ECE171A: Linear Control System Theory
Lecture 12: Root Locus

Nikolay Atanasov

natanasov@ucsd.edu

UC San Diego

JACOBS SCHOOL OF ENGINEERING

Electrical and Computer Engineering

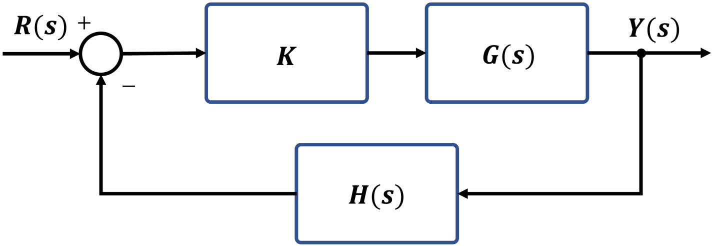
Outline

Outline

Root Locus Overview

- The response of a control system is determined by the locations of the poles of its transfer function in the complex domain
- Feedback control can be used to move the poles of the transfer function by choosing appropriate controller **type** and **gains**
- The **root locus** provides all possible closed-loop pole locations as a system parameter, e.g., the gain k of a proportional controller, varies
- **Root locus plot**
  - By computer: find the roots of the closed-loop characteristic polynomial at different values of the parameter
  - By hand: sketch the root locus shape by following rules determined by the feedback-loop pole and zero locations and phases
- Besides adjusting the proportional gain k of the controller, it is important to understand how to manipulate the root locus by changing the controller type

Root Locus: Example 1

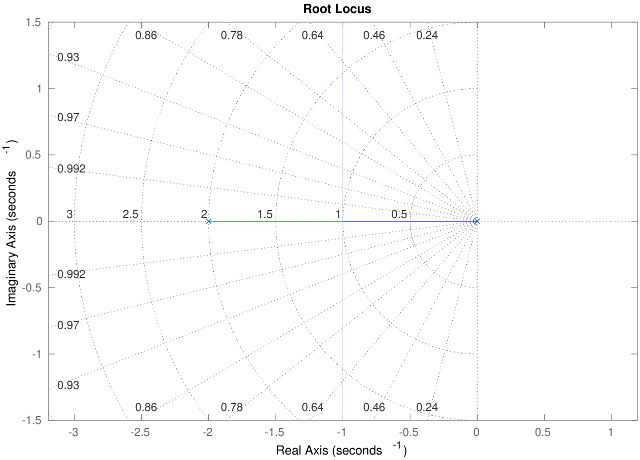


- Consider a feedback control system
  - Controller $F(s) = k$
  - Plant $G(s) = \frac{1}{s(s+2)}$
  - Sensor $H(s) = 1$
- Transfer function: $T(s) = \dfrac{Y(s)}{R(s)} = \dfrac{k}{s^2 + 2s + k}$
- Root locus: how do the transfer function poles vary as a function of k?

Root Locus: Example 1

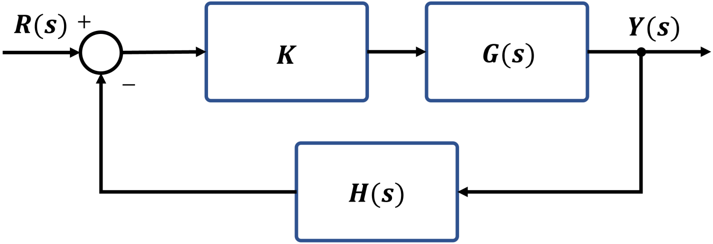
- Root locus of $G(s)H(s) = \frac{1}{s(s+2)}$

```
rlocus(tf([1],[1 2 0]));
sgrid; axis equal;
```



- Closed-loop characteristic polynomial $s^2 + 2s + k$ has roots $p_{1,2} = -1 \pm \sqrt{1-k}$

Root Locus: Example 2

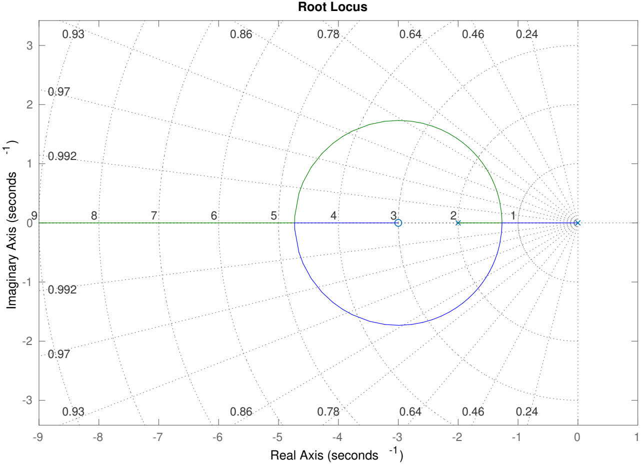


- Add a **left-half-plane zero** to the plant:
  - Controller $F(s) = k$
  - Plant $G(s) = \frac{(s+3)}{s(s+2)}$
  - Sensor $H(s) = 1$
- Transfer function: $T(s) = \dfrac{Y(s)}{R(s)} = \dfrac{k(s+3)}{s^2 + (s+k)s + 3k}$

Root Locus: Example 2

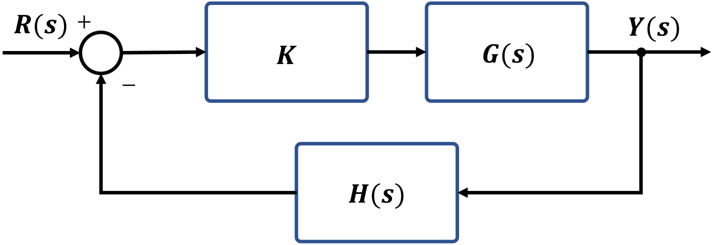
- Root locus of $G(s)H(s) = \frac{(s+3)}{s(s+2)}$

```
rlocus(tf([1 3],[1 2 0]));
sgrid; axis equal;
```



- **Adding a stable zero increases the relative stability of the system by attracting the branches of the root locus**

Root Locus: Example 3

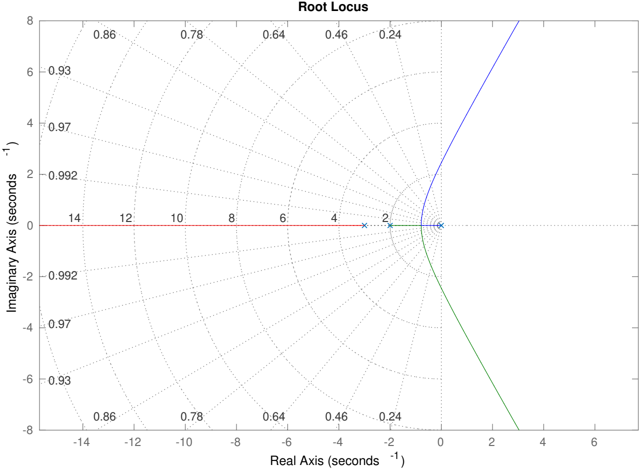


- Add a **left-half-plane pole** to the plant:
  - Controller $F(s) = k$
  - Plant $G(s) = \frac{1}{s(s+2)(s+3)}$
  - Sensor $H(s) = 1$
- Transfer function: $T(s) = \dfrac{Y(s)}{R(s)} = \dfrac{k}{s^3 + 5s^2 + 6s + k}$

Root Locus: Example 3

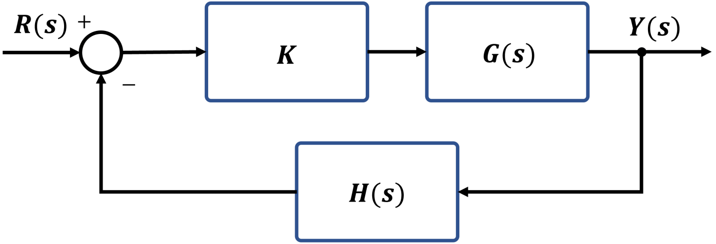
- Root locus for $G(s) = \frac{1}{s(s+2)(s+3)}$

```
rlocus(tf([1],[1 5 6 0]));
sgrid; axis equal;
```



- **Adding a stable pole decreases the relative stability of the system by repelling the branches of the root locus**

Root Locus Definition



- Transfer function: $T(s) = \frac{Y(s)}{R(s)} = \frac{kG(s)}{1 + kG(s)H(s)}$

- The poles of the closed-loop transfer function satisfy:

$$\Delta(s) = 1 + kG(s)H(s) = 0 \qquad \Leftrightarrow \qquad G(s)H(s) = -\frac{1}{k}$$

- **Root locus**: a graph of the roots of $\Delta(s)$ as the gain k varies from 0 to ∞

Positive vs Negative Root Locus

- **Root locus**: points s such that:

$$1 + kG(s)H(s) = 0 \qquad \Leftrightarrow \qquad G(s)H(s) = -\frac{1}{k}$$

- **Positive root locus**: for $k \geq 0$, the points s on the root locus satisfy:
  - **Magnitude condition**: $|G(s)H(s)| = \frac{1}{k}$
  - **Phase condition**: $\underline{/G(s)H(s)} = (1 + 2l)180^\circ$ for $l = 0, \pm 1, \pm 2, \ldots$

- **Negative root locus**: for $k \leq 0$, the points s on the root locus satisfy:
  - **Magnitude condition**: $|G(s)H(s)| = -\frac{1}{k}$
  - **Phase condition**: $\underline{/G(s)H(s)} = (2l)180^\circ$ for $l = 0, \pm 1, \pm 2, \ldots$

Outline

Positive Root Locus

- Consider the zeros and poles of $G(s)H(s)$ explicitly:

$$G(s)H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = \frac{b_m}{a_n} \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- **Positive root locus**: for $k \geq 0$, the points s on the root locus satisfy:
  - **Magnitude condition**: used to determine the gain k corresponding to a point s on the root locus:

$$|G(s)H(s)| = \left|\frac{b_m}{a_n}\right| \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{i=1}^{n} |s - p_i|} = \frac{1}{k}$$

  - **Phase condition**: used to check if a point s is on the root locus:

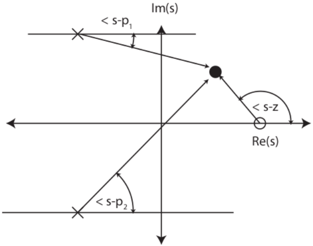
$$\angle G(s)H(s) = \angle \tfrac{b_m}{a_n} + \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = (1 + 2l)180^\circ,$$

    where $l \in \{0, \pm 1, \pm 2, \ldots\}$

Phase Condition Example

- Consider $G(s)H(s) = \frac{s+4}{s((s+1)^2+1)} = \frac{s+4}{s(s+1+j)(s+1-j)}$
- The phase condition allows checking if a point s is on the root locus
- Is the point $s = -3$ on the root locus?

$$\angle G(s)H(s) = \angle 1 - \angle -3 - \angle -2+j - \angle -2-j$$
$$= 0 - 180^\circ - 0 = -180^\circ$$



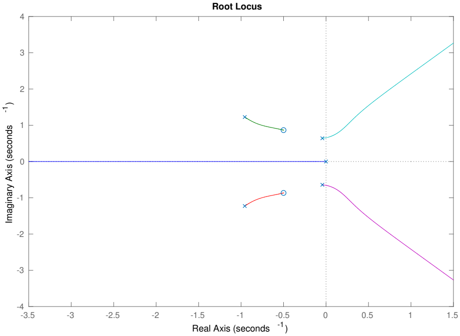
- Is the point $s = -4 + j$ on the root locus?

$$\angle G(s)H(s) = \angle j - \angle -4+j - \angle -3+j2 - \angle -3$$
$$= 90^\circ - \left(180^\circ - \tan^{-1}\left(\frac{1}{4}\right)\right) - \left(180^\circ - \tan^{-1}\left(\frac{2}{3}\right)\right) - 180^\circ$$
$$\approx -450^\circ + 47.7^\circ$$

- Using this method to determine all points on the root locus is cumbersome
- We need more general rules

Root Locus Symmetry

- The closed-loop poles are either real or complex conjugate pairs
- The root locus is **symmetric** about the real axis and the axes of symmetry of the pole-zero configuration of $G(s)H(s)$
- We can divide the root locus into:
  - points on the real axis
  - symmetric parts off the real axis

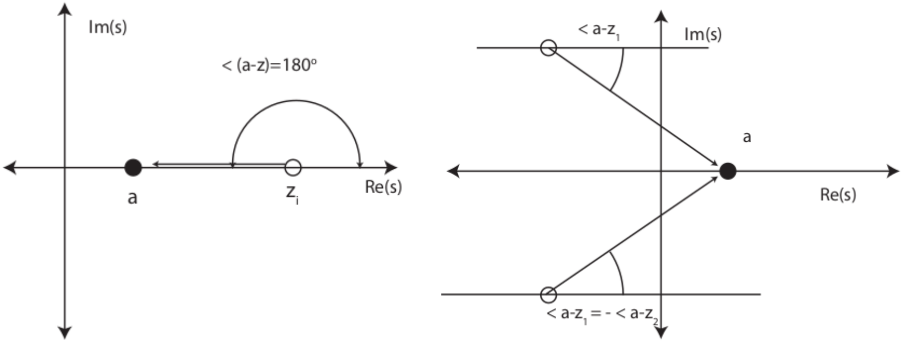


Points on the Real Axis

- Phase condition:

$$\angle G(s)H(s) = \angle \frac{b_m}{a_n} + \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = (1 + 2l)180^\circ, \quad l \in \{0, \pm 1, \pm 2, \ldots\}$$

- For real $s = a$:



(a) A zero to the right contributes 180°

(b) A conjugate pair of zeros does not contribute since the phases sum to zero

Points on the Real Axis

- Phase condition:

$$\angle G(s)H(s) = \angle \frac{b_m}{a_n} + \sum_{i=1}^{m} \angle (s - z_i) - \sum_{i=1}^{n} \angle (s - p_i) = (1 + 2l)180^\circ, \quad l \in \{0, \pm 1, \pm 2, \ldots\}$$

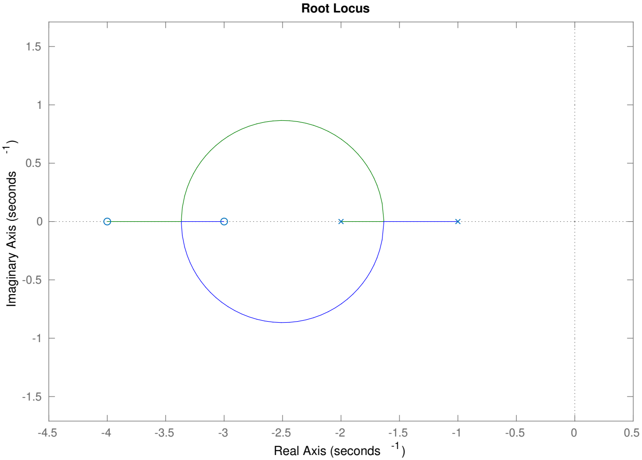
- If s is real:
  - Each zero to the right of s contributes 180°
  - Each pole to the right of s contributes -180°
  - A pole or zero to the left of s does not contribute since its phase is 0°
  - Pairs of complex conjugate poles or zeros do not contribute since their phases sum to zero

- **Rule**: The positive root locus contains all points on the real axis that are to the left of an odd number of zeros or poles

Points on the Real Axis: Example 1

- Determine the real axis portions of the root locus of

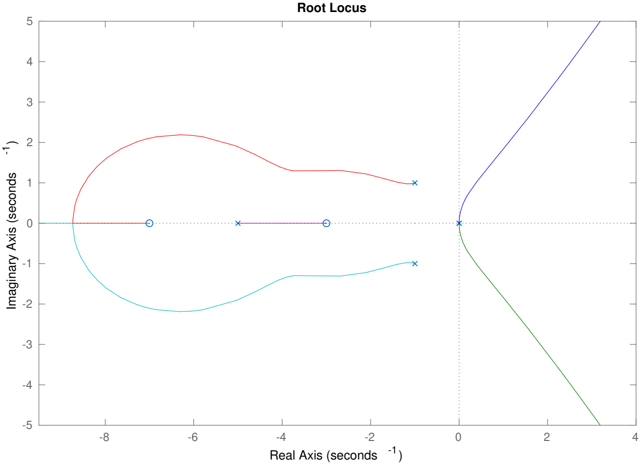
$$G(s)H(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)}$$



Points on the Real Axis: Example 2

- Determine the real axis portions of the root locus of

$$G(s)H(s) = \frac{(s+3)(s+7)}{s^2((s+1)^2+1)(s+5)}$$



Departure and Arrival Points

- **Root locus**: graphs the roots of the closed-loop characteristic polynomial:

$$\Delta(s) = 1 + kG(s)H(s) = 0 \qquad \Rightarrow \qquad a(s) + kb(s) = 0,$$

  where $a(s)$ is n-degree polynomial, $b(s)$ is m-degree polynomial

- Since $n \geq m$, $a(s) + kb(s)$ is an n-degree polynomial and has n roots

- **The root locus has n branches**

- **Departure points**:
  - if $k = 0$, the roots of $a(s) + kb(s)$ are roots of $a(s)$, i.e., **poles** of $G(s)H(s)$

- **Arrival points**:
  - if $k \to \infty$, the solutions of $\frac{b(s)}{a(s)} = -\frac{1}{k}$ are roots of $b(s)$, i.e., **zeros** of $G(s)H(s)$

- **Rule**: The n root locus branches begin at the **poles** of $G(s)H(s)$ (when $k = 0$), and m of the branches end at the zeros of $G(s)H(s)$ (as $k \to \infty$)

Asymptotic Behavior

- The root locus has n branches starting at the poles of $G(s)H(s)$ and m of them terminate at the zeros of $G(s)H(s)$
- What happens with the remaining $n - m$ branches?
- As $k \to \infty$, $G(s)H(s) = -\frac{1}{k} \to 0$

$$
\begin{aligned}
G(s)H(s) = \frac{b(s)}{a(s)} &= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\
&= \frac{b_m \frac{1}{s^{n-m}} + b_{m-1}\frac{1}{s^{n-m+1}} + \cdots + b_1 \frac{1}{s^{n-1}} + b_0 \frac{1}{s^n}}{a_n + a_{n-1}\frac{1}{s} + \cdots + a_1 \frac{1}{s^{n-1}} + a_0 \frac{1}{s^n}}
\end{aligned}
$$

- The numerator of $G(s)H(s)$ goes to zero if $|s| \to \infty$, i.e., there are $n - m$ **zeros at infinity**
- As $k \to \infty$, m branches go to the zeros of $G(s)H(s)$ and the remaining $n - m$ branches go off to infinity along asymptotes

Asymptotic Behavior

- Phase condition:

$$\underline{/G(s)H(s)} = \underline{/\tfrac{b_m}{a_n}} + \sum_{i=1}^{m} \underline{/(s - z_i)} - \sum_{i=1}^{n} \underline{/(s - p_i)} = (1 + 2l)180^\circ, \quad l \in \{0, \pm 1, \pm 2, \ldots\}$$

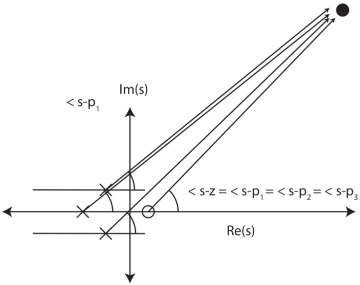
- As $|s| \to \infty$, all angles become the same:

$$\begin{aligned}\theta &\approx \underline{/(s - z_1)} \approx \cdots \approx \underline{/(s - z_m)} \\ &\approx \underline{/(s - p_1)} \approx \cdots \approx \underline{/(s - p_n)}\end{aligned}$$

- **Asymptote angles**:

$$\theta_l = \frac{(1 + 2l)}{|n - m|} 180^\circ - \underline{/\tfrac{b_m}{a_n}},$$

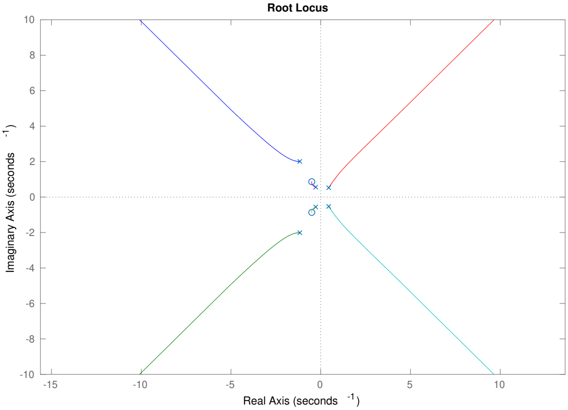
for $l \in \{0, \ldots, |n - m| - 1\}$



Asymptotic Behavior: Example

- Determine the root locus asymptotes of $G(s)H(s) = \frac{s^2+s+1}{s^6+2s^5+5s^4-s^3+2s^2+1}$
- There are $m = 2$ zeros and $n = 6$ poles and hence $n - m = 4$ asymptotes with angles:

$$\frac{\pi}{4} \qquad \frac{3\pi}{4} \qquad \frac{5\pi}{4} \qquad \frac{7\pi}{4}$$



Asymptotic Behavior

- Where do the asymptote lines start?
- If we consider a point s with very large magnitude, the poles and zeros of $G(s)H(s)$ will appear clustered at one point α on the real axis
- The **asymptote centroid** is a point α such that as $k \to \infty$:

$$G(s)H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \approx \frac{b_m}{a_n(s-\alpha)^{n-m}}$$

- Recall the Binomial theorem:

$$(s-\alpha)^{n-m} = s^{n-m} - \alpha(n-m)s^{n-m-1} + \cdots$$

- Recall polynomial long division:

$$\frac{s^n + \frac{a_{n-1}}{a_n}s^{n-1} + \cdots + \frac{a_1}{a_n}s + \frac{a_0}{a_n}}{s^m + \frac{b_{m-1}}{b_m}s^{m-1} + \cdots + \frac{b_1}{b_m}s + \frac{b_0}{b_m}} = s^{n-m} + \left(\frac{a_{n-1}}{a_n} - \frac{b_{m-1}}{b_m}\right)s^{n-m-1} + \cdots$$

Asymptotic Behavior

- Matching the coefficients of s^{n-m-1} shows the asymptote centroid:

$$\alpha = \frac{1}{n-m}\left(\frac{b_{m-1}}{b_m} - \frac{a_{n-1}}{a_n}\right)$$

- Recall Vieta's formulas:

$$\sum_{i=1}^{n} p_i = -\frac{a_{n-1}}{a_n} \qquad \sum_{i=1}^{m} z_i = -\frac{b_{m-1}}{b_m}$$

- **Rule**: the $n-m$ branches of the root locus that go to infinity approach asymptotes with angles θ_l coming out of the centroid $s = \alpha$, where:
  - **Angles**:

$$\theta_l = \frac{(1+2l)}{|n-m|}180^\circ - \angle\frac{b_m}{a_n}, \qquad l \in \{0, \ldots, |n-m|-1\}$$

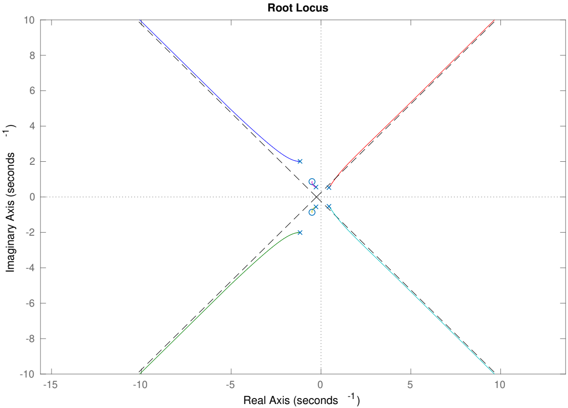
  - **Centroid**:

$$\alpha = \frac{1}{n-m}\left(\frac{b_{m-1}}{b_m} - \frac{a_{n-1}}{a_n}\right) = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

Asymptotic Behavior: Example

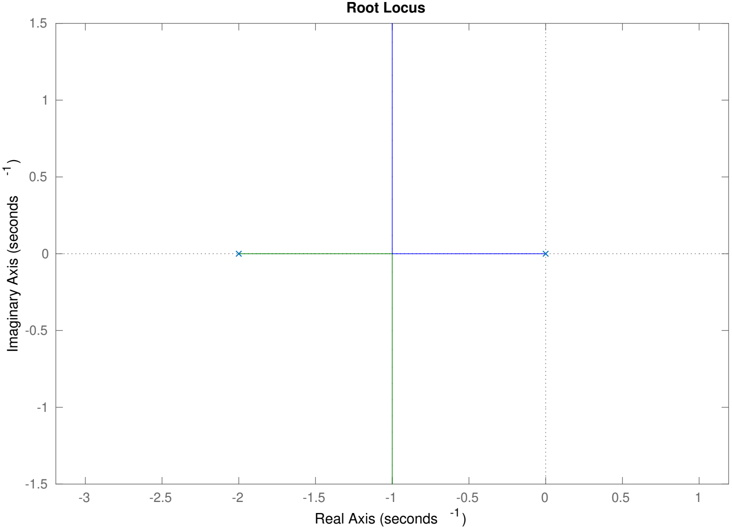
- Determine the root locus asymptotes of $G(s)H(s) = \frac{s^2+s+1}{s^6+2s^5+5s^4-s^3+2s^2+1}$
- There are 4 asymptotes with angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$ and centroid:

$$\alpha = \frac{1}{4}\left(\frac{1}{1} - \frac{2}{1}\right) = -\frac{1}{4}$$



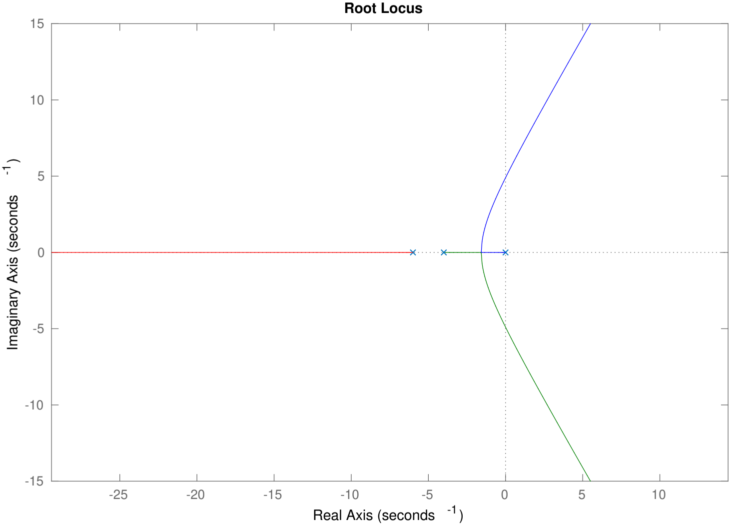
Positive Root Locus: Example 1

- Determine the real axis portions and the asymptotes of the positive root locus of $G(s)H(s) = \frac{1}{s(s+2)}$



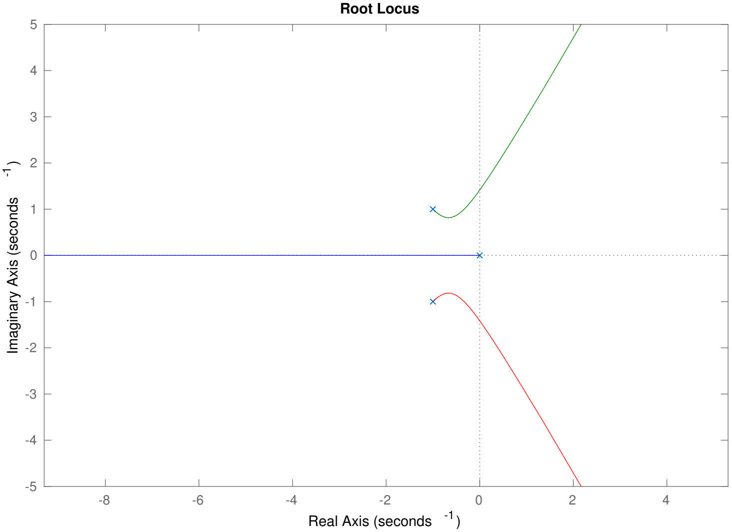
Positive Root Locus: Example 4

- Determine the real axis portions and the asymptotes of the positive root locus of $G(s)H(s) = \frac{1}{s(s+4)(s+6)}$



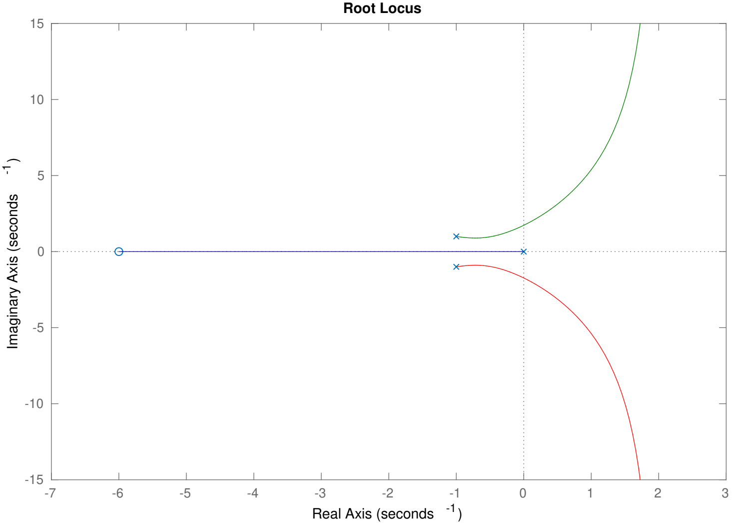
Positive Root Locus: Example 5

- Determine the real axis portions and the asymptotes of the positive root locus of $G(s)H(s) = \frac{1}{s((s+1)^2+1)}$



Positive Root Locus: Example 6

- Determine the real axis portions and the asymptotes of the positive root locus of $G(s)H(s) = \frac{s+6}{s((s+1)^2+1)}$



Breakaway Points

- The root locus leaves the real axis at **breakaway points** s_b where two or more branches meet
- The characteristic polynomial $\Delta(s) = a(s) + kb(s) = 0$ has repeated roots at the breakaway points:

$$\Delta(s) = (s - s_b)^q \bar{\Delta}(s) \qquad \text{for } q \geq 2$$

- Since s_b is a root of multiplicity $q \geq 2$:

$$\begin{aligned} \Delta(s_b) &= \ a(s_b) + \ k\ b(s_b) \ = 0 \\ \frac{d\Delta}{ds}(s_b) &= \frac{da}{ds}(s_b) + k\frac{db}{ds}(s_b) = 0 \end{aligned}$$

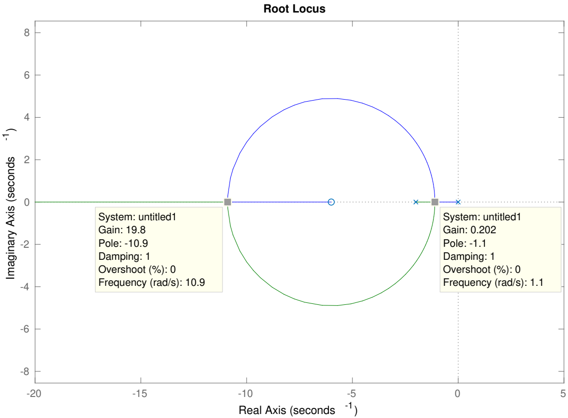
- **Rule**: The positive root locus breakaway points s_b occur when both:
  - $-\dfrac{a(s_b)}{b(s_b)} = k$ is a positive real number
  - $b(s_b)\frac{da}{ds}(s_b) - a(s_b)\frac{db}{ds}(s_b) = 0$

Breakaway Points: Example 1

- Determine the root locus breakaway points of $G(s)H(s) = \frac{b(s)}{a(s)} = \frac{s+6}{s(s+2)}$

$$b(s)\frac{da}{ds}(s) - a(s)\frac{db}{ds}(s) = 2(s+6)(s+1) - s(s+2) = s^2 + 12s + 12 = 0$$

$$\Rightarrow \quad s_b = -6 \pm 2\sqrt{6} \quad \Rightarrow \quad -\frac{a(s_b)}{b(s_b)} = \frac{-48 \pm 20\sqrt{6}}{\pm 2\sqrt{6}} = 10 \mp 4\sqrt{6} > 0$$



Breakaway Points: Example 2

- Determine the root locus breakaway points of

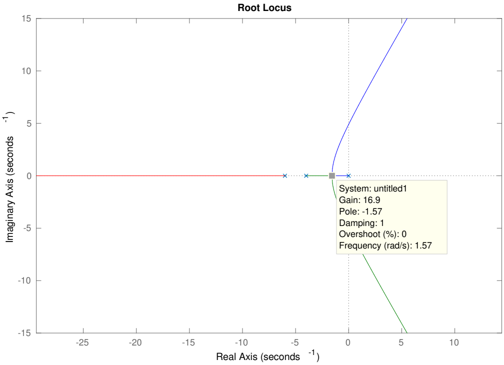
$$G(s)H(s) = \frac{1}{s(s+4)(s+6)} = \frac{1}{s^3 + 10s^2 + 24s}$$

- Breakaway points:

$$\begin{aligned} 0 &= b(s)\frac{da}{ds}(s) - a(s)\frac{db}{ds}(s) \\ &= -3s^2 - 20s - 24 \end{aligned}$$

$$s_b = \frac{-10 \pm 2\sqrt{7}}{3} = \begin{cases} -1.57 \\ -5.10 \end{cases}$$

$$-\frac{a(s_b)}{b(s_b)} = \begin{cases} 16.90 \\ -5.05 \end{cases}$$



Breakaway Points: Example 3

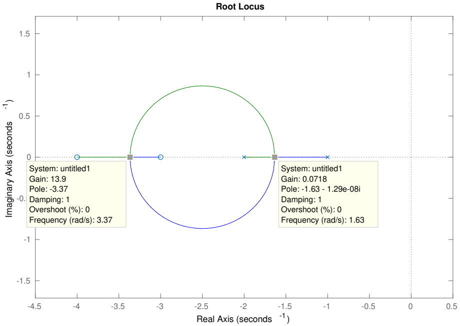
- Determine the root locus breakaway points of

$$G(s)H(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)} = \frac{s^2+7s+12}{s^2+3s+2}$$

- Breakaway points:

$$\begin{aligned} 0 &= b(s)\frac{da}{ds}(s) - a(s)\frac{db}{ds}(s) \\ &= (s^2+3s+2)(2s+7) \\ &\qquad - (2s+3)(s^2+7s+12) \\ &= -4s^2 - 20s - 22 \end{aligned}$$

$$s_b = \begin{cases} -1.634 \\ -3.366 \end{cases}$$



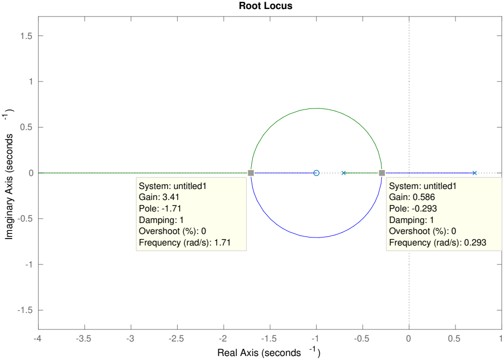
Breakaway Points: Example 4

- Determine the root locus breakaway points of $G(s)H(s) = \frac{s+1}{s^2-0.5}$

- Breakaway points:

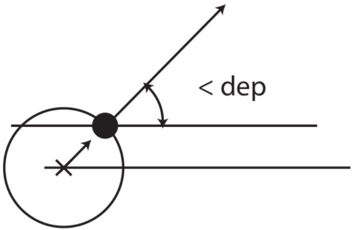
$$
\begin{aligned}
0 &= b(s)\frac{da}{ds}(s) - a(s)\frac{db}{ds}(s) \\
&= (s^2 - 0.5) - 2s(1+s) \\
&= -s^2 - 2s - 0.5
\end{aligned}
$$

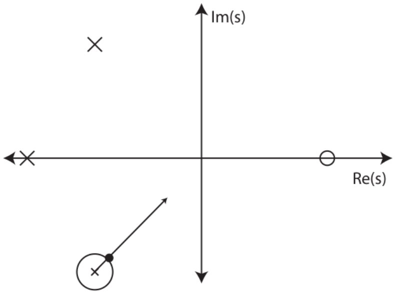
$$
s_b = \begin{cases} -0.293 \\ -1.707 \end{cases}
$$



Angle of Departure

- The root locus starts at the poles of $G(s)H(s)$. At what angles does the root locus depart from the poles?
- To determine the **departure angle**, look at a small region around a pole

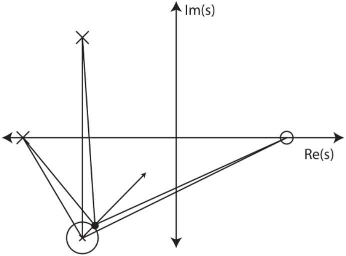




Angle of Departure

- Phase condition:

$$\underline{/G(s)H(s)} = \underline{/\tfrac{b_m}{a_n}} + \sum_{i=1}^{m} \underline{/(s - z_i)} - \sum_{i=1}^{n} \underline{/(s - p_i)} = (1 + 2l)180^\circ$$



- Consider s very close to a pole p_j:
  - $\angle_{\text{dep}} = \underline{/(s - p_j)}$
  - $\underline{/(s - z_i)} \approx \underline{/(p_j - z_i)}$ for all i
  - $\underline{/(s - p_i)} \approx \underline{/(p_j - p_i)}$ for $i \neq j$
  - $\underline{/(p_j - p_j)} = 0$

- **Angle of departure** at p_j:

$$\begin{aligned}\underline{/G(s)H(s)} &= \underline{/\tfrac{b_m}{a_n}} + \sum_{i=1}^{m} \underline{/(s - z_i)} - \sum_{i=1}^{n} \underline{/(s - p_i)} \\ &\approx \underline{/\tfrac{b_m}{a_n}} + \sum_{i=1}^{m} \underline{/(p_j - z_i)} - \sum_{i=1}^{n} \underline{/(p_j - p_i)} - \angle_{\text{dep}} \\ &= \underline{/G(p_j)H(p_j)} - \angle_{\text{dep}} = (1 + 2l)180^\circ\end{aligned}$$

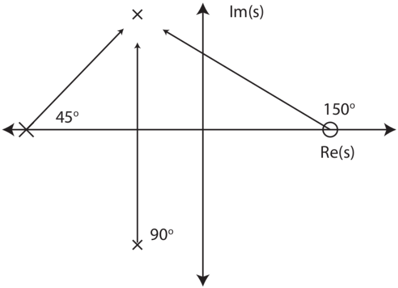
Angle of Departure

- **Angle of departure at a pole** p: $\angle_{\text{dep}} = \underline{/G(p)H(p)} + 180^\circ$
- **Angle of departure at a pole** p **with multiplicity** μ:

$$\mu\angle_{\text{dep}} = \underline{/G(p)H(p)} + 180^\circ$$

- Example:

$$\begin{aligned}\angle_{\text{dep}} &= \underline{/G(p)H(p)} + 180^\circ \\ &= 150^\circ - 90^\circ - 45^\circ + 180^\circ = 195^\circ\end{aligned}$$



Angle of Departure: Example

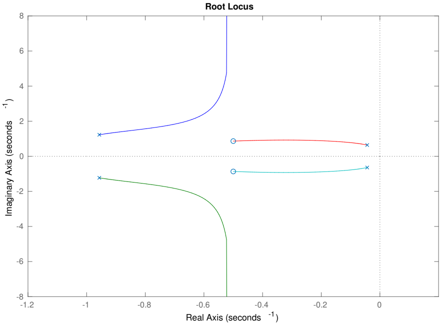
- Consider:

$$G(s)H(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1}$$

- Poles:

$$p_{1,2} = -0.96 \pm j1.23$$
$$p_{3,4} = -0.04 \pm j0.64$$

- Zeros: $z_{1,2} = -0.50 \pm j0.87$

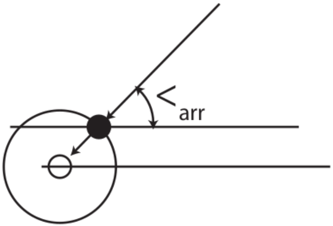


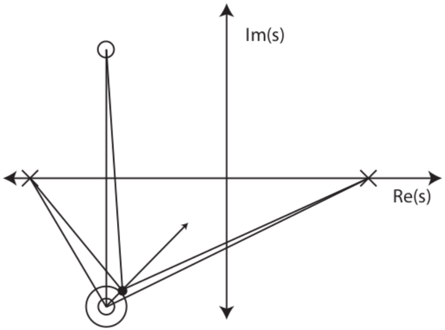
- Angle of departure at p_1:

$$\begin{aligned}\angle_{\text{dep}} &= \underline{/G(p_1)H(p_1)} + 180^\circ \\ &= \underline{/(p_1 - z_1)} + \underline{/(p_1 - z_2)} - \underline{/(p_1 - p_2)} - \underline{/(p_1 - p_3)} - \underline{/(p_1 - p_4)} + 180^\circ \\ &\approx 141.5^\circ + 102.3^\circ - 90^\circ - 147.2^\circ - 116.0^\circ + 180^\circ \\ &= 70.6^\circ\end{aligned}$$

Angle of Arrival

- The root locus ends at the zeros of $G(s)H(s)$. At what angles does the root locus arrive at the zeros?
- To determine the arrival angle, look at a small region around a zero

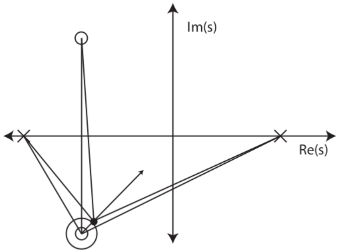




Angle of Arrival

- Phase condition:

$$\underline{/G(s)H(s)} = \underline{/\tfrac{b_m}{a_n}} + \sum_{i=1}^{m} \underline{/(s - z_i)} - \sum_{i=1}^{n} \underline{/(s - p_i)} = (1 + 2l)180^\circ$$



- Consider s very close to a zero z_j:
  - $\angle_{\text{arr}} = \underline{/(s - z_j)}$
  - $\underline{/(s - z_i)} \approx \underline{/(z_j - z_i)}$ for $i \neq j$
  - $\underline{/(s - p_i)} \approx \underline{/(z_j - p_i)}$ for all i
  - $\underline{/(z_j - z_j)} = 0$

- **Angle of arrival** at z_j:

$$\begin{aligned}
\underline{/G(s)H(s)} &= \underline{/\tfrac{b_m}{a_n}} + \sum_{i=1}^{m} \underline{/(s - z_i)} - \sum_{i=1}^{n} \underline{/(s - p_i)} \\
&\approx \angle_{\text{arr}} + \underline{/\tfrac{b_m}{a_n}} + \sum_{i=1}^{m} \underline{/(z_j - z_i)} - \sum_{i=1}^{n} \underline{/(z_j - p_i)} \\
&= \angle_{\text{arr}} + \underline{/G(z_j)H(z_j)} = (1 + 2l)180^\circ
\end{aligned}$$

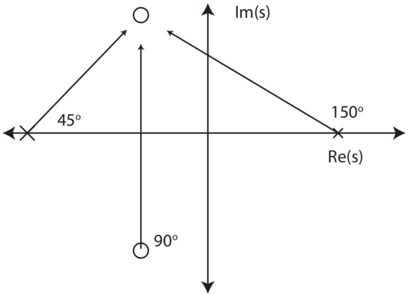
Angle of Arrival

- **Angle of arrival at a zero** z: $\angle_{\text{arr}} = 180^\circ - \underline{/G(z)H(z)}$
- **Angle of arrival at a zero** z **with multiplicity** μ:

$$\mu\angle_{\text{arr}} = 180^\circ - \underline{/G(z)H(z)}$$

- Example:

$$\begin{aligned}\angle_{\text{arr}} &= 180^\circ - \underline{/G(z)H(z)} \\ &= 180^\circ - 90^\circ + 45^\circ + 150^\circ = 285^\circ\end{aligned}$$



Positive Root Locus Summary

- **Positive root locus** of

$$G(s)H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = \frac{b_m}{a_n} \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

- **Step 1**: determine the **departure and arrival points**
  - The departure points are at the n poles of $G(s)H(s)$ (where $k = 0$)
  - The arrival points are at the m zeros of $G(s)H(s)$ (where $k = \infty$)

- **Step 2**: determine the **real-axis root locus**
  - The positive root locus contains all points on the real axis that are to the left of an **odd** number of zeros or poles

- **Step 3**: The root locus is **symmetric** about the real axis and the axes of symmetry of the pole-zero configuration of $G(s)H(s)$

Positive Root Locus Summary

- **Step 4**: determine the $|n-m|$ **asymptotes** as $|s| \to \infty$
  - Centroid: $\alpha = \frac{1}{n-m}\left(\frac{b_{m-1}}{b_m} - \frac{a_{n-1}}{a_n}\right) = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$
  - Angles: $\theta_l = \frac{(1+2l)}{|n-m|}180^\circ - \angle\frac{b_m}{a_n}, \qquad l \in \{0, \ldots, |n-m|-1\}$
- **Step 5**: determine the **breakaway points** where the root locus leaves the real axis
  - The breakaway points s_b are roots of $\Delta(s) = a(s) + kb(s)$ with non-unity multiplicity such that:
    - $-\dfrac{a(s_b)}{b(s_b)} = k$ is a positive real number
    - $b(s_b)\frac{da}{ds}(s_b) - a(s_b)\frac{db}{ds}(s_b) = 0$
  - Arrival/departure angle at breakaway point of q root locus branches: $\theta = \frac{\pi}{q}$

Positive Root Locus Summary

- **Step 6**: determine the **complex pole/zero angle of departure/arrival**
  - Departure angle: if s is close to a pole p with multiplicity μ:

$$\underline{/G(s)H(s)} \approx \underline{/G(p)H(p)} - \mu\angle_{\text{dep}} = (1+2l)180^\circ \quad \Rightarrow \quad \mu\angle_{\text{dep}} = \underline{/G(p)H(p)} + 180^\circ$$

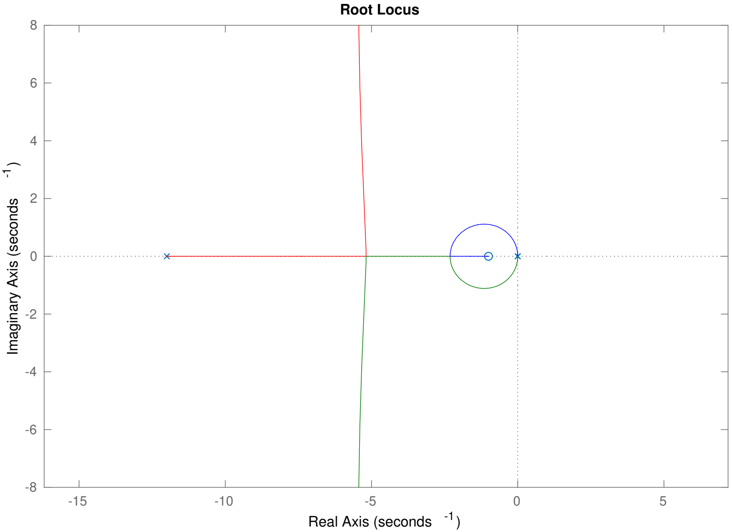
  - Arrival angle: if s is close to a zero z with multiplicity μ:

$$\underline{/G(s)H(s)} \approx \underline{/G(z)H(z)} + \mu\angle_{\text{arr}} = (1+2l)180^\circ \quad \Rightarrow \quad \mu\angle_{\text{arr}} = 180^\circ - \underline{/G(z)H(z)}$$

- **Step 7**: determine **crossover points** where the root locus crosses the $j\omega$ axis
  - A Routh table is used to obtain the auxiliary polynomial $A(s)$ and gain k
  - The crossover points are the roots of $A(s) = 0$

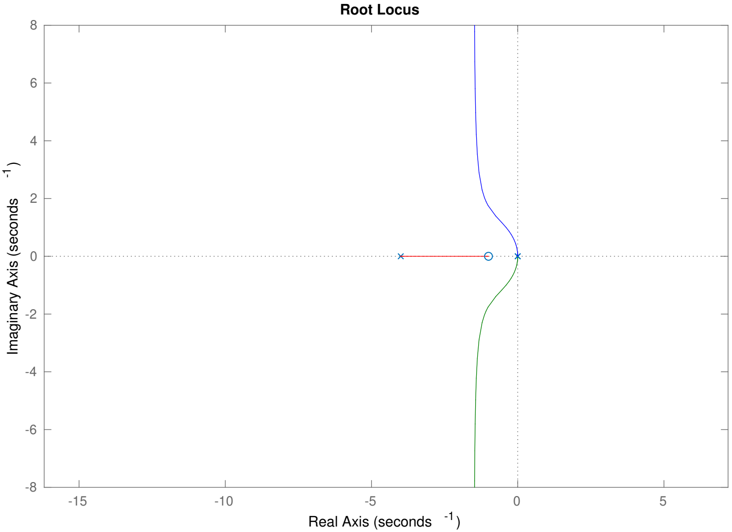
Positive Root Locus: Example 7

- Determine the positive root locus of $G(s)H(s) = \frac{s+1}{s^2(s+12)}$



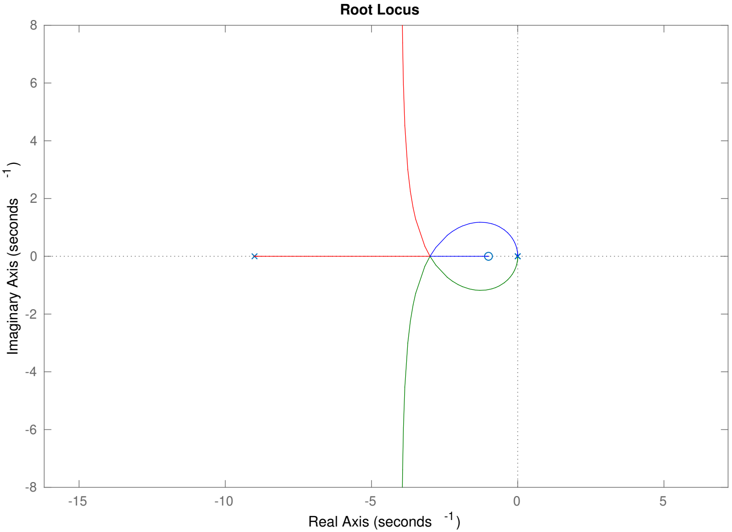
Positive Root Locus: Example 8

- Determine the positive root locus for $G(s)H(s) = \frac{s+1}{s^2(s+4)}$



Positive Root Locus: Example 9

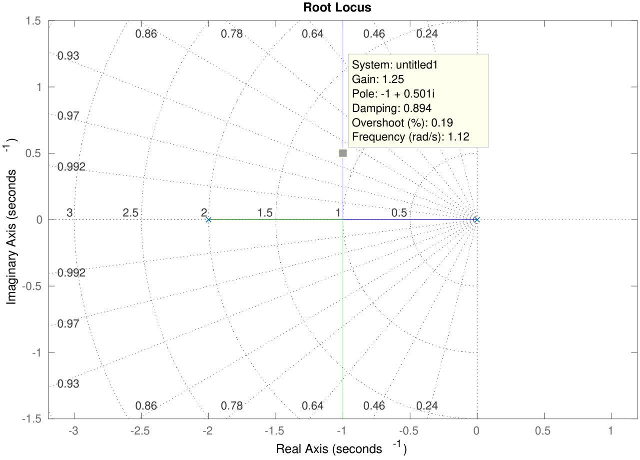
- Determine the positive root locus for $G(s)H(s) = \frac{s+1}{s^2(s+9)}$



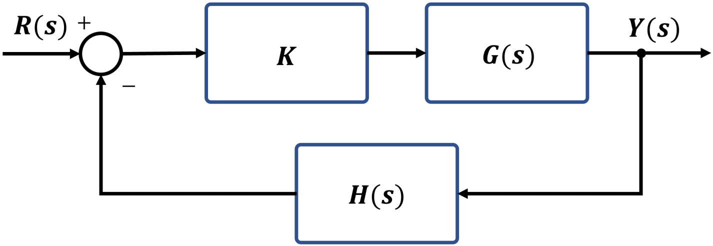
Positive Root Locus: Example 10

- Let $G(s)H(s) = \frac{1}{s^2+2s}$. Find the gain k that results in the closed-loop system having a peak time of at most 2π seconds.

$$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \leq 2\pi \quad \Rightarrow \quad \omega_n\sqrt{1-\zeta^2} \geq 0.5 \quad \Rightarrow \quad k \geq \left|1 + j\frac{1}{2}\right|\left|-1 + j\frac{1}{2}\right| = 1.25$$



Positive Root Locus: Example 11



- Consider a feedback control system with:

$$G(s) = \frac{1}{s\left(\frac{s^2}{2600} + \frac{s}{26} + 1\right)} \qquad H(s) = \frac{1}{1 + 0.04s}$$

- Choose k to obtain a stable closed-loop system with percent overshoot of at most 20% and steady-state error to a step reference of at most 5%

Positive Root Locus: Example 11

$$G(s)H(s) = \frac{65000}{s(s^2+100s+2600)(s+25)} = \frac{65000}{s^4+125s^3+5100s^2+65000s}$$

- **Poles** of $G(s)H(s)$: $p_1 = 0$, $p_2 = -25$, $p_{3,4} = -50 \pm j10$
- The positive root locus contains 4 **asymptotes** with:
  - angles: $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$
  - centroid: $\alpha = -\frac{1}{4}(125) = -31.25$
- **Breakaway point**: should be to the right of $(p_1+p_2)/2 = -12.5$ since the poles $p_{3,4} = -50 \pm j10$ repel the root locus branches

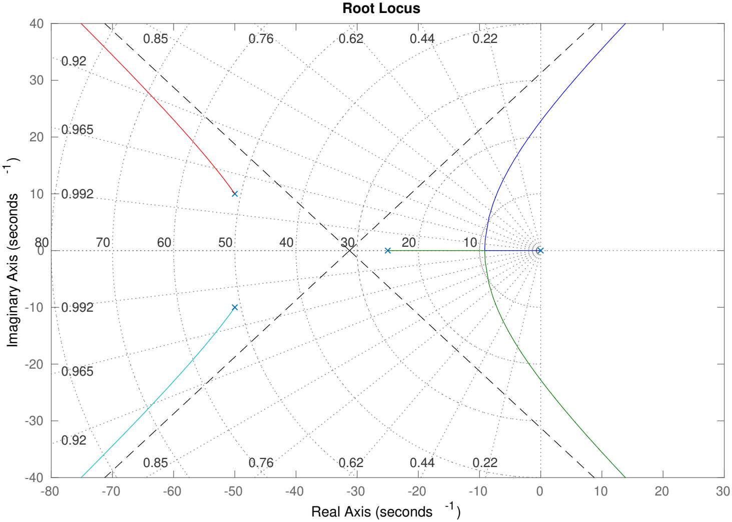
$$65000(4s^3+375s^2+10200s+65000) = 0$$

- **Departure angle** at p_3:

$$\begin{aligned}\angle_{\text{dep}} &= 180^\circ + \underline{/G(p_3)H(p_3)} = 180^\circ - \underline{/p_3 - p_1} - \underline{/p_3 - p_2} - \underline{/p_3 - p_4} \\ &= 180^\circ - 168.7^\circ - 158.2^\circ - 90^\circ = -236.9^\circ \quad \Rightarrow \quad \angle_{\text{dep}} = 123.1^\circ\end{aligned}$$

Positive Root Locus: Example 11

- Positive root locus of $G(s)H(s) = \frac{65000}{s(s^2+100s+2600)(s+25)}$



Positive Root Locus: Example 11

- Closed-loop transfer function characteristic polynomial:

$$\Delta(s) = a(s) + kb(s) = s^4 + 125s^3 + 5100s^2 + 65000s + 65000k$$

- Routh-Hurwitz table:

s^4	1	5100	$65000k$
s^3	1	520	0
s^2	4580	$65000k$	0
s^1	$520 - \frac{3250}{229}k$	0	0
s^0	$65000k$	0	0

- Necessary and sufficient condition for **BIBO stability**: $520 - \frac{3250}{229}k > 0$ and $65000k > 0$:

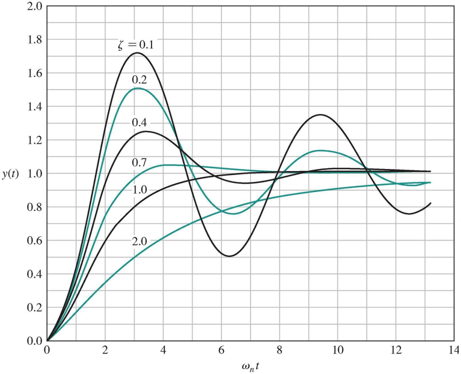
$$0 < k < \frac{916}{25} \approx 36.64$$

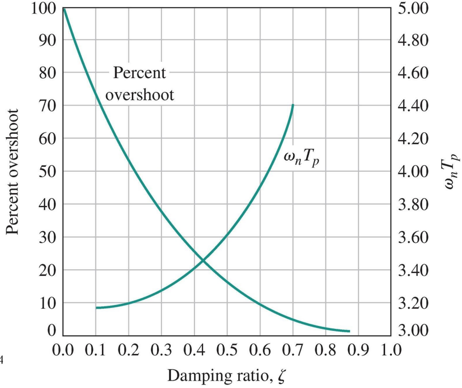
- Auxiliary polynomial at $k = 916/25$ and **crossover points**:

$$A(s) = s^2 + 520 \qquad s_{1,2} = \pm j22.8$$

Positive Root Locus: Example 11

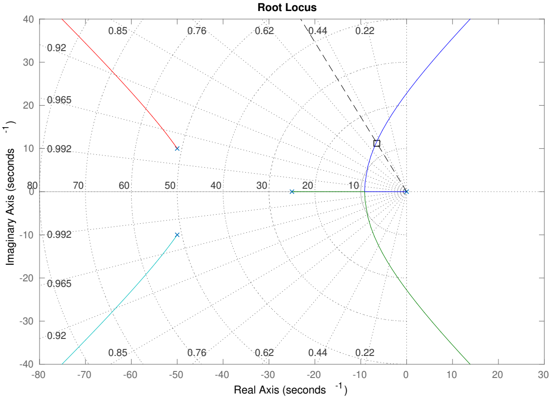
- Determine **dominant pole damping** to ensure percent overshoot $\leq 20\%$
- Pick a larger damping ratio, e.g., $\zeta \geq 0.5$, to ensure that the true fourth-order system satisfies the percent overshoot requirement





Positive Root Locus: Example 11

- Determine the dominant pole locations for $\zeta = 0.5$: $s_{1,2} = -6.6 \pm j11.3$



- Use the magnitude condition to obtain k:

$$\frac{1}{k} = \frac{65000}{|s_1||s_1+25||s_1+50-j10||s_1+50+j10|} \quad \Rightarrow \quad k \approx 9.1$$

Positive Root Locus: Example 11

- To determine the other two closed-loop poles $s_{3,4} = -\sigma \pm j\omega$ at $k = 9.1$, use Vieta's formulas:

$$\sum_{i=1}^{4} s_i = -2\sigma - 2(6.6) = -125 \qquad \Rightarrow \qquad \sigma \approx 55.9$$

- The imaginary part of $s_{3,4} = -55.9 \pm j\omega$ can be obtained from the root locus plot: $\omega \approx 18$

- Closed-loop poles for $k \approx 9.1$:

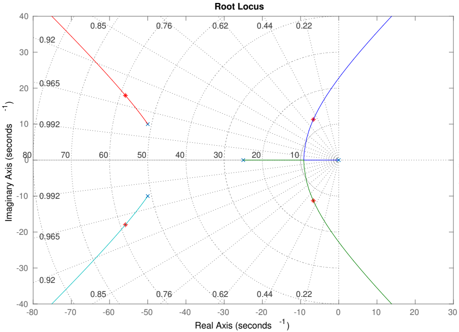
$$s_{1,2} \approx -6.6 \pm j11.3 \qquad\qquad s_{3,4} \approx -56 \pm j18$$

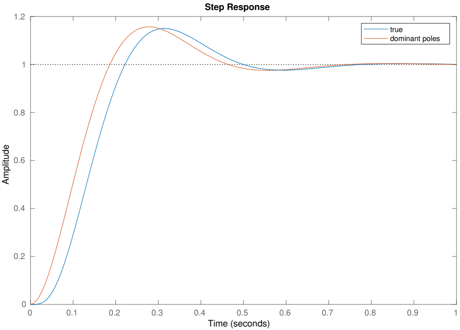
- The steady-state error to a step $R(s) = 1/s$ is:

$$\begin{aligned}\lim_{s\to 0} sE(s) = \lim_{s\to 0} s(R(s) - T(s)R(s)) = \lim_{s\to 0}(1 - T(s)) = \lim_{s\to 0} \frac{\Delta(s) - 65000k}{\Delta(s)} \\ = \lim_{s\to 0} \frac{s^4 + 125s^3 + 5100s^2 + 65000s}{s^4 + 125s^3 + 5100s^2 + 65000s + 65000k} = 0\end{aligned}$$

Positive Root Locus: Example 11

- Final design with $k \approx 9.1$
- The closed-loop system is stable
- The percent overshoot is less than 20%
- The steady-state error to a step input is less than 5%





Outline

Negative Root Locus Summary

- **Negative root locus**: set of points s in the complex plane such that:
  - **Magnitude condition**: $|G(s)H(s)| = -\frac{1}{k}$ for $k \leq 0$
  - **Phase condition**: $\angle G(s)H(s) = (2l)180^\circ$, where l is any integer

- Negative root locus construction procedure for

$$G(s)H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = \kappa \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

- **Step 1**: determine the **departure and arrival points**
  - The departure points are at the n poles of $G(s)H(s)$ (where $k = 0$)
  - The arrival points are at the m zeros of $G(s)H(s)$ (where $k = -\infty$)

Negative Root Locus Summary

- **Step 2**: determine the **real-axis root locus**
  - The negative root locus contains all points on the real axis that are to the left of an **even** number of zeros or poles
- **Step 3**: The root locus is **symmetric** about the real axis and the axes of symmetry of the pole-zero configuration of $G(s)H(s)$
- **Step 4**: determine the $|n-m|$ **asymptotes** as $|s| \to \infty$
  - Centroid: $\alpha = \frac{1}{n-m}\left(\frac{b_{m-1}}{b_m} - \frac{a_{n-1}}{a_n}\right) = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$
  - Angles: $\theta_l = \frac{2l}{|n-m|}180^\circ - \angle\frac{b_m}{a_n}, \qquad l \in \{0, \ldots, |n-m|-1\}$
- **Step 5**: determine the **breakaway points**
  - The breakaway points s_b are roots of $\Delta(s) = a(s) + kb(s)$ with non-unity multiplicity such that:
    - $\dfrac{a(s_b)}{b(s_b)} = -k$ is a positive real number
    - $b(s_b)\frac{da}{ds}(s_b) - a(s_b)\frac{db}{ds}(s_b) = 0$
  - Arrival/departure angle at breakaway point of q root locus branches: $\theta = \frac{\pi}{q}$

Negative Root Locus Summary

- **Step 6**: determine the **complex pole/zero angle of departure/arrival**
  - Departure angle: if s is close to a pole p with multiplicity μ:

$$\underline{/G(s)H(s)} \approx \underline{/G(p)H(p)} - \mu\angle_{\text{dep}} = (2l)180^\circ \quad \Rightarrow \quad \mu\angle_{\text{dep}} = \underline{/G(p)H(p)}$$

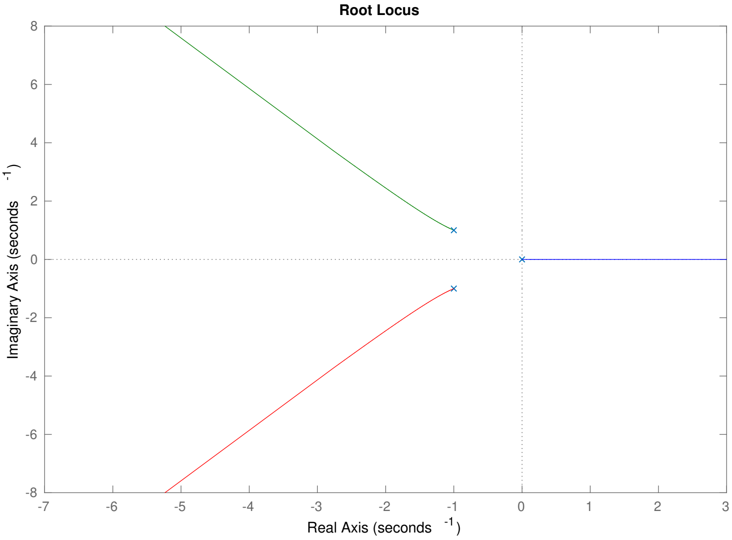
  - Arrival angle: if s is close to a zero z with multiplicity μ:

$$\underline{/G(s)H(s)} \approx \underline{/G(z)H(z)} + \mu\angle_{\text{arr}} = (2l)180^\circ \quad \Rightarrow \quad \mu\angle_{\text{arr}} = -\underline{/G(z)H(z)}$$

- **Step 7**: determine **crossover points** where the root locus crosses the $j\omega$ axis
  - A Routh table is used to obtain the auxiliary polynomial $A(s)$ and gain k
  - The crossover points are the roots of $A(s) = 0$

Negative Root Locus: Example

- Determine the negative root locus of $G(s)H(s) = \frac{1}{s((s+1)^2+1)}$



Negative Root Locus: Example

- Determine the complete (positive and negative) root locus of $G(s)H(s) = \frac{1}{s((s+1)^2+1)}$