ECE171A: Linear Control System Theory
Lecture 3: System Modeling

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Outline

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System Modeling

- A **model** is a mathematical representation of a dynamical system
- Models allow us to make predictions about how a system will behave
- There may be multiple models for a single dynamical system
- All models are approximations of the real system behavior
- Whether we choose a simple coarse model or a complex precise model depends on the questions we wish to answer

System Modeling

- Dynamic behavior can be described in several ways:
  - **ordinary differential equations** (ODEs) in continuous time
  - **partial differential equations** (PDEs) when the system behavior is determined by other variables in addition to time
  - **difference equations** (DEs) in discrete time
- The relationships among the variables and their derivatives in these equations may be **linear** or **nonlinear**
- The coefficients of these equations may be **invariant** or **varying**
- We will focus on linear time-invariant (LTI) ordinary differential equations (ODEs)

Why LTI ODEs?

- Many practically relevant systems can be modeled as LTI ODEs:
  - Electric circuits (e.g., RLC circuits).
  - Mechanical systems (e.g., spring-mass systems)
- Many techniques have been developed for LTI ODE analysis and design:
  - Classical control analysis tools: step, impulse, and frequency response
  - Classical control design tools: Bode/Nyquist/Nichols plots, gain/phase margins, loop shaping
  - Optimal estimation and control: Kalman filter and linear quadratic regulator (LQR)
  - Robust control design: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design and μ analysis for structural uncertainty.
- LTI ODEs provide a foundation for nonlinear system analysis and control (e.g., via linearization)

Differential Equations

- A **differential equation** is any equation involving a function and its derivatives
  - Example: $\frac{d}{dt}y(t) = -y(t)$
- A **solution to a differential equation** is any function that satisfies the equation
  - Example: a solution to the differential equation above is:
$$y(t) = e^{-t}$$
  - Another solution is
$$y(t) = 2e^{-t}$$
  - A general solution is
$$y(t) = e^{-t}y(0).$$
where $y(0) \in \mathbb{R}$ is the initial value of $y(t)$ at $t = 0$.
- When the variable is time t, we will use short-hand derivative notation:

$$\frac{d}{dt}y(t) \equiv \dot{y}(t) \qquad \frac{d^2}{dt^2}y(t) \equiv \ddot{y}(t) \qquad \cdots \qquad \frac{d^n}{dt^n}y(t) \equiv y^{(n)}(t)$$

Ordinary Differential Equations

- An n**th-order linear time-invariant ordinary differential equation** is:

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1\frac{d}{dt}y(t) + a_0 y(t) = u(t)$$

- If $u(t) \equiv 0$, then the nth-order linear ODE is called **homogeneous**
- A **particular solution** is a solution $y(t)$ that contains no arbitrary constants
- A **general solution** is a solution $y(t)$ that contains n arbitrary constants
- An **initial value problem** is an LTI ODE with initial value constraints:

$$y(t_0) = y_0, \quad \dot{y}(t_0) = y_1, \quad \ldots, \quad y^{(n-1)}(t_0) = y_{n-1}.$$

Theorem: Existence and Uniqueness of Solutions

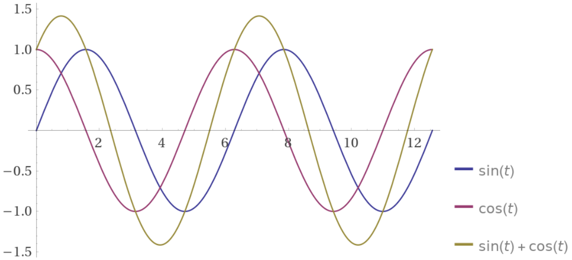
Let $u(t)$ be continuous on an interval $\mathcal{I} = [t_1, t_2]$. Then, for any $t_0 \in \mathcal{I}$, a solution $y(t)$ of the initial value problem exists on $\mathcal{I}$ and is unique.

