ECE171A: Linear Control System Theory
Lecture 5: Transfer Function

Nikolay Atanasov

natanasov@ucsd.edu

JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

LTI ODE Solution

- Consider the LTI ODE system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- The system output satisfies the **convolution equation**:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

- **Observations**:
  - Using the convolution equation directly for control design can be challenging
  - A simpler relationship between $\mathbf{u}(t)$ and $\mathbf{y}(t)$ can be obtained by transforming the LTI ODE from the time domain to the complex domain using a **Laplace transform**

Laplace Transform

The **Laplace transform** $\mathcal{L}$ maps a real function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ to a complex function $F : \mathbb{C} \mapsto \mathbb{C}$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$

- The Laplace transform $\mathcal{L}$ converts an LTI ODE in the time domain into a linear algebraic equation in the complex domain
- Example:

$$\begin{array}{ccc} \ddot{y}(t) + y(t) = 0 & \xrightarrow{\mathcal{L}} & s^2 Y(s) - sy(0) - \dot{y}(0) + Y(s) = 0 \\ & & \downarrow \\ y(t) = y(0)\cos(t) + \dot{y}(0)\sin(t) & \xleftarrow{\mathcal{L}^{-1}} & Y(s) = \dfrac{sy(0) + \dot{y}(0)}{s^2 + 1} \end{array}$$

Outline

Complex Numbers and Rational Functions

Laplace Transform

Transfer Function

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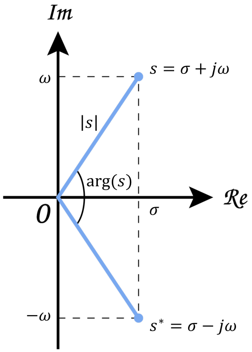
Complex Numbers $\mathbb{C}$

- The **space of real numbers** is denoted by $\mathbb{R}$
- The **space of complex numbers** is denoted by $\mathbb{C}$
- A **complex number** has the form:

$$s = \sigma + j\omega,$$

  where $\sigma, \omega \in \mathbb{R}$ and $j = \sqrt{-1}$

- Cartesian coordinates: $s = \sigma + j\omega$
  - The **real part** of s is $\text{Re}(s) = \sigma$
  - The **imaginary part** of s is $\text{Im}(s) = \omega$
- Polar coordinates: $s = re^{j\theta} = r(\cos(\theta) + j\sin(\theta))$
  - The **magnitude** of s is $|s| = r = \sqrt{\sigma^2 + \omega^2}$
  - The **phase** of s is $\arg(s) = \angle s = \theta = \text{atan2}(\omega, \sigma)$
- The **complex conjugate** of $s = \sigma + j\omega$ is $s^* = \sigma - j\omega$



Complex Polynomial

- A **complex polynomial** of order n is a function $a : \mathbb{C} \mapsto \mathbb{C}$:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_2 s^2 + a_1 s + a_0$$

  where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ are constants.

- A **root** of a complex polynomial $a(s)$ is a number $\lambda \in \mathbb{C}$ such that:

$$a(\lambda) = 0$$

- A root λ of **multiplicity** m of a complex polynomial $a(s)$ satisfies:

$$\lim_{s \to \lambda} \frac{a(s)}{(s - \lambda)^m} < \infty$$

- **Fundamental theorem of algebra**: a complex polynomial $a(s)$ of degree n has exactly n roots, counting multiplicities, and can be factorized as:

$$a(s) = a_n s^n + \ldots + a_0 = a_n (s - \lambda_1) \cdots (s - \lambda_n)$$

  where $\lambda_1, \ldots, \lambda_n$ are the n roots of $a(s)$

Complex Polynomial with Real Coefficients

- A complex polynomial of order n with real coefficients is a function:

$$a(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_2 s^2 + a_1 s + a_0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are constants.

- The roots of a complex polynomial with real coefficients are either real, $\lambda = \sigma$, or come in complex conjugate pairs, $\lambda = \sigma \pm j\omega$.

- Every complex polynomial with real coefficients can be factorized into polynomials of degree one or two:

$$a(s) = a_n s^n + \ldots + a_0 = a_n \prod_{i=1}^{n_1}(s - \lambda_i) \prod_{k=1}^{n_2} \left(s^2 + 2\zeta_k \omega_k s + \omega_k^2\right)$$

where n_1 and n_2 are the numbers of real roots and complex conjugate pairs.

