

# ECE171A: Linear Control System Theory

## Lecture 11: Performance Measures

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# Outline

Stability Margins

Frequency Domain Performance Specifications

Closed-loop Control from Open-loop Frequency Response

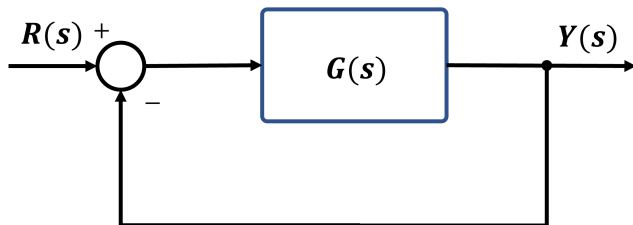
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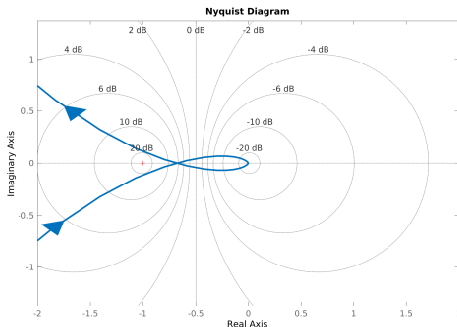
## Stability Margins from a Nyquist Plot



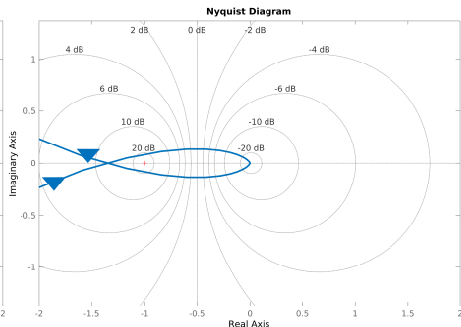
- ▶ Consider an open-loop transfer function:  $G(s) = k \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}$
- ▶ Increasing  $k$  increases the magnitude of all points on the Nyquist plot of  $G(s)$ , i.e., pushes the contour  $G(C)$  further away from the origin

## Stability Margins from a Nyquist Plot: Example

- ▶ Nyquist plot of  $G(s) = \frac{k}{s(s+1)(s+10)}$



(a)  $k = 75$

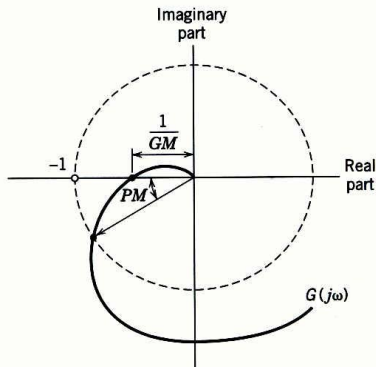


(b)  $k = 150$

- ▶ The closed-loop system is stable for small  $k$  and unstable for large  $k$
- ▶ In practice, it is not enough that the system is stable. There must also be a stability margin allowing robustness to disturbances.
- ▶ **Stability margin:** quantifies how far the Nyquist plot  $G(C)$  is from the critical point  $-1$

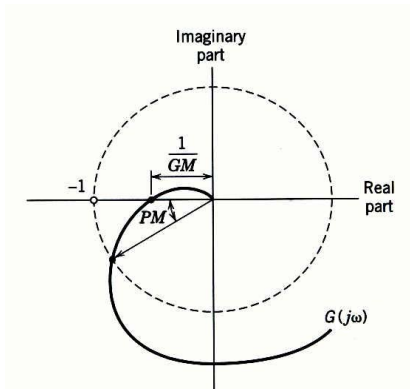
## Gain Margin

- ▶ **Gain Margin (GM):**
  - ▶ the factor by which the open-loop gain can be increased before a stable closed-loop system becomes unstable
  - ▶ the factor by which the open-loop gain should be decreased until an unstable system becomes stable
- ▶ Nyquist plot: GM is the inverse of the distance from the origin to the first point where  $G(C)$  crosses the real axis



## Phase Margin

- ▶ **Phase Margin (PM):**
  - ▶ the amount by which the open-loop phase can be decreased before a stable closed-loop system becomes unstable
  - ▶ the amount by which the open-loop phase should be increased before an unstable system becomes stable
- ▶ Nyquist plot: PM is the smallest angle on the unit circle between  $-1$  and  $G(C)$



## Algebraic Definitions of Gain Margin and Phase Margin

- ▶ **Phase-Crossover Frequency:**  $\omega_p$  at which  $G(j\omega)$  crosses the real axis:

$$\angle G(j\omega_p) = -180^\circ$$

- ▶ **Gain Margin:** the inverse of the open-loop gain at  $\omega_p$ :

$$GM = 20 \log \frac{1}{|G(j\omega_p)|} = -20 \log |G(j\omega_p)| \text{ dB}$$

- ▶ **Gain-Crossover Frequency:**  $\omega_g$  at which  $G(j\omega)$  crosses the unit circle:

$$20 \log |G(j\omega_g)| = 0 \text{ dB}$$

- ▶ **Phase Margin:** amount by which the open-loop phase at  $\omega_g$  exceeds  $-180^\circ$ :

$$PM = \angle G(j\omega_g) + 180^\circ$$



## Gain Margin and Phase Margin

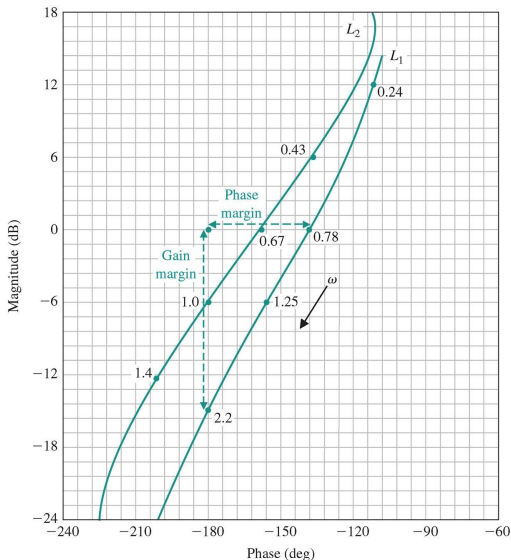
- ▶ For a stable minimum-phase system both  $GM$  and  $PM$  are positive. Larger gains mean larger relative stability.
- ▶ When  $\omega_g = \omega_p = \omega_*$ , there are closed-loop poles on the imaginary axis and instability starts to occur:

$$|G(j\omega_*)| = 1, \quad \underline{\angle G(j\omega_*)} = -180^\circ \quad \Rightarrow \quad 1 + G(j\omega_*) = 0$$

- ▶ **Bode plot** and **magnitude-phase** plot provide  $|G(j\omega)|$  and  $\underline{\angle G(j\omega)}$  and hence  $\omega_p$ ,  $\omega_g$ ,  $GM$ , and  $PM$  can all be seen
- ▶ **Caution:** the Bode plot or magnitude-phase plot interpretation of  $GM$  and  $PM$  to determine stability can be incorrect if the system is non-minimum phase or has delays. Only the Nyquist stability criterion should be used to determine stability.

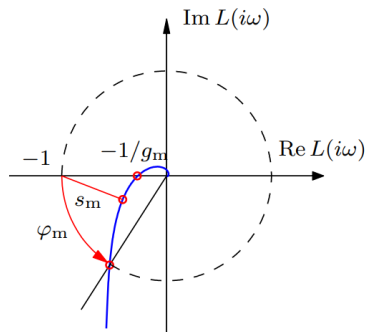
## Gain Margin and Phase Margin on a Magnitude-Phase Plot

- Magnitude-phase plot of  $G_1(s) = \frac{1}{s(s+1)(s/5+1)}$  and  $G_2(s) = \frac{1}{s(s+1)^2}$

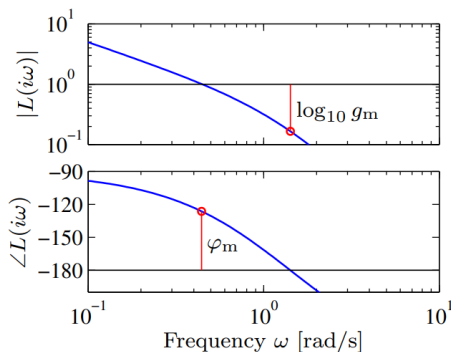


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## Gain Margin and Phase Margin on a Bode Plot



(a) Nyquist plot

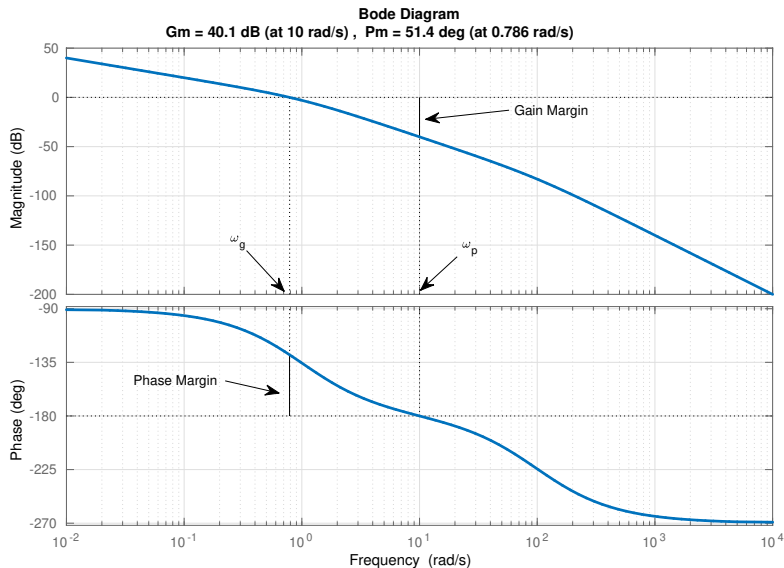


(b) Bode plot

- ▶ **Stability margin:** shortest distance  $s_m$  from Nyquist plot  $G(C)$  to  $-1$
- ▶ **Gain margin:** inverse gain  $g_m$  at phase-crossover  $\omega_p$
- ▶ **Phase margin:** phase distance  $\varphi_m$  from  $-180^\circ$  at gain-crossover  $\omega_g$

## Gain Margin and Phase Margin on a Bode Plot

- Bode plot of  $G(s) = \frac{k}{s(s+1)(s/100+1)}$  with  $k = 1$



## Gain Margin and Phase Margin on a Bode Plot

- ▶ If  $k > 0$ , it has no effect on the phase and shifts the magnitude up or down by  $20 \log k$ . This changes the gain-crossover frequency  $\omega_g$  but not the phase-crossover frequency  $\omega_p$ .
- ▶ Some closed-loop poles lie on the imaginary axis when  $\omega_g = \omega_p$
- ▶ Choose  $k \approx 100$  to shift the magnitude up by  $\sim 40$  dB, making  $\omega_g \approx \omega_p$
- ▶ The imaginary axis crossing can be determined from the Bode plot but we do not know if we are going from stability to instability or vice versa
- ▶ Assuming that the system is stable initially (can only be verified by Nyquist or Routh-Hurwitz stability criteria), we expect the region of stability to be  $0 < K < 100$

## Gain Margin and Phase Margin on a Bode Plot

- ▶ Use Routh-Hurwitz to verify the region of stability for:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{k}{s(s+1)(s/100+1) + k} = \frac{100k}{s^3 + 101s^2 + 100s + 100k}$$

- ▶ Characteristic polynomial  $a(s) = s^3 + 101s^2 + 100s + 100k$
- ▶ The Routh table is:

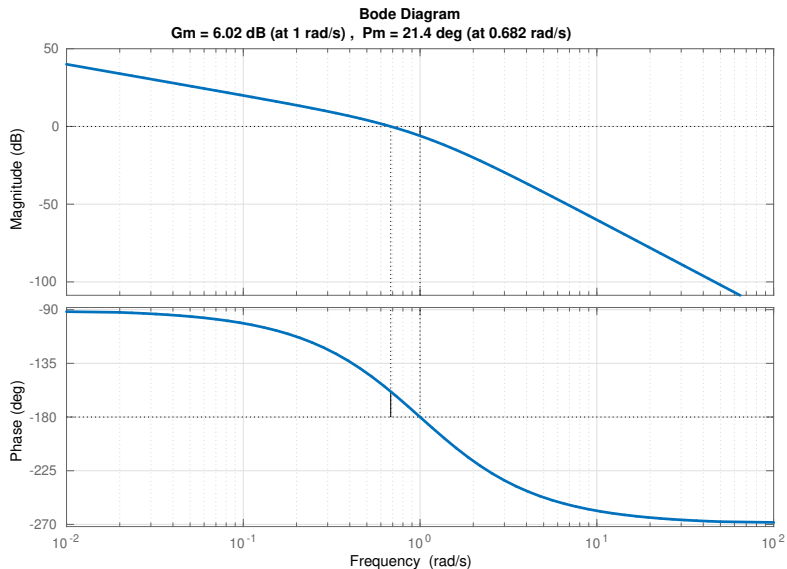
$s^3$	1	100
$s^2$	101	$100k$
$s^1$	$100 - \frac{100k}{101}$	0
$s^0$	$100k$	0

- ▶ Stability region:  $0 < k < 101$
- ▶ Auxiliary polynomial roots for  $k = 101$ :

$$A(s) = 101(s^2 + 100) \quad \Rightarrow \quad s = \pm j10$$

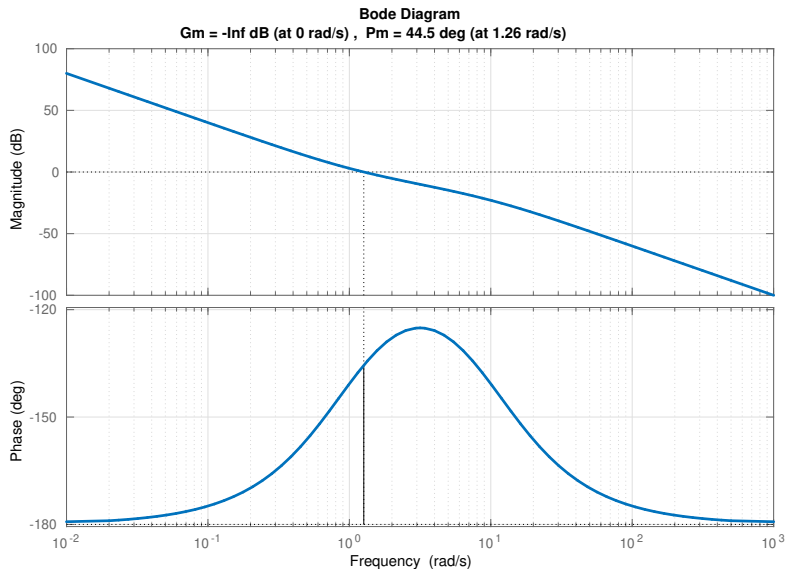
## Stability Margins: Example 1

- ▶ What are the gain margin and phase margin of  $G(s) = \frac{1}{s(s+1)^2}$ ?



## Stability Margins: Example 2

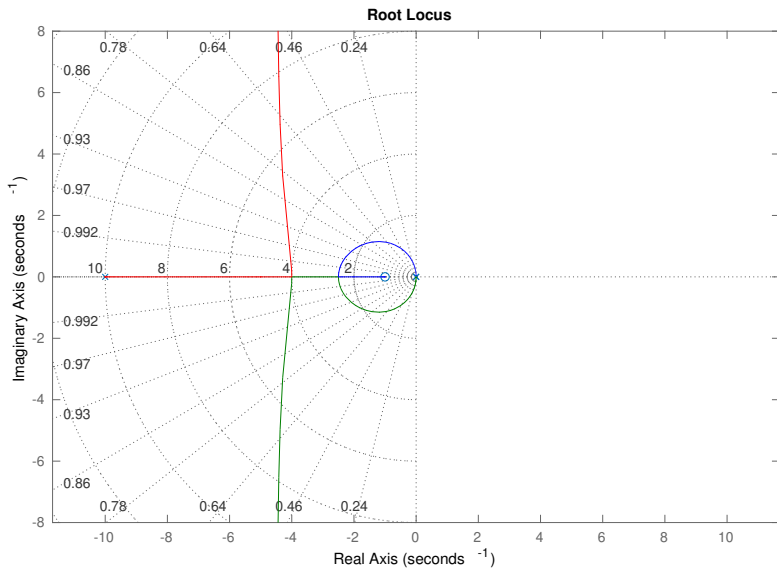
- What are the gain margin and phase margin of  $G(s) = \frac{(s+1)}{s^2(s/10+1)}$ ?





## Stability Margins: Example 2

- ▶ Root locus of  $G(s) = \frac{(s+1)}{s^2(s/10+1)}$



## Stability Margins: Example 2

- ▶ What are the gain margin and phase margin of  $G(s) = \frac{k(s+1)}{s^2(s/10+1)}$ ?
- ▶ The gain margin is  $\infty$  since the phase hits  $-180^\circ$  at  $\omega_p = \infty$
- ▶ As  $k \rightarrow \infty$ , the gain-crossover frequency  $\omega_g$  moves to the right and the **phase margin decreases**
- ▶ As  $k \rightarrow \infty$ , a pair of closed-loop poles moves vertically on the root locus and the **damping ratio  $\zeta$  decreases**
- ▶ There is a relationship between **phase margin PM** and **damping ratio  $\zeta$**
- ▶ We will analyze a second-order system to determine this and establish a relationship between **frequency response** and **transient step response**

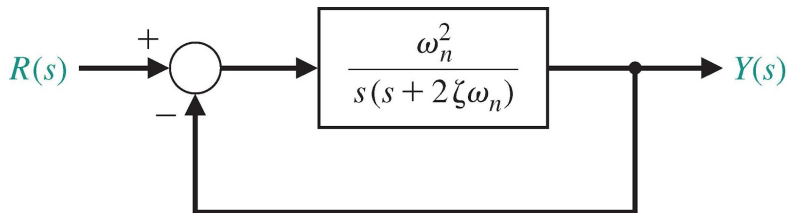
# Outline

Stability Margins

Frequency Domain Performance Specifications

Closed-loop Control from Open-loop Frequency Response

## Frequency Domain Performance Specifications



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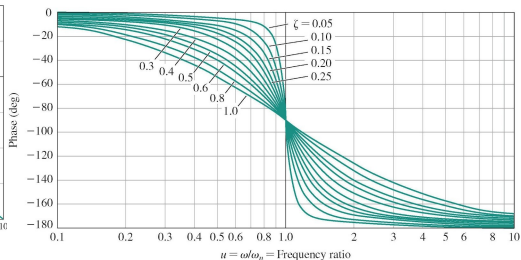
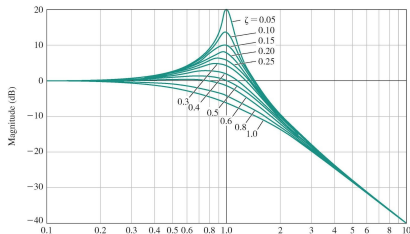
- ▶ Consider a second-order system:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1}$$

- ▶ How does the closed-loop **frequency response**  $T(j\omega)$  relate to the **transient step response** (rise time, overshoot, settling time)?

# Frequency Response of a Second-order System

- ▶ Bode plot of  $T(s) = \frac{1}{\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1}$



- ▶ The **damping ratio**  $\zeta$  is related to the **resonant peak**  $\max_{\omega} |T(j\omega)|$
- ▶ The **natural frequency**  $\omega_n$  and **rise time**  $t_r$  are related to the **bandwidth**  $\omega_b$  (frequency range  $(0, \omega_b)$  over which the system tracks an input signal well)

## Frequency Domain Performance Specifications

- ▶ **Low-frequency (DC) gain:** the magnitude of the transfer function  $|T(j\omega)|$  for low frequencies  $\omega \rightarrow 0$  is equal to the steady-state step response
- ▶ **Bandwidth:** the frequency  $\omega_b$  at which the transfer function magnitude drops 3 dB below the DC gain:

$$|T(j\omega_b)| = \frac{1}{\sqrt{2}}|T(0)|$$

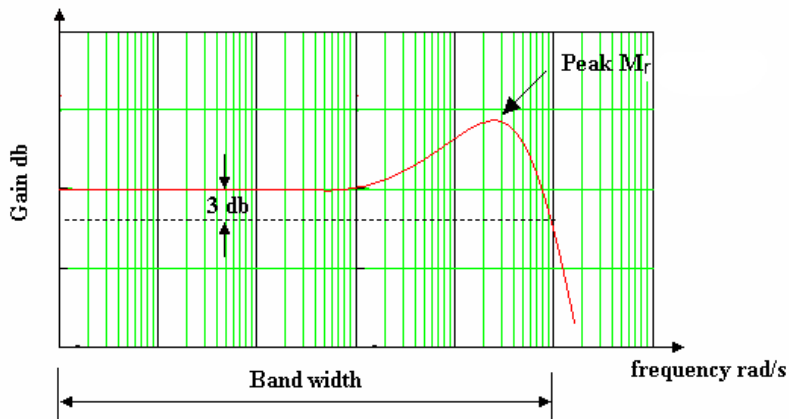
- ▶ **Resonant frequency:**  $\omega_r$  where the transfer function magnitude is maximized:

$$\omega_r = \arg \max_{\omega} |T(j\omega)|$$

- ▶ **Resonant peak:** the maximum value of the transfer function magnitude:

$$M_r = |T(j\omega_r)|$$

## Frequency Domain Performance Specifications



## Frequency Response of a Second-order System

- ▶ Consider a second-order system:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1}$$

- ▶ **Transfer function magnitude** at  $s = j\omega$ :

$$|T(j\omega)| = \frac{1}{\left| -\frac{\omega^2}{\omega_n^2} + 2\zeta\frac{\omega}{\omega_n}j + 1 \right|} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

- ▶ **Transfer function phase** at  $s = j\omega$ :

$$\angle T(j\omega) = \angle \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)j + 1} = -\arctan\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$



## Resonant Frequency of a Second-order System

- ▶ Resonant frequency:

$$\frac{d|T(j\omega)|}{d\omega} = 0 \quad \Rightarrow \quad \omega_r = 0 \quad \text{or} \quad \omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

- ▶ Resonant peak:

- ▶ Case 1:  $\zeta \leq \frac{1}{\sqrt{2}}$ :

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \qquad M_r = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$$

- ▶ Case 2:  $\zeta > \frac{1}{\sqrt{2}}$ :

$$\omega_r = 0 \qquad M_r = 1$$

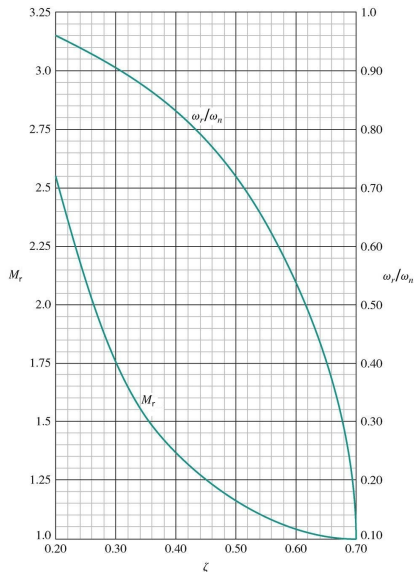
## Resonant Frequency of a Second-order System

► Plot of  $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$  and  $\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$  as a function of  $\zeta$

► The **resonant peak**  $M_r$  is related to the **percent overshoot** via  $\zeta$

► Example:

- The resonant peak of the closed-loop system should be less than 1.75 ( $\approx 5$  dB)
- Equivalent to  $\zeta$  should be greater than 0.3
- Equivalent to p.o. should be less than 37%



## Bandwidth of a Second-order System

- ▶ **Bandwidth:** the low frequency range  $(0, \omega_b)$  over which the closed-loop system tracks an input signal well

$$|T(j\omega_b)| = \frac{1}{\sqrt{2}}|T(0)|$$

- ▶ **Relationship between  $\omega_b$ ,  $\omega_n$ , and  $\zeta$ :** with  $u = \omega_b/\omega_n$ :

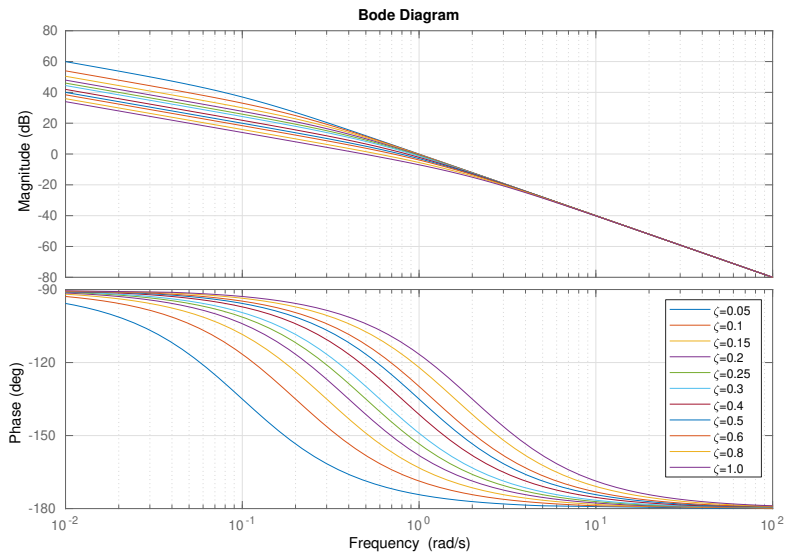
$$u^4 + 2(\zeta^2 - 1)u^2 + 1 = 2 \quad \Rightarrow \quad u^2 = (1 - 2\zeta^2) \pm \sqrt{4\zeta^4 - 4\zeta^2 + 2}$$

$$\omega_b = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

- ▶ Bandwidth  $\omega_b$  and rise time  $t_r \approx \frac{2.16\zeta + 0.6}{\omega_n}$  are inversely proportional:
  - ▶ If  $\omega_n \uparrow$ , then  $\omega_b \uparrow$  and  $t_r \downarrow$
  - ▶ If  $\zeta \uparrow$ , then  $\omega_b \downarrow$  and  $t_r \uparrow$
- ▶ Adding a zero to  $G(s)$  increases  $\omega_b$  of the closed-loop transfer function  $T(s)$
- ▶ Adding a pole to  $G(s)$  decreases  $\omega_b$  of the closed-loop transfer function  $T(s)$

## Stability Margins of a Second-order System

- Bode plot of  $G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$



## Stability Margins of a Second-order System

- ▶ The phase plot of  $G(s)$  shows that the **phase-crossover frequency** is:

$$\omega_p = \infty$$

- ▶ The **gain margin** is:

$$GM = \infty$$

- ▶ Set  $|G(j\omega)|$  to 1 to obtain the gain-crossover frequency  $\omega_g$ :

$$1 = |G(j\omega_g)| = \frac{\omega_n^2}{|j\omega_g||j\omega_g + 2\zeta\omega_n|} = \frac{\omega_n^2}{\omega_g \sqrt{4\zeta^2\omega_n^2 + \omega_g^2}}$$

- ▶ The **gain-crossover frequency** is:

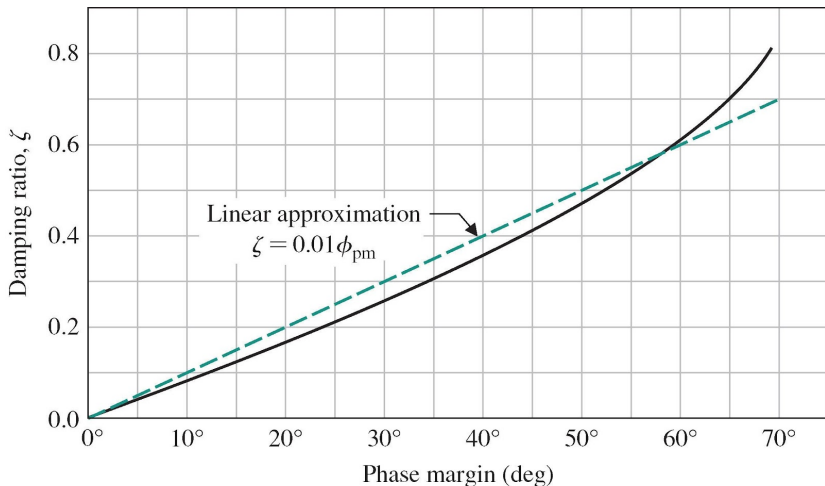
$$\omega_g = \omega_n \sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}$$

- ▶ The **phase margin** is:

$$PM = \angle G(j\omega_g) + \pi = \tan^{-1} \left( \frac{2\zeta}{\sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}} \right)$$

## Phase Margin of a Second-order System

- ▶ The phase margin of a second-order system is a function of  $\zeta$  but not  $\omega_n$
- ▶ The relationship between  $PM$  and  $\zeta$  can be approximated well by a straight line for small values of  $\zeta$



## Phase Margin of a Second-order System

- ▶ For  $0 \leq \zeta \leq 0.7$ , the phase margin  $PM$  (in degrees) and the damping ratio  $\zeta$  of a second-order system are related by:

$$PM \approx 100\zeta$$

- ▶ The relationship between  $\zeta$  and  $PM$  can be used to design control systems in the frequency domain meeting time-domain specifications
- ▶ Poles that are ignored in a dominant-pole-pair approximation contribute phase lag so it is important to keep a large phase margin
- ▶ For  $0.2 \leq \zeta \leq 0.8$ , the gain-crossover frequency  $\omega_g$  of  $G(s)$  is related to the closed-loop system bandwidth  $\omega_b$ :

$$\omega_b \approx 1.8\omega_g$$

## Frequency Domain Control Design

- ▶ Consider proportional control design with gain  $k$
- ▶ To obtain low steady-state error, we want large gain  $k$
- ▶ To obtain fast transient response we want large  $\omega_g$  since  $\omega_b \uparrow, t_r \downarrow$
- ▶ Increasing  $k$ , increases  $\omega_g$  but decreases the phase margin and the system becomes less stable and might exhibit oscillatory behavior
- ▶ More complicated control design may be needed to simultaneously provide good phase margin, good gain-crossover frequency, and good steady state tracking



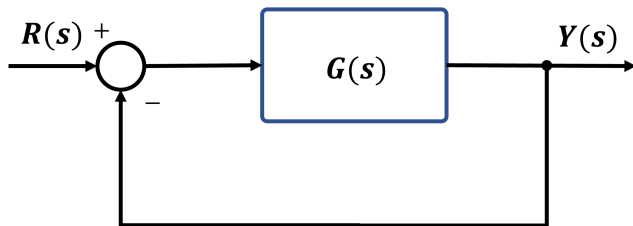
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## Frequency Domain Performance Specifications



- ▶ Feedback control system with control gain  $k$  and open-loop transfer function:

$$G(s) = k \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}$$

- ▶ How can the closed-loop frequency-domain performance specifications (resonant peak  $M_r$ , resonant frequency  $\omega_r$ , bandwidth  $\omega_b$ ) be related to the open-loop frequency response ( $G(j\omega)$ )?
- ▶ How can the gain  $k$  be adjusted to meet frequency-domain performance specifications?

## Closed-loop Transfer Function Magnitude

- ▶ Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

- ▶ Closed-loop transfer function magnitude:

$$M(s) = |T(s)| = \frac{|G(s)|}{|1 + G(s)|}$$

- ▶ Obtain  $M(s)$  as a function of the real and imaginary parts of  $G(s) = x(s) + jy(s)$ :

$$M = \frac{\sqrt{x^2 + y^2}}{\sqrt{(1 + x)^2 + y^2}}$$

- ▶ This equation turns out to be a circle on a Nyquist plot

## Constant Magnitude Circles

- ▶ Relationship between the magnitude of the closed-loop transfer function  $M$  and the real part  $x$  and imaginary part  $y$  of the open-loop transfer function:

$$M^2(1+x)^2 + M^2y^2 = x^2 + y^2$$
$$M^2 = (1 - M^2)x^2 - 2M^2x + (1 - M^2)y^2$$

- ▶ Assume  $M \neq 1$  and divide both sides by  $(1 - M^2)$ :

$$x^2 - 2\frac{M^2}{1 - M^2}x + y^2 = \frac{M^2}{1 - M^2}$$

- ▶ Add  $M^4/(1 - M^2)^2$  to both sides to complete the square for  $x$ :

$$\left(x - \frac{M^2}{1 - M^2}\right)^2 + y^2 = \frac{M^2}{(1 - M^2)^2}$$

## Constant Magnitude Circles

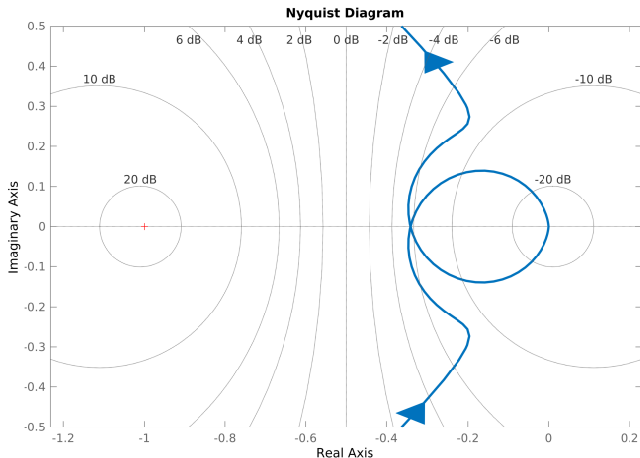
- ▶  **$M$  circle**: a circle of constant closed-loop transfer function magnitude on a polar/Nyquist plot:

$$\left(x - \frac{M^2}{1 - M^2}\right)^2 + y^2 = \frac{M^2}{(1 - M^2)^2}$$

- ▶ An  $M$  circle is centered at  $\left(\frac{M^2}{1 - M^2}, 0\right)$  with radius  $\frac{M}{|1 - M^2|}$
- ▶ As  $M \rightarrow \infty$ , the  $M$  circle is centered at  $(-1, 0)$  with radius 0
- ▶ For  $1 < M < \infty$ , the  $M$  circle center moves to the left of  $(-1, 0)$ , while the radius increases
- ▶ As  $M \rightarrow 0$ , the  $M$  circle is centered at  $(0, 0)$  with radius 0
- ▶ For  $0 < M < 1$ , the  $M$  circle center moves to the right of  $(0, 0)$ , while the radius increases
- ▶ At  $M = 1$ , we get a degenerate circle at  $(\pm\infty, 0)$  with radius  $\infty$

## Constant Magnitude Circles on a Nyquist Plot

- ▶ Nyquist plot of  $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$
- ▶ If the frequencies  $\omega$  along the polar plot of  $G(s)$  are available, we can construct a closed-loop Bode plot using the  $M$  circles

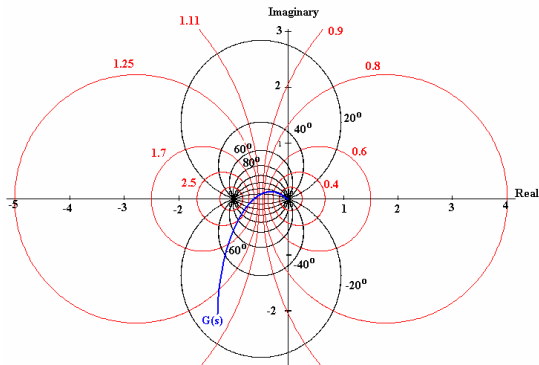


## Constant Phase Circles

- **$N$  circle:** a circle of constant  $N = \tan \angle T(s)$  on a polar/Nyquist plot:

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4} \left(1 + \frac{1}{N^2}\right)$$

- An  $N$  circle is centered at  $(-0.5, 0.5/N)$  with radius  $0.5\sqrt{1 + 1/N^2}$
- $N$  circles are orthogonal to  $M$  circles, i.e., intersect at  $90^\circ$



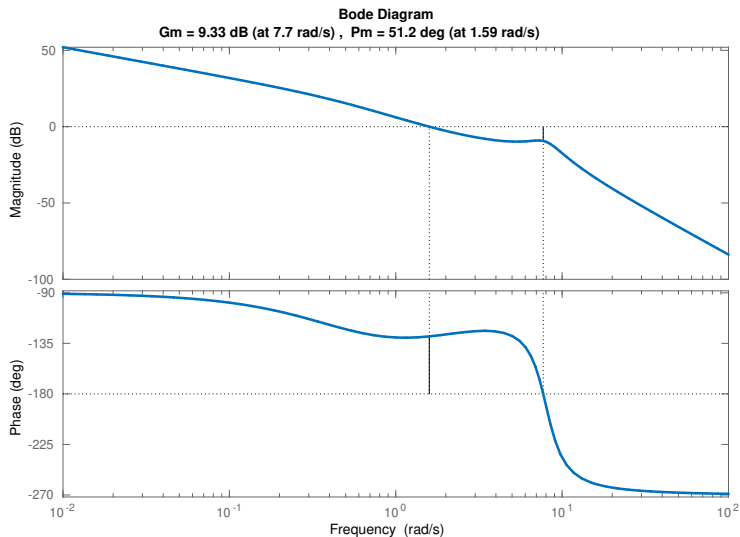
# Frequency Domain Performance Specifications

- ▶ Given the frequency response of an open-loop transfer function  $G(s)$ , we can verify stability and frequency domain performance metrics
- ▶ **Stability:**
  - ▶ Determine using the Nyquist criterion
  - ▶ What if  $k < 0$ ? Rotate the Nyquist plot clockwise by  $180^\circ$ .
- ▶ **Gain margin  $GM$  and phase margin  $PM$ :**
  - ▶ Can be obtained from a Nyquist plot, Bode plot, or magnitude-phase plot
- ▶ **Resonant peak  $M_r$ , resonant frequency  $\omega_r$ , and bandwidth  $\omega_B$ :**
  - ▶ Use the  $M$  circles on a Nyquist plot



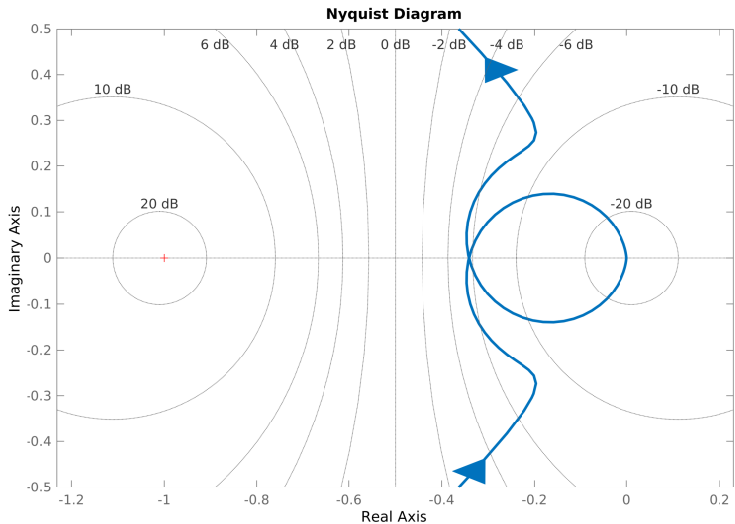
# Open-loop Bode Plot

- Open-loop Bode plot for  $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$



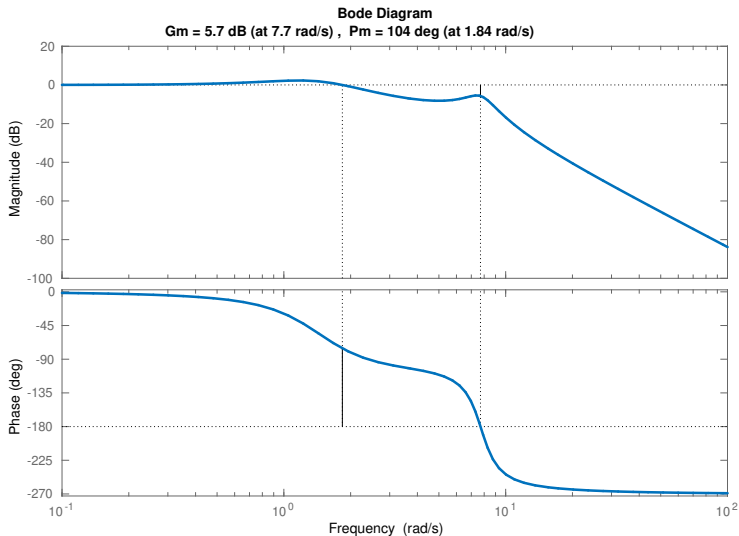
# Nyquist Plot

- Nyquist plot for  $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$



## Closed-loop Bode Plot

- Closed-loop Bode plot for  $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$



## Frequency Domain Control Design

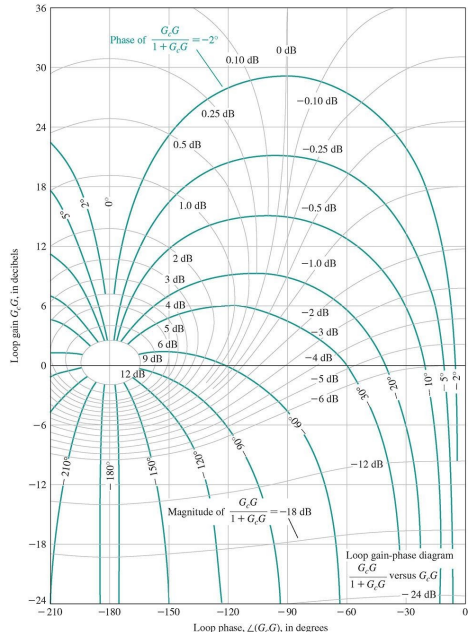
- ▶ How should  $k$  be adjusted to meet desired closed-loop frequency domain specifications?
  - ▶ It is difficult to determine how much to change  $k$  to meet a resonant peak specification on a Nyquist plot
  - ▶ It is difficult to tell where the Nyquist plot would become tangent to the desired  $M$  circle
- ▶ **Nathaniel Nichols** proposed to transform the  $M$  and  $N$  circles from a Nyquist plot to a magnitude-phase plot
- ▶ On a magnitude-phase plot, the  $M$  and  $N$  contours are no longer circles
- ▶ If  $k$  changes, a magnitude-phase plot only moves up or down, which is much easier to interpret than the change of the shape on a Nyquist plot



N. Nichols

# Nichols Plot

- ▶ **Nichols plot:** a magnitude-phase plot with overlaid  $M$  and  $N$  contours of constant closed-loop transfer-function magnitude and phase
- ▶ The gain margin and phase margin can be obtained
- ▶ The resonant peak  $M_r$  and bandwidth  $\omega_b$  can be obtained
- ▶ A change in the gain  $k$  moves the response up or down and can be used to meet closed-loop frequency domain specifications



## Nichols Plot

- Nichols plot of  $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$