Example

- Consider the homogeneous linear ODE: $\frac{d^2}{dt^2}y(t) + y(t) = 0$
- Two particular solutions are:

$$y_1(t) = \cos(t) \quad \Rightarrow \quad \frac{d^2}{dt^2}\cos(t) = -\cos(t)$$
$$y_2(t) = \sin(t) \quad \Rightarrow \quad \frac{d^2}{dt^2}\sin(t) = -\sin(t)$$

- In fact, any linear combination $y(t) = c_1 y_1(t) + c_2 y_2(t)$ wiht $c_1, c_2 \in \mathbb{R}$ is also a solution



Superposition Principle for Homogeneous Linear ODEs

Let y_1, y_2, ..., y_k be solutions to a homogeneous nth-order linear ODE on an interval $\mathcal{I}$. Then, any linear combination:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_k y_k(t)$$

is also a solution, where c_1, c_2, ..., c_k are constants.

Superposition Principle for Nonhomogeneous Linear ODEs

For $i = 1, \ldots, k$, let $y_{p_i}(t)$ denote particular solutions to the linear ODEs:

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1\frac{d}{dt}y(t) + a_0 y(t) = u_i(t).$$

Then, $y_p(t) = c_1 y_{p_1}(t) + c_2 y_{p_2}(t) + \ldots + c_k y_{p_k}(t)$ is a particular solution of:

$$\begin{aligned}\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1\frac{d}{dt}y(t) + a_0 y(t)\\ = c_1 u_1(t) + c_2 u_2(t) + \ldots + c_k u_k(t),\end{aligned}$$

where c_1, c_2, ..., c_k are constants.

Outline

State-Space Model

- An nth-order LTI ODE can be reformulated into a first-order vector LTI ODE of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

- Define variables:

$$x_1(t) = y(t), \qquad x_2(t) = \frac{d}{dt}y(t), \qquad \dots, \qquad x_n(t) = \frac{d^{n-1}}{dt^{n-1}}y(t)$$

- The nth-order linear ODE specifies the following relationships:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
&\vdots \\
\dot{x}_{n-1}(t) &= x_n(t) \\
\dot{x}_n(t) &= -a_0 x_1(t) - a_1 x_2(t) - \cdots - a_{n-1} x_n(t) + u(t)
\end{aligned}$$

State-Space Model

- Let $\mathbf{x}(t) := \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}^\top$ be a vector called system **state**
- The forcing function $u(t)$ is called system **control input**
- A **state-space model** of the nth-order linear ODE is obtained by rewriting the equations in vector-matrix form:

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}}_{\mathbf{A}} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{b}} u(t)$$

- The system **output** $y(t)$ can be obtained from the state $\mathbf{x}(t)$ as:

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}}_{\mathbf{c}^\top} \mathbf{x}(t)$$

State-Space Model

- An **LTI ODE state-space model** of a dynamical system is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

  with:
  - **state**: $\mathbf{x} \in \mathbb{R}^n$
  - **input**: $\mathbf{u} \in \mathbb{R}^m$
  - **output**: $\mathbf{y} \in \mathbb{R}^p$
  - **parameters**: $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, $\mathbf{D} \in \mathbb{R}^{p \times m}$

- **Single-input single-output (SISO) system**: $m = p = 1$

- **Multi-input multi-output (MIMO) system**: $m, p > 1$

State-Space Model Variables

- State-space model variables:
  - **State**: consists of variables that capture information from the past motion of the system sufficient to predict the future motion
  - **Input**: consists of external effects acting on the system
  - **Output**: consists of measured variables
  - **Parameters**: describe the state evolution in the form of an update rule
- The choice of state is not unique:
  - There may be many choices of variables that are sufficient to describe the system evolution
- The choice of input and output depends on the point of view
  - Inputs in one model might be outputs of another model (e.g., the output of a cruise controller provides the input to the vehicle model)
  - Outputs are variables (often states) that can be measured and depend on what components of the system interact with external system components

Historical Perspectives

- In the 1940s, when control theory emerged as a discipline, the modeling approach was strongly influenced by *input-output models* used in electrical engineering
- An algebraic relationship, called **transfer function**, between the input and the output of an LTI ODE system can be obtained by transforming it from the time domain to the complex domain via a **Laplace transform**
- In the 1950s, a second wave of control developments, inspired by mechanics, focused on *state-space models*
- Both perspectives provide useful and often distinct information about the system behavior and offer different tools for control analysis and design
- State-space techniques generalize more directly and are easier to use for MIMO systems

Nonlinear Systems

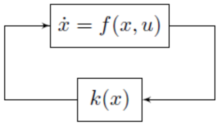
- **Nonlinear State-space Model**:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases} \quad \text{v.s.} \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