- **Vieta's formulas** relate the coefficients a_i to the roots λ_i:

$$\sum_{i=1}^{n} \lambda_i = -\frac{a_{n-1}}{a_n} \qquad \prod_{i=1}^{n} \lambda_i = (-1)^n \frac{a_0}{a_n} \qquad \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j} = (-1)^k \frac{a_{n-k}}{a_n}$$

Rational Function

- A **rational function** $F : \mathbb{C} \mapsto \mathbb{C}$ is a ratio of complex polynomials:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

- Rational functions remain rational functions under addition, subtraction, multiplication, division (except by 0)
- The **characteristic equation** of a rational function $F(s) = \frac{b(s)}{a(s)}$ is:

$$a(s) = 0$$

- A **zero** $z \in \mathbb{C}$ of a rational function $F(s)$ is a root of the numerator: $b(z) = 0$
- A **pole** $p \in \mathbb{C}$ of a rational function $F(s)$ is a root of the characteristic equation: $a(p) = 0$

Pole-Zero Map

- The **pole-zero form** of a rational function $F(s)$ is:

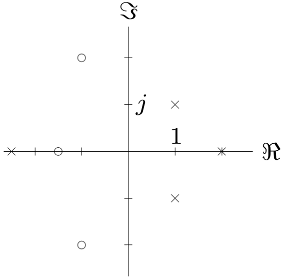
$$F(s) = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0} = k \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

  where $k = b_m / a_n$, $z_1, \ldots, z_m$ are the zeros of $F(s)$, and $p_1, \ldots, p_n$ are the poles of $F(s)$

- A **pole-zero map** is a plot of the poles and zeros of $F(s)$ in the s-domain:
  - Example:

$$F(s) = k \frac{(s + 1.5)(s + 1 + 2j)(s + 1 - 2j)}{(s + 2.5)(s - 2)(s - 1 - j)(s - 1 + j)}$$

  - $\times$ = pole; $\circ$ = zero; k = not available



Example: Zeros and Poles

- Consider $F(s) = \frac{2s+1}{3s^2+2s+1}$
- $F(s)$ has one zero: $z = -\frac{1}{2}$
- The roots of a quadratic polynomial $a(s) = a_2 s^2 + a_1 s + a_0$ are:

$$s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

- $F(s)$ has two conjugate poles: $p_1 = -\frac{1}{3} + j\frac{\sqrt{2}}{3}$ and $p_2 = -\frac{1}{3} - j\frac{\sqrt{2}}{3}$
- Pole-zero form of $F(s)$:

$$F(s) = \frac{2(s-z)}{3(s-p_1)(s-p_2)}$$

Partial Fraction Expansion (no repeated poles)

- Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

  is **strictly proper** ($m < n$) and has no repeated poles (all roots of $a(s)$ have multiplicity one)

- The **residue** r_i associated with pole p_i is:

$$r_i = \lim_{s \to p_i} (s - p_i) F(s)$$

- The **partial fraction expansion** of $F(s)$ is:

$$F(s) = \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n}$$

  where $p_1, \ldots, p_n$ and $r_1, \ldots, r_n$ are the poles and residues of $F(s)$

Example: Residues

- Consider $F(s) = \frac{2s+1}{3s^2+2s+1}$ with zero $z = -\frac{1}{2}$ and poles $p_{1,2} = -\frac{1}{3} \pm j\frac{\sqrt{2}}{3}$
- The residue associated with p_1 is:

$$r_1 = \lim_{s \to p_1} (s - p_1)F(s) = \lim_{s \to p_1} \frac{2(s-z)}{3(s-p_2)} = \frac{2(p_1 + 1/2)}{3(p_1 - p_2)}$$
$$= \frac{2(p_1 + 1/2)}{j2\sqrt{2}} = -j\frac{\sqrt{2}}{2}\left(\frac{1}{6} + j\frac{\sqrt{2}}{3}\right) = \frac{1}{3} - j\frac{\sqrt{2}}{12}$$

- Residues associated with complex conjugate poles are also complex conjugate!
- The residue associated with $p_2 = p_1^*$ is $r_2 = r_1^* = \frac{1}{3} + j\frac{\sqrt{2}}{12}$
- The partial fraction expansion of $F(s)$ is:

$$F(s) = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)}$$

Partial Fraction Expansion (repeated poles)

- Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n (s - p_1)^{m_1} \cdots (s - p_k)^{m_k}}$$

  is **strictly proper** and has poles $p_1, \ldots, p_k$ with multiplicities $m_1, \ldots, m_k$

- The **residue** r_{i,m_i-j} associated with pole p_i of multiplicity m_i is:

$$r_{i,m_i-j} = \lim_{s \to p_i} \frac{1}{j!} \frac{d^j}{ds^j} \left[(s - p_i)^{m_i} F(s) \right], \qquad j = 0, \ldots, (m_i - 1)$$

- The **partial fraction expansion** of $F(s)$ is:

$$\begin{aligned} F(s) = & \frac{r_{1,m_1}}{(s-p_1)^{m_1}} + \frac{r_{1,m_1-1}}{(s-p_1)^{m_1-1}} + \cdots + \frac{r_{1,1}}{s-p_1} \\ & + \frac{r_{2,m_2}}{(s-p_2)^{m_2}} + \frac{r_{2,m_2-1}}{(s-p_2)^{m_2-1}} + \cdots + \frac{r_{2,1}}{s-p_2} \\ & + \cdots \\ & + \frac{r_{k,m_k}}{(s-p_k)^{m_k}} + \frac{r_{k,m_k-1}}{(s-p_k)^{m_k-1}} + \cdots + \frac{r_{k,1}}{s-p_k} \end{aligned}$$

Partial Fraction Expansion (improper rational function)

- Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

  is not strictly proper ($m \geq n$)

- The numerator $b(s)$ can be divided by the denominator $a(s)$ to obtain:

$$F(s) = \frac{b(s)}{a(s)} = c(s) + \frac{d(s)}{a(s)}$$

  where $c(s)$ is of order $m - n$ and $d(s)$ is of order $k < n$

- $\frac{d(s)}{a(s)}$ is now strictly proper and has a partial fraction expansion

Outline

Complex Numbers and Rational Functions

Laplace Transform

Transfer Function

Laplace Transform and Inverse Laplace Transform

- The **Laplace transform** $F(s)$ of a function $f(t)$ is:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,$$

where $s = \sigma + j\omega$ is a complex number.

- The **inverse Laplace transform** $f(t)$ of a function $F(s)$ is:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds,$$

where σ is greater than the real part of all singularities of $F(s)$.

- **Cauchy's Residue Theorem**: If $F(s)$ is a strictly proper rational function:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \sum_{s \text{ is a pole of } F(s)} \left(\text{residue of } F(s)e^{st} \text{ at } s\right)$$

Laplace Transform Properties

- The Laplace transform is **linear**:

$$\begin{aligned}\mathcal{L}\left\{\alpha f(t)+\beta g(t)\right\} &= \int_0^{\infty}(\alpha f(t)+\beta g(t))e^{-st}dt\\ &= \alpha\int_0^{\infty} f(t)e^{-st}dt+\beta\int_0^{\infty} g(t)e^{-st}dt\\ &= \alpha\mathcal{L}\left\{f(t)\right\}+\beta\mathcal{L}\left\{g(t)\right\}\end{aligned}$$

- **Convolution**: for $f(t)$, $g(t)$ supported on $t \in [0,\infty)$:

$$(f*g)(t) = \int_0^{t} f(\tau)g(t-\tau)d\tau$$

- Convolution in time domain becomes multiplication in the complex domain:

$$\begin{aligned}\mathcal{L}\left\{(f*g)(t)\right\} &= \int_0^{\infty}\int_0^{\infty} f(\tau)g(t-\tau)e^{-st}d\tau dt\\ &= \int_0^{\infty}\int_{-\tau}^{\infty} f(\tau)g(\mu)e^{-s\tau}e^{-s\mu}d\mu d\tau\\ &\overset{g(\mu)=0,\mu<0}{=\!=\!=\!=\!=} \int_0^{\infty} f(\tau)e^{-s\tau}d\tau\int_0^{\infty} g(\mu)e^{-s\mu}d\mu\\ &= \mathcal{L}\left\{f(t)\right\}\mathcal{L}\left\{g(t)\right\}\end{aligned}$$

Laplace Transform Properties

- **Differentiation**:

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = s\mathcal{L}\{x(t)\} - x(0)$$

  - Proof:

$$\int_0^\infty \frac{d}{dt}\left(x(t)e^{-st}\right)dt = x(t)e^{-st}\Big|_0^\infty = -x(0)$$

$$\int_0^\infty \frac{d}{dt}\left(x(t)e^{-st}\right)dt = \int_0^\infty \left(\frac{d}{dt}x(t)\right)e^{-st}dt + \int_0^\infty x(t)\left(\frac{d}{dt}e^{-st}\right)dt$$

$$= \mathcal{L}\left\{\frac{d}{dt}x(t)\right\} - s\mathcal{L}\{x(t)\}$$

- **Integration**:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

  - Note that $\frac{d}{dt}\left(\int_0^t f(\tau)d\tau\right) = f(t)$

Laplace Transform Properties

- Laplace transform of e^{at}:

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = -\frac{1}{(s-a)}e^{-(s-a)t}\Big|_{t=0}^{t=\infty}$$

$$\underset{\text{Re}(s)>a}{\overset{\text{require}}{=\!=\!=}} 0 - \left(-\frac{1}{(s-a)}e^0\right) = \frac{1}{s-a}$$

- **Delta function** (Impulse):

$$\delta_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/\epsilon & \text{if } 0 \leq t < \epsilon \\ 0 & \text{if } t \geq \epsilon \end{cases} \qquad \delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

- **Sifting property**: for any $f(t)$ continuous at $\tau \in (a, b)$:

$$\int_a^b f(t)\delta(t-\tau)dt = f(\tau)$$

- Laplace transform of $\delta(t)$:

$$\mathcal{L}\left\{\delta(t)\right\} = \int_0^\infty \delta(t)e^{-st}dt = e^{-st}\Big|_{t=0} = 1$$

Laplace Transform Properties

- **Heaviside step function**:

$$H(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad \Rightarrow \qquad \mathcal{L}\left\{H(t)\right\} = \frac{1}{s}$$

- **Ramp function**:

$$tH(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad \Rightarrow \qquad \mathcal{L}\left\{H(t)\right\} = \frac{1}{s^2}$$

- **Parabola function**:

$$\frac{t^2}{2} H(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad \Rightarrow \qquad \mathcal{L}\left\{H(t)\right\} = \frac{1}{s^3}$$

Laplace Transform Properties

	t domain	s domain
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
convolution	$(f * g)(t)$	$F(s)G(s)$
multiplication	$f(t)g(t)$	$\frac{1}{2\pi j} \int_{Re(\sigma)-j\infty}^{Re(\sigma)+j\infty} F(\sigma)G(s-\sigma)d\sigma$
scaling, $a > 0$	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
s-domain derivative	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
time-domain derivative	$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
s-domain integarion	$\frac{1}{t} f(t)$	$\int_s^\infty F(\sigma)d\sigma$
time-domain integarion	$\int_0^t f(\tau)d\tau = (H * f)(t)$	$\frac{1}{s} F(s)$
s-domain shift	$e^{at} f(t)$	$F(s-a)$
time-domain shift, $a > 0$	$f(t-a)H(t-a)$	$e^{-as} F(s)$

- Heaviside step function $H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$
- Convolution: $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

Laplace Transform Properties

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$

Laplace Transform Properties

$f(t)$	$F(s)$
$\int_{-\infty}^{t} f(t)\,dt$	$\frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} f(t)\,dt$
Impulse function $\delta(t)$	1
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{1}{\omega}[(\alpha-a)^2+\omega^2]^{1/2}e^{-at}\sin(\omega t+\phi)$, $\phi = \tan^{-1}\frac{\omega}{\alpha-a}$	$\frac{s+\alpha}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}t, \zeta<1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$\frac{1}{a^2+\omega^2}+\frac{1}{\omega\sqrt{a^2+\omega^2}}e^{-at}\sin(\omega t-\phi)$, $\phi = \tan^{-1}\frac{\omega}{-a}$	$\frac{1}{s[(s+a)^2+\omega^2]}$
$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$, $\phi = \cos^{-1}\zeta, \zeta<1$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
$\frac{\alpha}{a^2+\omega^2}+\frac{1}{\omega}\left[\frac{(\alpha-a)^2+\omega^2}{a^2+\omega^2}\right]^{1/2}e^{-at}\sin(\omega t+\phi)$. $\phi = \tan^{-1}\frac{\omega}{\alpha-a}-\tan^{-1}\frac{\omega}{-a}$	$\frac{s+\alpha}{s[(s+a)^2+\omega^2]}$

Initial and Final Value Theorems

Initial Value Theorem

Suppose that $f(t)$ has a Laplace transform $F(s)$. Then:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

Final Value Theorem

Suppose that $f(t)$ has a Laplace transform $F(s)$. Suppose that every pole of $F(s)$ is either in the open left-half plane or at the origin of $\mathbb{C}$. Then:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Example: Spring-Mass-Damper

- Consider a spring-mass-damper system:

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = 0$$

- This is an example of a second-order system with **natural frequency** $\omega_n = \sqrt{k/M}$ and **damping ratio** $\zeta = b/(2\sqrt{kM})$:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = 0$$

- Laplace transform:

$$(s^2 Y(s) - sy(0) - \dot{y}(0)) + 2\zeta\omega_n(sY(s) - y(0)) + \omega_n^2 Y(s) = 0$$

- Natural response:

$$Y(s) = \frac{(s + 2\zeta\omega_n)y(0) + \dot{y}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Example: Spring-Mass-Damper

- Consider the natural response with $\omega_n^2 = k/M = 2$ and $2\zeta\omega_n = b/M = 3$:

$$Y(s) = \frac{(s+3)y(0) + \dot{y}(0)}{s^2 + 3s + 2} = \frac{(s+3)y(0) + \dot{y}(0)}{(s+1)(s+2)}$$
$$= \frac{2y(0) + \dot{y}(0)}{s+1} - \frac{y(0) + \dot{y}(0)}{s+2}$$

- Poles: $p_1 = -1$ and $p_2 = -2$
- Zeros: $z_1 = -\frac{\dot{y}(0)}{y(0)} - 3$
- Residues:

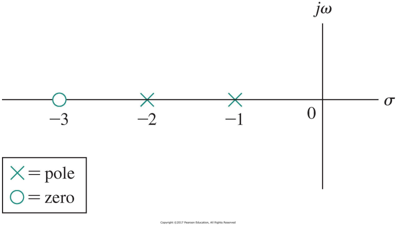
$$r_1 = \left.\frac{(s+3)y(0) + \dot{y}(0)}{(s+2)}\right|_{s=-1} = 2y(0) + \dot{y}(0)$$

$$r_2 = \left.\frac{(s+3)y(0) + \dot{y}(0)}{(s+1)}\right|_{s=-2} = -y(0) - \dot{y}(0)$$

Example: Spring-Mass-Damper

- **Spring-Mass-Damper Pole-Zero Map**
- Let the initial conditions be $y(0) = 1$ and $\dot{y}(0) = 0$
- The poles and zeros are:

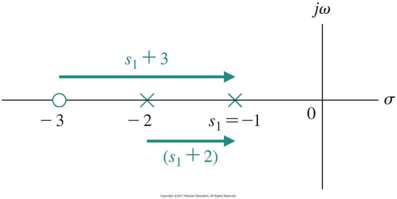
$$p_1 = -1, \quad p_2 = -2, \quad z_1 = -3$$



- The residues are:

$$r_1 = \left.\frac{(s+3)}{(s+2)}\right|_{s=-1} = 2$$

$$r_2 = \left.\frac{(s+3)}{(s+1)}\right|_{s=-2} = -1$$



Example: Spring-Mass-Damper

- The time-domain **natural response** of the spring-mass-damper system can be obtained using an inverse Laplace transform:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2y(0)+\dot{y}(0)}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{y(0)+\dot{y}(0)}{s+2}\right\} \\ &= (2y(0)+\dot{y}(0))\,e^{-t} - (y(0)+\dot{y}(0))\,e^{-2t} \end{aligned}$$

- The **steady-state** response can be obtained via the Final Value Theorem:

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0} \frac{(s^2+3s)y(0)+s\dot{y}(0)}{s^2+3s+2} = 0 \end{aligned}$$

Example: Spring-Mass-Damper

- The poles of the system are the roots of the characteristic equation:

$$a(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

- The natural response is determined by the poles:
  - **Overdamped** $(\zeta > 1)$: the poles are real:

$$p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \qquad p_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

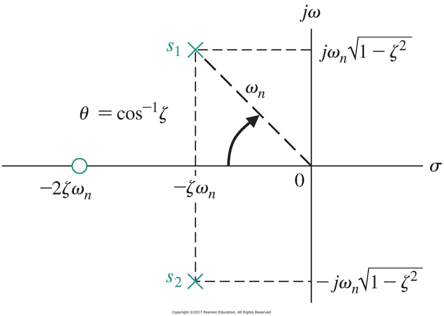
  - **Critically damped** $(\zeta = 1)$: the poles are repeated and real:

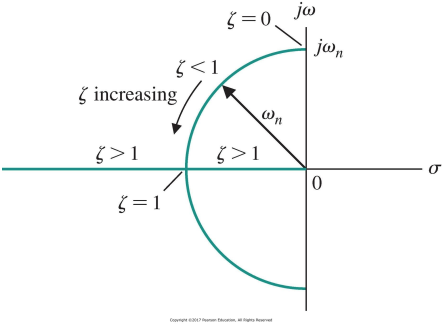
$$p_1 = p_2 = -\omega_n$$

  - **Underdamped** $(\zeta < 1)$: the poles are complex:

$$p_1 = -\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2} \qquad p_2 = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$$

Example: Spring-Mass-Damper Locus of Roots



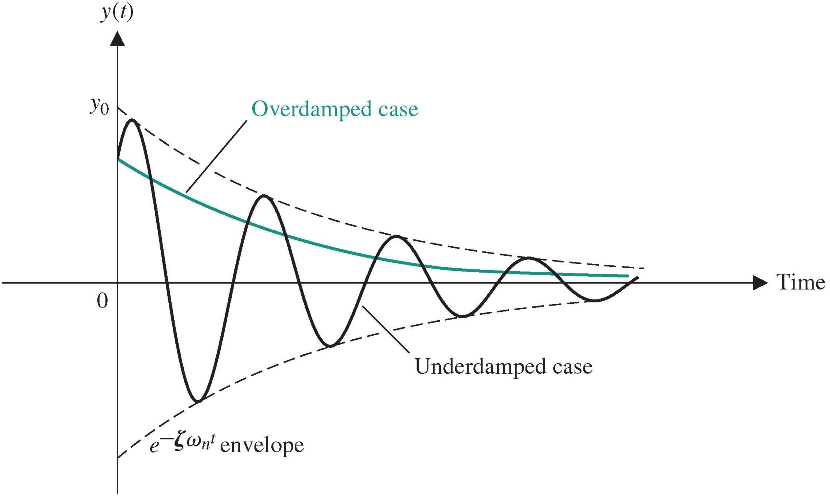


- s-domain plot of the poles ($\times$) and zeros ($\circ$) of $Y(s)$ with $\dot{y}(0) = 0$
- For constant ω_n, as ζ varies, the complex conjugate roots follow a circular locus
- The poles and zeros can be expressed either in Cartesian coordinates or Polar coordinates (e.g., magnitude ω_n and angle $\theta = \cos^{-1}(\zeta)$)

Example: Spring-Mass-Damper Response

- The time-domain natural response can be obtained by determining the residues and applying an inverse Laplace transform:
  - **Overdamped** ($\zeta > 1$):
    $$y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t}$$
    where $p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$, $p_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$, $r_1 = \frac{p_2 y(0) + \dot{y}(0)}{p_2 - p_1}$, and $r_2 = -\frac{p_1 y(0) + \dot{y}(0)}{p_2 - p_1}$
  - **Critically damped** ($\zeta = 1$):
    $$y(t) = y(0)e^{-\omega_n t} + (\dot{y}(0) + \omega_n y(0))te^{-\omega_n t}$$
  - **Underdamped** ($\zeta < 1$):
    $$y(t) = e^{-\zeta\omega_n t}\left(c_1 \cos(\omega_n\sqrt{1-\zeta^2}t) + c_2 \sin(\omega_n\sqrt{1-\zeta^2}t)\right)$$
    where $c_1 = y(0)$ and $c_2 = \frac{\dot{y}(0) + \zeta\omega_n y(0)}{\omega_n\sqrt{1-\zeta^2}}$

Example: Spring-Mass-Damper Natural Response with $\dot{y}(0) = 0$



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Outline

Complex Numbers and Rational Functions

Laplace Transform

Transfer Function

Laplace Transform of LTI ODE

- Consider an LTI ODE with zero initial conditions:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_0 u$$

- Let $Y(s) = \mathcal{L}\{y(t)\}$ and $U(s) = \mathcal{L}\{u(t)\}$

- Recall that $\mathcal{L}\left\{\frac{d^n}{dt^n} y(t)\right\} = s^n Y(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1}}{dt^{k-1}} y(t)\Big|_{t=0}$

- Laplace transform of the LTI ODE:

$$\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right) Y(s) = \left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0\right) U(s)$$

- **Transfer function**: ratio of Laplace transform of output to Laplace transform of input with zero initial conditions:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

Transfer Function

Transfer Function

The transfer function $G(s)$ of a single-input single-output LTI ODE is the ratio of the Laplace transform $Y(s)$ of the output $y(t)$ to the Laplace transform $U(s)$ of the input $u(t)$ with zero initial conditions:

$$G(s) = \frac{Y(s)}{U(s)}$$

Relative Degree

The relative degree of a single-input single-output LTI ODE with transfer function $G(s)$ is the difference $r = n - m$ between the number of poles n and number of zeros m of $G(s)$.

- If $r > 0$, the transfer function is called *strictly proper*.
- If $r \geq 0$, the transfer function is called *proper*.
- If $r < 0$, the transfer function is called *improper* (there is no state space realization).

Example

- A vehicle with position $p(t)$ and acceleration input $u(t)$ satisfies:

$$m\ddot{p}(t) = u(t)$$

- The transfer function of this system is:

$$G(s) = \frac{P(s)}{U(s)} = \frac{1}{ms^2}$$

- The transfer function is strictly proper with relative degree $r = 2$

Example: Second-order LTI ODE

- Consider a second-order system with natural frequency ω_n, damping ratio ζ, and input $u(t)$:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = u(t)$$

- Laplace transform:

$$(s^2 Y(s) - sy(0) - \dot{y}(0)) + 2\zeta\omega_n(sY(s) - y(0)) + \omega_n^2 Y(s) = U(s)$$

- Transfer function (set $y(0) = \dot{y}(0) = 0$):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Total response:

$$Y(s) = \underbrace{\frac{(s + 2\zeta\omega_n)y(0) + \dot{y}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{\text{natural response}} + \underbrace{G(s)U(s)}_{\text{forced response}}$$

Transfer Function of State-space Model

- Consider an LTI ODE system in state-space:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- Laplace transform:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

- The response $\mathbf{Y}(s)$ of LTI ODE system consists of **natural response** due to the initial conditions $\mathbf{x}(0)$ and **forced response** due to the input $\mathbf{U}(s)$:

$$\mathbf{Y}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0) + \underbrace{\left(\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right)}_{\mathbf{G}(s)}\mathbf{U}(s)$$

The transfer function of an LTI ODE system in state-space form is:

$$\mathbf{G}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$$

Response to Periodic Signals

- The idea of a transfer function comes from looking at the response of an LTI ODE system to periodic input signals with fundamental frequency ω_{f}:

$$u(t) = \sum_{k=0}^{\infty} (a_k \sin(k\omega_{\text{f}} t) + b_k \cos(k\omega_f t))$$

- **Euler's formula**: $e^{j\omega} = \cos\omega + j\sin\omega$
- The exponential function e^{st} with $s = j\omega$ can represent periodic signals:

$$\sin(\omega t) = \text{Im}(e^{j\omega t}) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$
$$\cos(\omega t) = \text{Re}(e^{j\omega t}) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

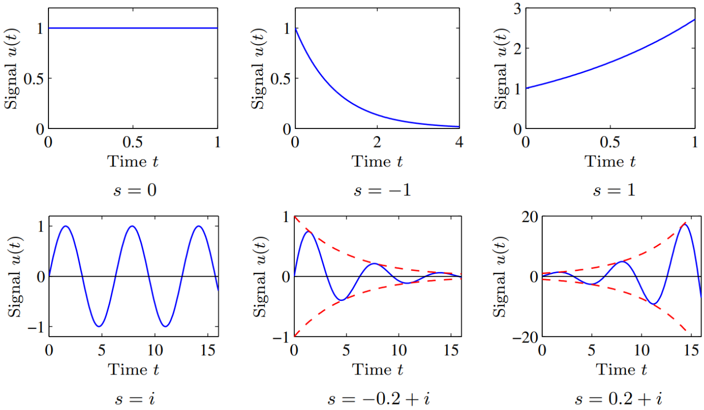
- Thanks to linearity (**superposition**), it suffices to compute the response to $u(t) = e^{st}$ and then reconstruct the response to a cosine or sine by combining the responses corresponding to $s = j\omega$ and $s = -j\omega$

Exponential Input e^{st}

- The exponential input e^{st} generalizes periodic signals to a broader class:

$$e^{st} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}(\cos(\omega t) + j \sin(\omega t))$$

- Examples of exponential signals:
  - Top row: exponential signals with a real exponent $s = \sigma$
  - Bottom row: exponential signals with a complex exponent $s = j\omega$



Frequency Domain Analysis

- Analyze LTI ODE response to sinusoidal and exponential signals
- State-space model:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

- Convolution equation:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

- SISO system with input $u(t) = e^{st}$ such that s is not an eigenvalue of $\mathbf{A}$:

$$\begin{aligned} y(t) &= \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0}_{\text{natural response}} + \underbrace{\mathbf{C}e^{\mathbf{A}t}(s\mathbf{I}-\mathbf{A})^{-1}\left(e^{(s\mathbf{I}-\mathbf{A})t} - \mathbf{I}\right)\mathbf{B} + \mathbf{D}e^{st}}_{\text{forced response}} \\ &= \underbrace{\mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}(0) - (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\right)}_{\text{transient response}} + \underbrace{\left(\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)e^{st}}_{\text{steady-state response}} \end{aligned}$$

Frequency Domain Analysis

- SISO LTI ODE response to $u(t) = e^{st}$:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}(0) - (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right)}_{\text{transient response}} + \underbrace{\left(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)e^{st}}_{\text{steady-state response}}$$

The transfer function from $u(t)$ to $y(t)$ of a SISO LTI ODE is the coefficient of the **steady-state response to an exponential input**:

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- The transfer function represents the system dynamics in terms of the generalized frequency s instead of time t
- Analyzing the system in the complex domain uncovers interesting properties

Example

- Consider a SISO LTI ODE with state-space model:

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{D} = 0$$

- Transfer function:

$$\begin{aligned} G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + a_1 & a_2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2 + a_1 s + a_2} \begin{bmatrix} s & -a_2 \\ 1 & s + a_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{s^2 + a_1 s + a_2}. \end{aligned}$$

Example

- Consider a Heaviside step input:

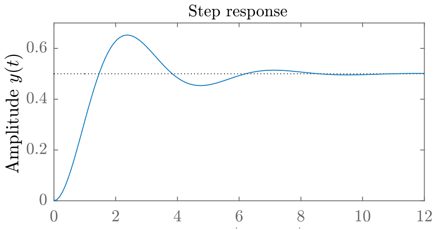
$$u(t) = H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

- Note that $u(t) = e^{st}$ with $s = 0$ for $t \geq 0$:

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}(0) + \mathbf{A}^{-1}\mathbf{B}\right) + G(0)u(t)$$

- Suppose $a_1 = 1$ and $a_2 = 2$: $G(s) = \frac{1}{s^2+s+2}$

- The steady-state response as $t \to \infty$ is $G(0) = \frac{1}{2}$



Controllable Canonical Form

- Consider a general n-th order transfer function (some of b_i may be 0):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

- To convert this transfer function to state-space form multiply by $Z(s)/Z(s)$:

$$G(s) = \frac{Y(s)/Z(s)}{U(s)/Z(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

- Time-domain LTI ODEs:

$$\begin{aligned} y &= b_n z^{(n)} + b_{n-1} z^{(n-1)} + \ldots + b_1 \dot{z} + b_0 z \\ u &= \quad z^{(n)} + a_{n-1} z^{(n-1)} + \ldots + a_1 \dot{z} + a_0 z \end{aligned}$$

- This suggests the following choice of state variables:

$$x_1 = z \qquad x_2 = \dot{z} \qquad \cdots \qquad x_n = z^{(n-1)}$$

Controllable Canonical Form

- Consider a general n-th order transfer function (some of b_i may be 0):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

- The **controllable canonical form** is a state-space model with the same transfer function:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} (b_0 - a_0 b_n) & (b_1 - a_1 b_n) & \cdots & (b_{n-1} - a_{n-1} b_n) \end{bmatrix} \mathbf{x} + b_n u$$

Zero Frequency Gain

- The features of the transfer function reveal important system properties
- **Zero frequency gain**: the magnitude $|G(0)|$ of the transfer function at $s = 0$
- Interpretation: the ratio of the steady-state output to a step input
- LTI ODE:

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0} \qquad \Rightarrow \qquad G(0) = \frac{b_0}{a_0}$$

- State-space model:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \qquad \Rightarrow \qquad G(0) = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}$$

- Integrator: $\dot{y} = u$

$$G(s) = \frac{1}{s} \qquad \Rightarrow \qquad G(0) \to \infty \quad \textbf{pole}$$

- Differentiator $y = \dot{u}$

$$G(s) = s \qquad \Rightarrow \qquad G(0) = 0 \quad \textbf{zero}$$

Transfer Function Poles

- Consider the LTI ODE:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_0 u$$

- The response $Y(s)$ consists of **natural response** due to the initial conditions $\mathbf{x}(0)$ and **forced response** due to the input $U(s)$:

$$Y(s) = \underbrace{\frac{c(s)}{a(s)}}_{\text{natural response}} + \underbrace{\frac{b(s)}{a(s)} U(s)}_{\text{forced response}}$$

- The transfer function $G(s) = \frac{b(s)}{a(s)}$ and the natural response have the same denominator:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$

- A pole p of the transfer function $G(s)$ is a solution to the characteristic equation $a(s) = 0$. If $u(t) \equiv 0$, then $y(t) = e^{pt}$ is a solution to the LTI ODE.

The poles p of a transfer function $G(s)$ correspond to the natural solutions $y(t) = e^{pt}$ of the LTI ODE called modes.

Transfer Function Zeros

- SISO LTI ODE response to an exponential input $u(t) = e^{st}$:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}(0) - (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right)}_{\text{transient response}} + \underbrace{\left(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)e^{st}}_{\text{steady-state response}}$$

- A zero z of the transfer function $G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ makes $G(z) = 0$ and hence the steady-state response to $u(t) = e^{zt}$ is zero

The zeros z of a transfer function $G(s)$ block transmission of an exponential input $u(t) = e^{zt}$.

Example: Vibration Damper

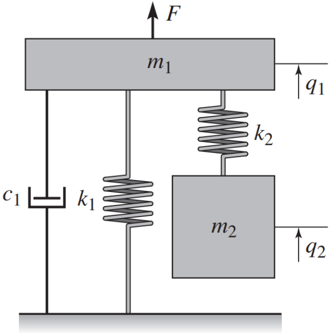


Figure: Vibrations of the mass m_1 can be damped by providing an auxiliary mass m_2, attached to m_1 by a spring with stiffness k_2. The parameters m_2 and k_2 are chosen so that the frequency $\sqrt{k_2/m_2}$ matches the frequency of vibration.

Example: Vibration Damper

- Vibration damper dynamics:

$$\begin{aligned} m_1\ddot{q}_1 + c_1\dot{q}_1 + k_1 q_1 + k_2(q_1 - q_2) &= f \\ m_2\ddot{q}_2 + k_2(q_2 - q_1) &= 0 \end{aligned}$$

- The Laplace transform with zero initial conditions is:

$$\begin{aligned} (m_1 s^2 + c_1 s + k_1)Q_1(s) + k_2(Q_1(s) - Q_2(s)) &= F(s) \\ m_2 s^2 Q_2(s) + k_2(Q_2(s) - Q_1(s)) &= 0 \end{aligned}$$

- The transfer function from $F(s)$ to $Q_1(s)$ is obtained by eliminating $Q_2(s)$:

$$G(s) = \frac{Q_1(s)}{F(s)} = \frac{m_2 s^2 + k_2}{m_1 m_2 s^4 + m_2 c_1 s^3 + (m_1 k_2 + m_2(k_1 + k_2))s^2 + k_2 c_1 s + k_1 k_2}$$

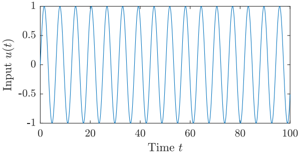
- **Blocking property**: the transfer function has **zeros** at $s = \pm j\sqrt{k_2/m_2}$

Example: Vibration Damper

- Blocking property with parameters

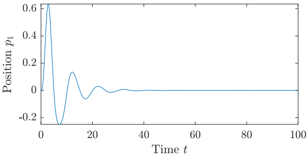
$$m_1 = 1, c_1 = 1, k_1 = 1, m_2 = 1, k_2 = 1$$

- **Case 1: external input** : $u = \sin(\omega t)$, with $\omega = 1$

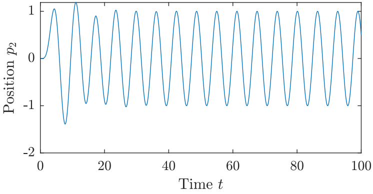


(a) Input $u = \sin(t)$

$\Rightarrow$



(b) Position of mass 1

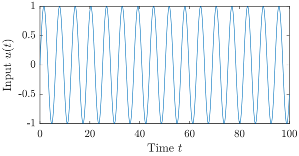


(c) Postion of mass 2

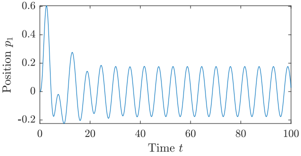
Example: Vibration Damper

- Other frequency responses

- **Case 2: external input** : $u = \sin(\omega t)$, with $\omega = 1.1$

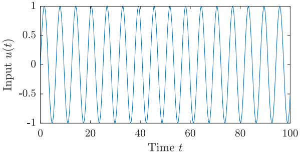


$\Rightarrow$

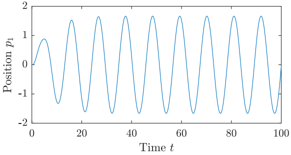


(a) Input $u = \sin(1.1t)$ (b) Position of mass 1

- **Case 3: external input** :$u = \sin(\omega t)$, with $\omega = 0.578$



$\Rightarrow$



(a) Input $u = \sin(1.1t)$ (b) Position of mass 1