Contour** C_2:
    - plot $G(re^{j\theta})$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
    - as $r \to \infty$, $s = re^{j\theta}$ dominates every factor it appears in
    - if $G(s)$ is strictly proper, then $G(re^{j\theta}) \to 0$
    - if $G(s)$ is not strictly proper, then $G(re^{j\theta}) \to \text{const}$
  - **Contour** C_3:
    - plot $G(j\omega)$ for $\omega \in (-\infty, 0)$ in the complex plane
    - $G(-jb)$ is the complex conjugate of $G(jb)$
    - $G(-jb)$ and $G(jb)$ have the same magnitude but opposite phases
    - $G(C_3)$ is a reflected version of $G(C_1)$ about the real axis

Nyquist Plot: Example 1

- Draw a Nyquist plot of $G(s) = \frac{s+1}{s+10}$

- **Contour** C_1: $s = j\omega$ with $\omega \in (0, \infty)$:
  - $\omega = 0$ and $\omega \to \infty$:

$$G(j0) = \frac{1}{10}\underline{/0^\circ} \qquad\qquad G(j\infty) = 1\underline{/0^\circ}$$

  - for $0 < \omega < \infty$:

$$|G(j\omega)| = \frac{1}{10}\frac{\sqrt{1+\omega^2}}{\sqrt{1+(\omega/10)^2}} \qquad \underline{/G(j\omega)} = \tan^{-1}(\omega) - \tan^{-1}(\omega/10)$$

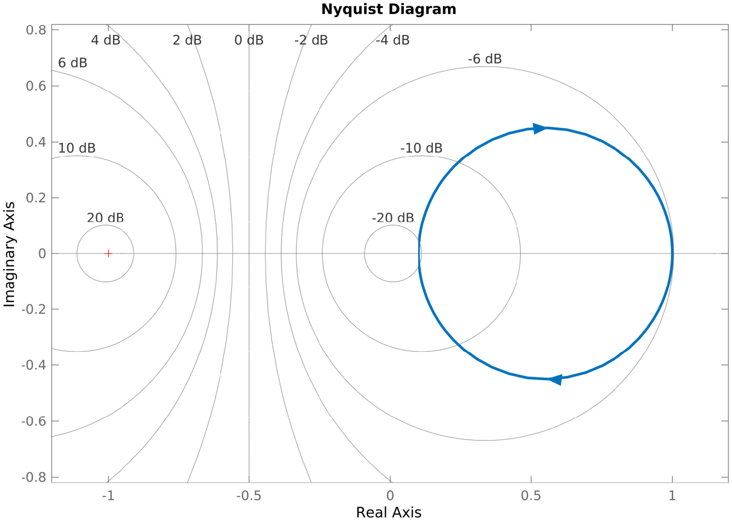
- **Contour** C_2: $s = re^{j\theta}$ with $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r\to\infty} G(re^{j\theta}) = \lim_{r\to\infty} \frac{re^{j\theta}+1}{re^{j\theta}+10} = 1\underline{/0^\circ}$$

- **Contour** C_3: $s = j\omega$ with $\omega \in (-\infty, 0)$:
  - $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

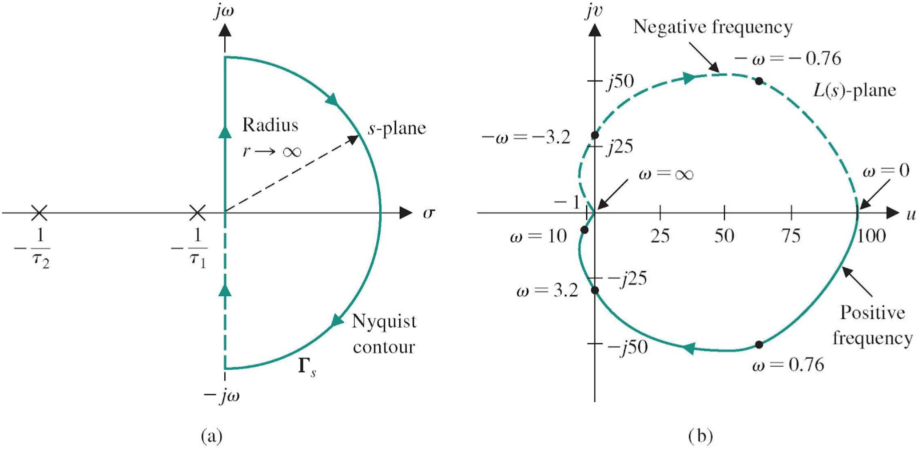
Nyquist Plot: Example 1

- Nyquist plot of $G(s) = \frac{s+1}{s+10}$
- Type 0 system as on Slide 56 of Lecture 9 with $\lim_{r \to \infty} G(re^{j\theta}) = 1$



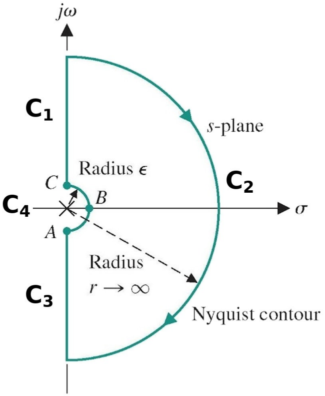
Nyquist Plot: Example 2

- Draw a Nyquist plot of $G(s) = \frac{\kappa}{(1+\tau_1 s)(1+\tau_2 s)} = \frac{100}{(1+s)(1+s/10)}$
- **Contour** C_1: $G(j0) = \kappa\underline{/0^\circ}$, $G(j\infty) = 0\underline{/-180^\circ}$
- **Contour** C_2: $\lim_{r\to\infty} G(re^{j\theta}) = 0$



Nyquist Plot: Poles on the Imaginary Axis

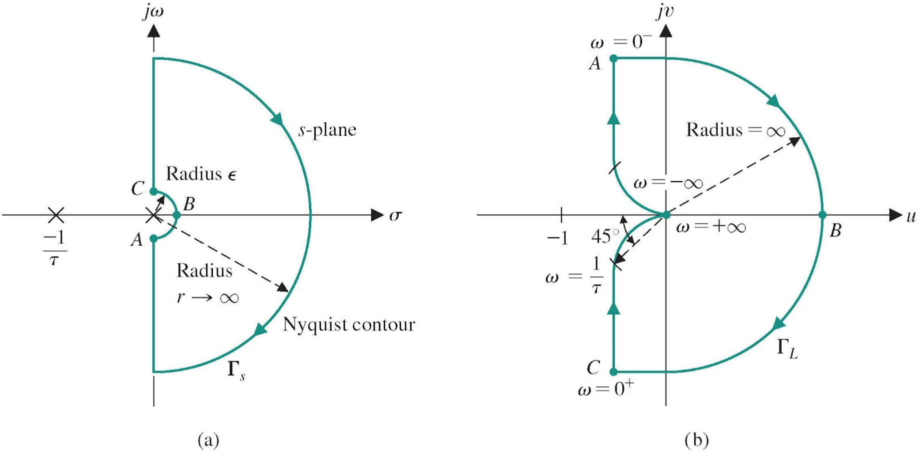
- The Principle of the Argument assumes C does not pass through zeros or poles of $\Delta(s)$
- There might be poles of $G(s)$ on the imaginary axis, which are poles of $\Delta(s)$
- The Nyquist contour needs to be modified to take a small detour around poles of $G(s)$ on the imaginary axis
- **Contour** C_4: avoid poles of $G(s)$ at origin:
  - plot $G(\epsilon e^{j\theta})$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$



- If $G(s)$ has other poles p on the imaginary axis, more contours need to be introduced. Substitute $s = p + \epsilon e^{j\theta}$ into $G(s)$ and examine what happens as $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Nyquist Plot: Example 3

- Draw a Nyquist plot of a type 1 system: $G(s) = \frac{\kappa}{s(1+\tau s)}$
- Since there is a pole at the origin, we need to use a modified Nyquist contour



Nyquist Plot: Example 3

- **Contour** C_1: $s = j\omega$ with $\omega \in (0, \infty)$: polar plot as on Slide 58 of Lecture 9:

$$G(j0^+) = \infty \underline{/-90^\circ}$$

$$G(j\infty) = \lim_{\omega\to\infty} \frac{\kappa}{j\omega(1 + j\omega\tau)} = \lim_{\omega\to\infty} \left|\frac{\kappa}{\tau\omega^2}\right| \underline{/-90^\circ - \tan^{-1}(\omega\tau)} = 0\underline{/-180^\circ}$$

- **Contour** C_2: $s = re^{j\theta}$ with $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r\to\infty} G(re^{j\theta}) = \lim_{r\to\infty} \left|\frac{\kappa}{\tau r^2}\right| e^{-2j\theta} = 0\underline{/-2\theta}$$

  - The phase of $G(s)$ changes from -180° at $\omega = \infty$ to 180° at $\omega = -\infty$

- **Contour** C_3: $s = j\omega$ with $\omega \in (-\infty, 0)$:
  - $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

- **Contour** C_4: $s = \epsilon e^{j\theta}$ with $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon\to 0} G(\epsilon e^{j\theta}) = \lim_{\epsilon\to 0} \frac{\kappa}{\epsilon e^{j\theta}(1 + \tau\epsilon e^{j\theta})} \overset{\frac{1}{1+\epsilon}\approx 1-\epsilon}{=\!=\!=\!=} -\kappa\tau + \lim_{\epsilon\to 0} \frac{\kappa}{\epsilon} e^{-j\theta} = \infty\underline{/-\theta}$$

  - $G(\epsilon e^{j\theta})$ approaches an asymptote at $-\kappa\tau$ as $\epsilon \to 0$
  - The phase of $G(s)$ changes from 90° at $\omega = 0^-$ to -90° at $\omega = 0^+$

Nyquist Plot: Example 4

- Draw a Nyquist plot of a type 1 system: $G(s) = \frac{\kappa}{s(1+\tau_1 s)(1+\tau_2 s)}$
- **Contour** C_4: $s = \epsilon e^{j\theta}$ with $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:
  - C_4 maps into a semicircle with infinite radius as in Example 3:

$$G(j0) = \infty \underline{/-\theta}$$

- **Contour** C_2: $s = re^{j\theta}$ with $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:
  - C_2 maps into a point at 0 with phase $\underline{/-3\theta}$
- **Contour** C_1: $s = j\omega$ with $\omega \in (0, \infty)$: polar plot as on Slide 59 of Lecture 9:

$$G(j\infty) = 0\underline{/-270^\circ}$$

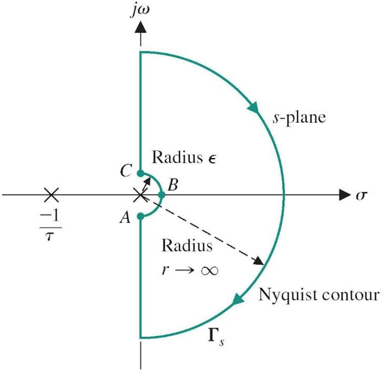
- **Contour** C_3: $G(C_3)$ is a reflection of $G(C_1)$ about the real axis

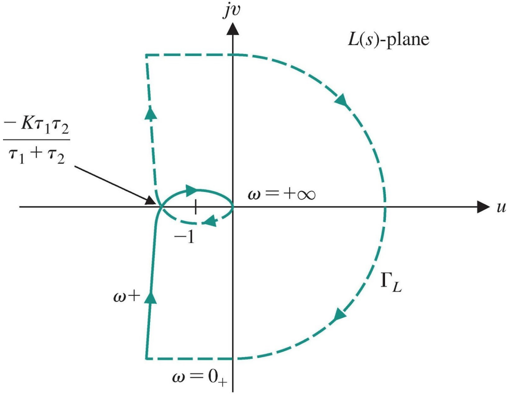
Nyquist Plot: Example 4

- **Contour** C_1 with $\omega \in (0, \infty)$:

$$G(j\omega) = \frac{\kappa}{j\omega(1+j\omega\tau_1)(1+j\omega\tau_2)} = \frac{-\kappa(\tau_1+\tau_2) - j\kappa(1-\omega^2\tau_1\tau_2)\omega}{1+\omega^2(\tau_1^2+\tau_2^2)+\omega^4\tau_1^2\tau_2^2}$$

$$= \frac{\kappa}{\sqrt{\omega^4(\tau_1+\tau_2)^2+\omega^2(1-\omega^2\tau_1\tau_2)^2}} \angle -90^\circ - \tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)$$





Nyquist Plot: Example 5

- Draw a Nyquist plot of a type 2 system: $G(s) = \frac{\kappa}{s^2(1+\tau s)}$
- Two poles at the origin $\Rightarrow$ need to use a modified Nyquist contour
- Magnitude and phase:

$$G(j\omega) = \frac{\kappa}{(j\omega)^2(1+j\omega\tau)} = \frac{|\kappa|}{\sqrt{\omega^4+\omega^6\tau^2}}\underline{/-180^\circ - \tan^{-1}(\omega\tau)}$$

- **Contour** C_1: $s = j\omega$ with $\omega \in (0,\infty)$:

$$\begin{aligned} G(j0^+) &= \infty\underline{/-180^\circ} \\ G(j\infty) &= \lim_{\omega\to\infty} \frac{\kappa}{(j\omega)^2(1+j\omega\tau)} = \lim_{\omega\to\infty} \left|\frac{\kappa}{\tau\omega^3}\right| \underline{/-180^\circ - \tan^{-1}(\omega\tau)} \\ &= 0\underline{/-270^\circ} \end{aligned}$$

- **Contour** C_3: $s = j\omega$ with $\omega \in (-\infty, 0)$:
  - $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

Nyquist Plot: Example 5

- Magnitude and phase:

$$G(j\omega) = \frac{\kappa}{(j\omega)^2(1+j\omega\tau)} = \frac{|\kappa|}{\sqrt{\omega^4+\omega^6\tau^2}} \underline{/-180^\circ - \tan^{-1}(\omega\tau)}$$

- **Contour** C_2: $s = re^{j\theta}$ with $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r\to\infty} G(s) = \lim_{r\to\infty} \frac{\kappa}{\tau s^3} = \lim_{r\to\infty} \left|\frac{\kappa}{\tau r^3}\right| e^{-3j\theta} = 0 \underline{/-3\theta}$$

  - The phase of $G(s)$ changes from -270° at $\omega = \infty$ to 270° at $\omega = -\infty$

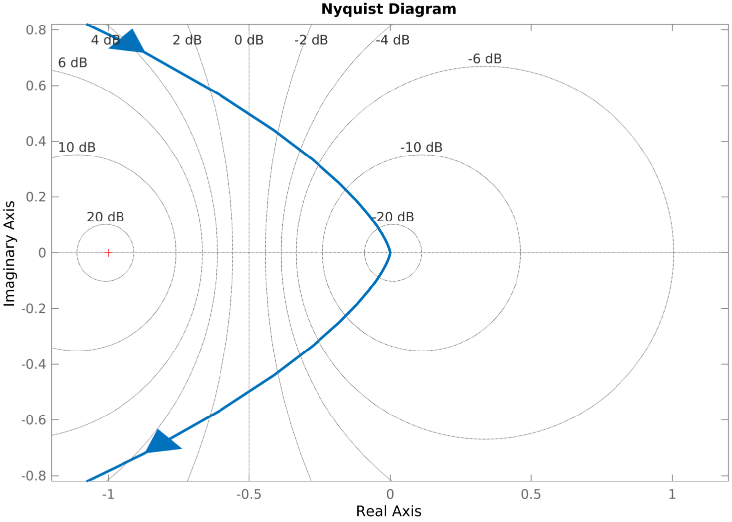
- **Contour** C_4: $s = \epsilon e^{j\theta}$ with $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon\to 0} G(s) = \lim_{\epsilon\to 0} \frac{\kappa}{s^2} = \lim_{\epsilon\to 0} \frac{\kappa}{\epsilon^2} e^{-2j\theta} = \infty \underline{/-2\theta}$$

  - The phase of $G(s)$ changes from 180° at $\omega = 0^-$ to -180° at $\omega = 0^+$

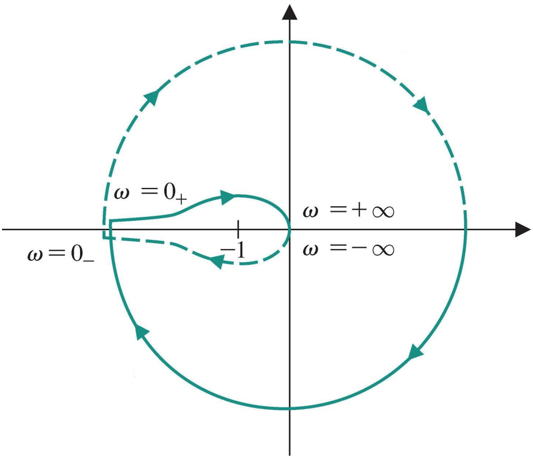
Nyquist Plot: Example 5

- Nyquist plot of a type 2 system: $G(s) = \frac{\kappa}{s^2(1+\tau s)} = \frac{1}{s^2(s+1)}$
- **Caution**: Matlab's *nyquistplot* does not generate $G(C_4)$



Nyquist Plot: Example 5

- Nyquist plot of a type 2 system: $G(s) = \frac{K}{s^2(1+\tau s)}$



Nyquist Plot: Example 6

- Draw a Nyquist plot of $G(s) = \frac{1}{(s+a)^3}$

$$G(j\omega) = \frac{1}{(j\omega + a)^3} = \frac{(a - j\omega)^3}{(a^2 + \omega^2)^3} = \frac{a^3 - 3a\omega^2}{(a^2 + \omega^2)^3} + j\frac{\omega^3 - 3a^2\omega}{(a^2 + \omega^2)^3}$$

- **Contour** C_1: $s = j\omega$ with $\omega \in (0, \infty)$:

$$G(j0) = \frac{1}{a^3}\underline{/0^\circ}, \qquad G(j\infty) = 0\underline{/-270^\circ}$$

- **Contour** C_2: $s = re^{j\theta}$ with $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$G(re^{j\theta}) = \frac{1}{(re^{j\theta} + a)^3} \to 0\underline{/-3\theta}$$

- **Contour** C_3: a reflection (complex conjugate) of $G(C_1)$ about the real axis

Nyquist Plot: Example 6

- Draw a Nyquist plot of $G(s) = \frac{1}{(s+0.6)^3}$

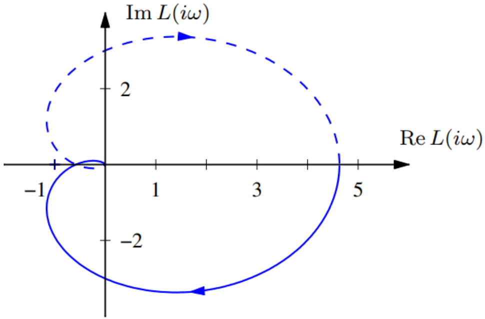
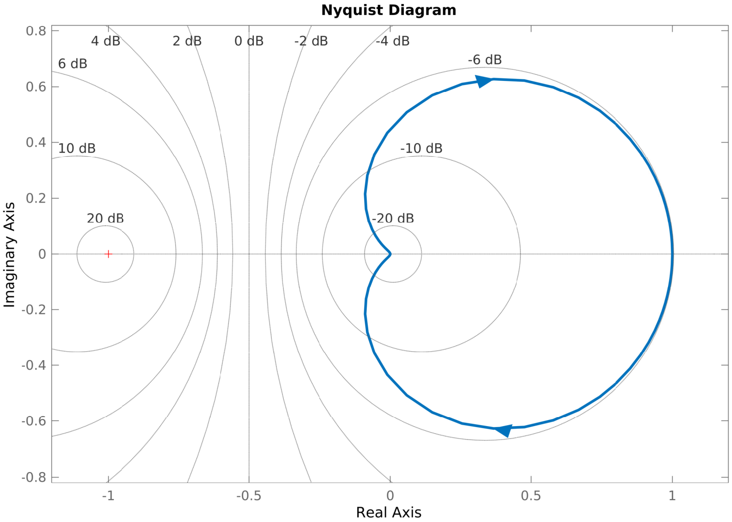


Figure 10.5: Nyquist plot for a third-order transfer function $L(s)$. The Nyquist plot consists of a trace of the loop transfer function $L(s) = 1/(s+a)^3$ with $a = 0.6$. The solid line represents the portion of the transfer function along the positive imaginary axis, and the dashed line the negative imaginary axis. The outer arc of the Nyquist contour Γ maps to the origin.

Nyquist Plot: Example 7

- Draw a Nyquist plot of $G(s) = \frac{s(s+1)}{(s+10)^2}$



Nyquist's Stability Criterion

- Consider the stability of the closed-loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{G(s)}{\Delta(s)}$$

- **Open-loop poles**: the poles of $\Delta(s)$ are the poles of $G(s)$
- **Closed-loop poles**: the zeros of $\Delta(s)$ are the poles of $T(s)$
- Principle of the Argument applied to $\Delta(s) = 1 + G(s)$:
  - Let C be a Nyquist contour
  - Let P be the number of poles of $\Delta(s)$ (open-loop poles) inside C
  - Let Z be the number of zeros of $\Delta(s)$ (closed-loop poles) inside C
  - Then, $\Delta(C)$ encircles the origin in clockwise direction $N = Z - P$ times

Nyquist's Stability Criterion

- From the Principle of the Argument applied to $\Delta(s)$, the number of closed-loop poles in the closed right half-plane is:

$$Z = N + P$$

where:

  - N: the clockwise encirclements of the origin by $\Delta(C)$ correspond to the clockwise encirclements of $-1 + j0$ by $G(C)$ and can be determined from a Nyquist plot of $G(s)$
  - P: the number of poles of $\Delta(s)$ inside C corresponds to the number of poles of $G(s)$ inside C and can be determined from $G(s)$ or its Bode plot

Nyquist's Stability Criterion

Consider a unity feedback control system with open-loop transfer function $G(s)$. Let C be a Nyquist contour. The system is stable if and only if the number of counterclockwise encirclements of $-1 + j0$ by $G(C)$ is equal to the number of poles of $G(s)$ inside C.

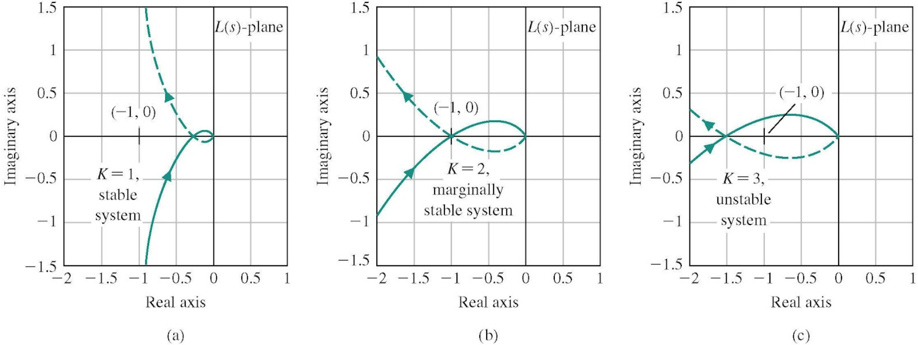
Nyquist Stability: Example 4

- Determine the closed-loop stability of $G(s) = \frac{\kappa}{s(1+\tau_1 s)(1+\tau_2 s)} = \frac{\kappa}{s(1+s)^2}$
- $G(C_1)$ crosses the real axis when:

$$G(j\omega) = \frac{-\kappa(\tau_1+\tau_2) - j\kappa(1-\omega^2\tau_1\tau_2)\omega}{1+\omega^2(\tau_1^2+\tau_2^2)+\omega^4\tau_1^2\tau_2^2} = \alpha + j0$$

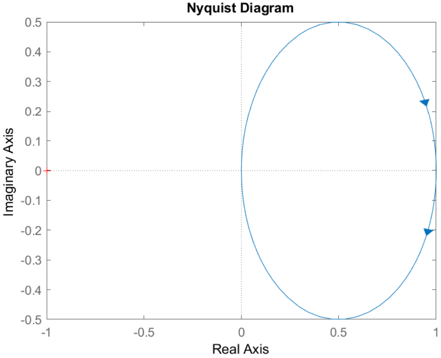
$$\Rightarrow \omega = \frac{1}{\sqrt{\tau_1\tau_2}} \qquad \alpha = -\frac{\kappa\tau_1\tau_2}{\tau_1+\tau_2}$$

- The system is stable when $\alpha = -\frac{\kappa\tau_1\tau_2}{\tau_1+\tau_2} \geq -1$



Nyquist Plot: Example 8

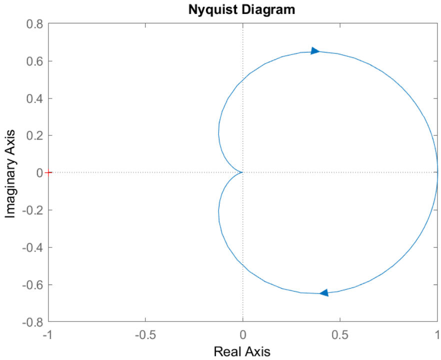
- Open-loop transfer function: $G(s) = \frac{1}{s+1}$
- Number of closed-loop poles in CRHP: $Z = N + P = 0$



- Closed-loop transfer function: $T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s+2}$

Nyquist Plot: Example 9

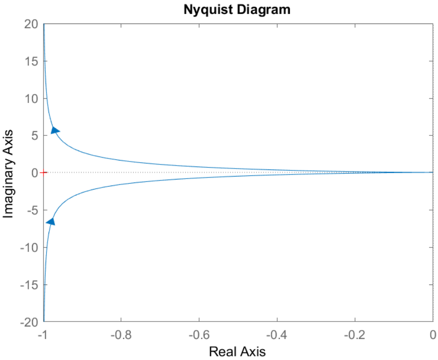
- Open-loop transfer function: $G(s) = \frac{1}{(s+1)^2}$
- Number of closed-loop poles in CRHP: $Z = N + P = 0$



- Closed-loop transfer function: $T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s^2+2s+2}$

Nyquist Plot: Example 10

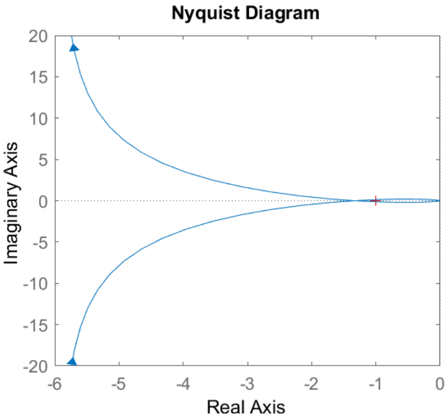
- Open-loop transfer function: $G(s) = \frac{1}{s(s+1)}$
- Number of closed-loop poles in CRHP: $Z = N + P = 0$



- Closed-loop transfer function: $T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s^2+s+1}$

Nyquist Plot: Example 11

- Open-loop transfer function: $G(s) = \frac{1}{s(s+1)(s+0.5)}$
- Number of closed-loop poles in CRHP: $Z = N + P = 2$



- Closed-loop transfer function: $T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s^3+1.5s^2+0.5s+1}$
- Closed-loop poles: $p_{1,2} = 0.0416 \pm j0.7937$ and $p_3 = -1.5832$