- Typical control problems: design a function $\mathbf{u} = \mathbf{k}(\mathbf{x})$, called **feedback control law**, such that:
  - **Regulation problem**: the state converges to zero: $\mathbf{x}(t) \to \mathbf{0}$
  - **Servo problem**: the state tracks a reference signal: $\mathbf{x}(t) \to \mathbf{r}(t)$

- **Closed-loop system**:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x})) = \mathbf{F}(\mathbf{x})$$



Discrete-time Systems

- It some situations, it is natural to describe the evolution of a system at discrete instants of time rather than continuously in time

- Time step: $k = 0, 1, 2, \ldots$

- **Discrete-time nonlinear system**: modeled by nonlinear difference equation:

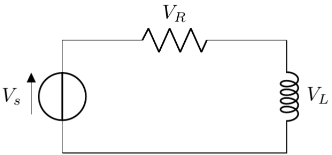
$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) \\ \mathbf{y}_k &= \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k) \end{aligned}$$

- **Discrete-time linear system**: modeled by linear difference equation:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k \end{aligned}$$

Outline

RL Circuit



$$R: \quad \text{Resistance}$$

$$L: \quad \text{Inductance}$$

$$V_R = Ri: \quad \text{Resistor}$$

$$V_L = L\frac{di}{dt}: \quad \text{Inductor}$$

- **Kirchhoff's voltage law**:

$$V_S - V_R - V_L = 0$$

- **System model**:

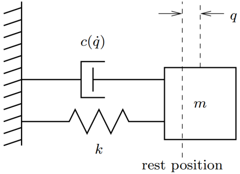
$$L\frac{di}{dt} = V_S - Ri$$

- **State-space model**:
    - Variables: $x = i$, $u = V_S$, $y = V_R$
    - Model:

$$\dot{x} = -\frac{R}{L}x + \frac{1}{L}u$$

$$y = Rx$$

Spring-mass System



m = mass

F = external force

c = friction (damper)

k = spring stiffness

q = deviation from rest position

- **System model**: from Newton's second law:

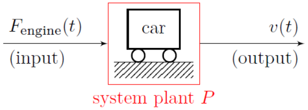
$$m\ddot{q} + c\dot{q} + kq = F$$

- **State-space model**:
  - Variables: $x_1 = q$, $x_2 = \dot{q}$, $y = x_1 = q$, $u = F$
  - Model:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}(-cx_2 - kx_1 + u) \end{bmatrix} \\ y = x_1 \end{cases} \Leftrightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u \end{cases}$$

Speech Control

$\Longrightarrow$



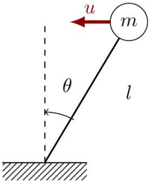
- **Variables**: position p, velocity v, engine force F_{engine}, mass m, gravity acceleration g, road slope θ
- **System model**: from Newton's second law:

$$\dot{p} = v$$
$$m\dot{v} = F_{engine} - mg \sin\theta$$

- **State-space model**:
  - Variables: $x_1 = p$, $x_2 = \dot{p}$, $y = x_2 = v$, $u_1 = F_{\text{engine}}$, $u_2 = g\sin(\theta)$
  - Model:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}u_1 - u_2 \end{bmatrix} \\ y = x_2. \end{cases} \Leftrightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m} & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{cases}$$

Inverted Pendulum



$m =$ mass

$l =$ length

$u =$ external force

$\theta =$ angle

- **Torque**: $T = mgl \sin\theta - ul\cos\theta$
- **Moment of inertia**: $J = ml^2$
- **System model**: from Newton's second law:

$$ml^2\ddot{\theta} = mgl\sin\theta - ul\cos\theta$$

- **State-space model**:
  - Variables: $x_1 = \theta$, $x_2 = \dot{\theta}$, $y = \theta$
  - Model (nonlinear):

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{mgl\sin(x_1) - ul\cos(x_1)}{ml^2}\end{bmatrix}$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + 0u$$

Population Dynamics

- Population growth is a complex dynamic process that involves the interaction of one or more species with their environment and the larger ecosystem

- **Variables**: $x(t)$: species population at time t, b: birth rate, d: death rate, $r = (b - d)$: differential birth rate, k: carying capacity of the environment

- **Logistic growth model**:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right), \quad x \geq 0$$

- Logistic growth model simulation with $r = 1.2$ and $k = 10$: