ECE171A: Linear Control System Theory
Lecture 6: Block Diagram and Signal Flow Graph

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Outline

Outline

Block Diagram

- The **Laplace transform** converts an LTI ODE in the time domain into a linear algebraic equation in the complex domain
- **Transfer function**: a description of the input-output relationship of a SISO LTI ODE system as a ratio of the output-to-input Laplace transforms with zero initial conditions:
$$G(s) = \frac{Y(s)}{U(s)}$$
- The transfer functions of system elements can be represented as blocks in a **block diagram** to obtain a powerful algebraic method to analyze complex LTI ODE systems

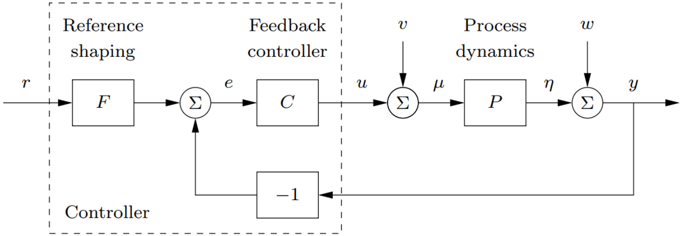
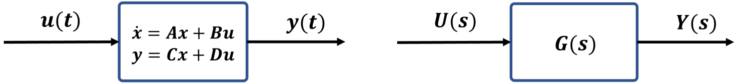


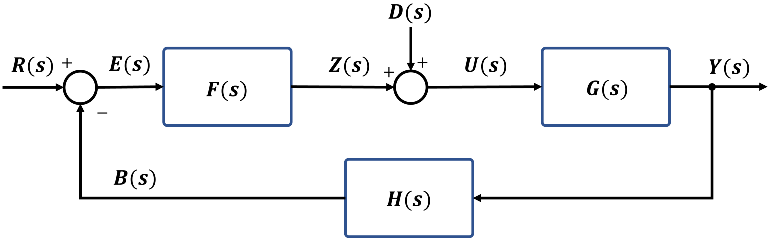
Figure: A block diagram for a feedback control system

Block Diagram

- **Block**: represents input-output relationship of a system component either in the time domain (**LTI ODE**) or in the complex domain (**transfer function**)



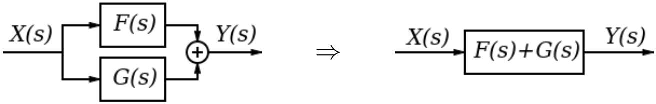
- **Block diagram**: interconnects blocks to represent a multi-element system



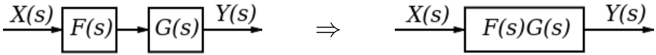
- **Summing point**: adds or subtracts two or more signals

Block Diagram Transformations

- A block diagram can be simplified using equivalent transformations
- **Parallel connection**: if two or more elements are connected in parallel, the total transfer function is the sum of the individual transfer functions:

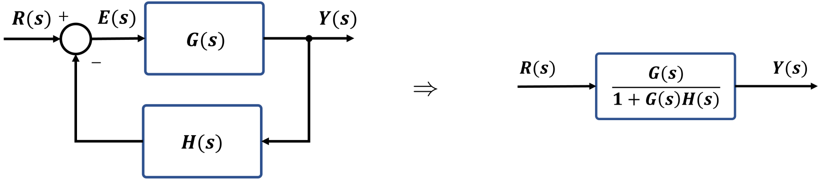


- **Series connection**: if two or more elements are connected in series, the total transfer function is the product of the individual transfer functions:



Block Diagram Transformations

- **Feedback connection**: two or more elements are connected in a loop



- Forward path:

$$Y(s) = G(s)E(s)$$

- Feedback path:

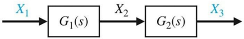
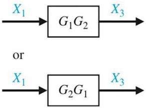
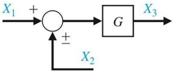
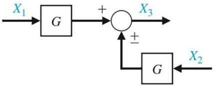
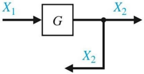
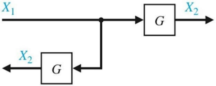
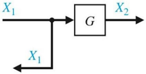
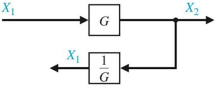
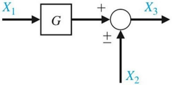
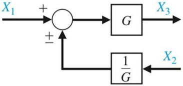
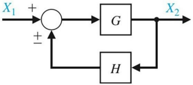
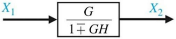
$$E(s) = R(s) - H(s)Y(s)$$

- Equivalent transfer function:

$$Y(s) = G(s)\left[R(s) - H(s)Y(s)\right] \qquad \Rightarrow \qquad \left[1 + G(s)H(s)\right]Y(s) = G(s)R(s)$$

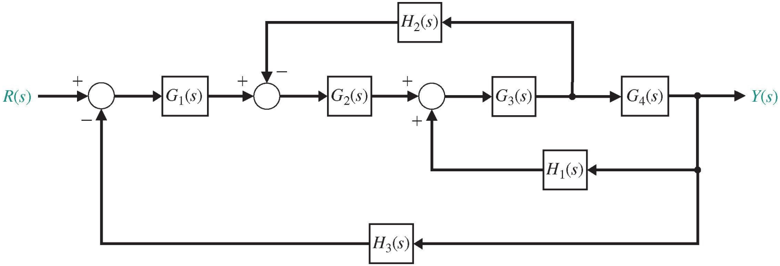
$$\Rightarrow \qquad Y(s) = \left[\frac{G(s)}{1 + G(s)H(s)}\right]R(s)$$

Table 2.5 Block Diagram Transformations

Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

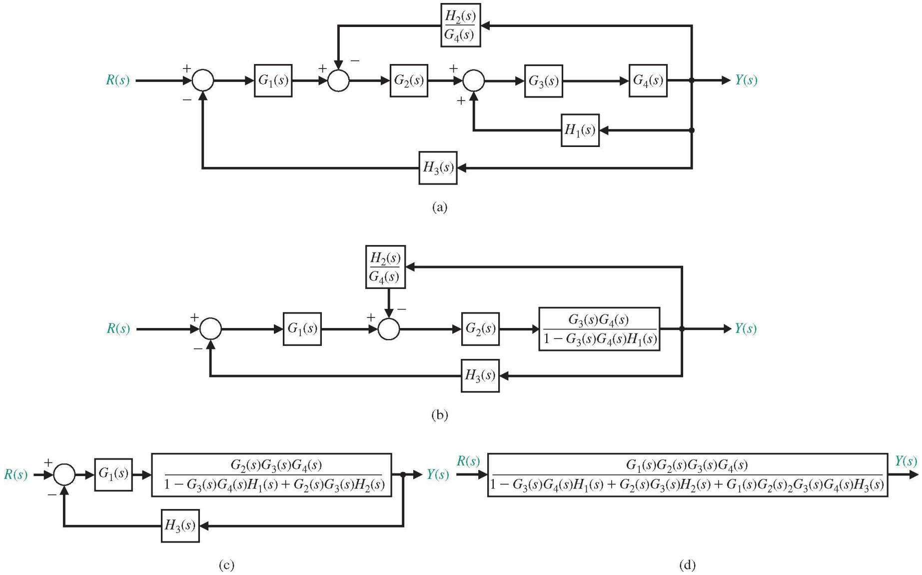
Example: Block Diagram Reduction

- Consider a multi-loop feedback control system:



- Apply equivalent transformations to eliminate the feedback loops and obtain the system transfer function $\frac{Y(s)}{R(s)}$

Example: Block Diagram Reduction

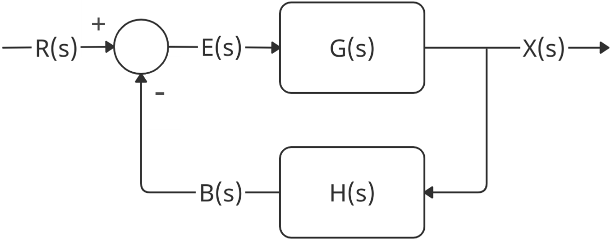


Outline

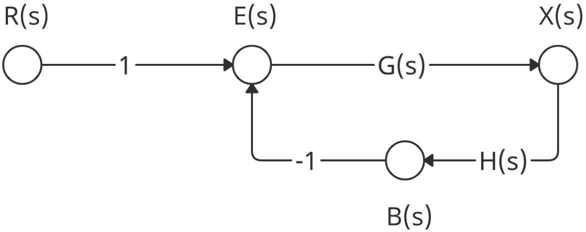
Signal Flow Graph

- **Signal Flow Graph**: a graphical representation of a control system consisting of nodes connected by branches
- **Node**: a junction point representing a signal variable as the sum of all signals entering it
- **Branch**: a directed line connecting two nodes with an associated transfer function
- **Path**: continuous succession of branches traversed in the same direction
- **Forward Path**: starts at an input node, ends at an output node, and no node is traversed more than once
- **Path Gain**: the product of all branch gains along the path
- **Loop**: a closed path that starts and ends at the same node and no node is traversed more than once
- **Non-touching Loops**: loops that do not contain common nodes

Block Diagram vs Signal Flow Graph



(a) Block Diagram



(b) Signal Flow Graph

Mason's Gain Formula

- A method for reducing a signal flow graph to a single transfer function
- The transfer function $T^{ij}(s)$ from **input** $X_i(s)$ to **any** variable $X_j(s)$ is:

$$T^{ij}(s) = \frac{X_j(s)}{X_i(s)} = \frac{\sum_k P_k^{ij}(s)\Delta_k^{ij}(s)}{\Delta(s)}$$

  where:
  - $\Delta(s)$: graph determinant
  - $P_k^{ij}(s)$: gain of the k-th forward path between $X_i(s)$ and $X_j(s)$
  - $\Delta_k^{ij}(s)$: graph determinant with the loops touching the k-th forward path between $X_i(s)$ and $X_j(s)$ removed
- The transfer function $T^{nj}(s)$ from **non-input** $X_n(s)$ to variable $X_j(s)$ is:

$$T^{nj}(s) = \frac{X_j(s)}{X_n(s)} = \frac{X_j(s)/X_i(s)}{X_n(s)/X_i(s)} = \frac{T^{ij}(s)}{T^{in}(s)} = \frac{\sum_k P_k^{ij}(s)\Delta_k^{ij}(s)}{\sum_k P_k^{in}(s)\Delta_k^{in}(s)}$$

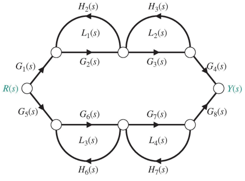
Mason's Gain Formula

- $L_n(s)$: gain of the n-th loop
- $\Delta(s)$: graph determinant

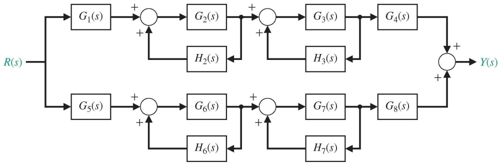
$$\begin{aligned}
\Delta(s) = 1 &- \sum \text{(individual loop gains)} \\
&+ \sum \prod \text{(gains of all 2 non-touching loop combinations)} \\
&- \sum \prod \text{(gains of all 3 non-touching loop combinations)} \\
&+ \cdots \\
= 1 &- \sum_{n} L_n(s) + \sum_{\substack{n,m \\ \text{nontouching}}} L_n(s)L_m(s) - \sum_{\substack{n,m,p \\ \text{nontouching}}} L_n(s)L_m(s)L_p(s) + \cdots
\end{aligned}$$

- $\Delta_k^{ij}(s)$: graph determinant with the loops touching the k-th forward path between $X_i(s)$ and $X_j(s)$ removed

Mason's Gain Formula Example 1



(a)



(b)

- Determine the transfer function $\frac{Y(s)}{R(s)}$ using Mason's gain formula
- Forward paths from $R(s)$ to $Y(s)$:

$$P_1(s) = G_1(s)G_2(s)G_3(s)G_4(s)$$
$$P_2(s) = G_5(s)G_6(s)G_7(s)G_8(s)$$

- Loop gains:

$$L_1(s) = G_2(s)H_2(s), \qquad L_2(s) = H_3(s)G_3(s),$$
$$L_3(s) = G_6(s)H_6(s), \qquad L_4(s) = G_7(s)H_7(s)$$

Mason's Gain Formula Example 1

- Determinant:

$$\Delta(s) = 1 - (L_1(s) + L_2(s) + L_3(s) + L_4(s)) \\ + (L_1(s)L_3(s) + L_1(s)L_4(s) + L_2(s)L_3(s) + L_2(s)L_4(s))$$

- Cofactor of path 1:

$$\Delta_1(s) = 1 - (L_3(s) + L_4(s))$$

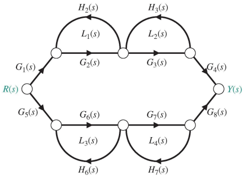
- Cofactor of path 2:

$$\Delta_2(s) = 1 - (L_1(s) + L_2(s))$$

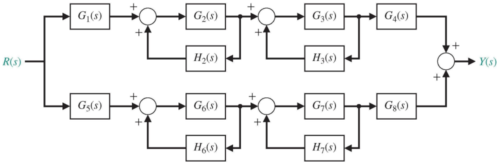
- Transfer function:

$$T(s) = \frac{P_1(s)\Delta_1(s) + P_2(s)\Delta_2(s)}{\Delta(s)}$$

Mason's Gain Formula Example 1



(a)

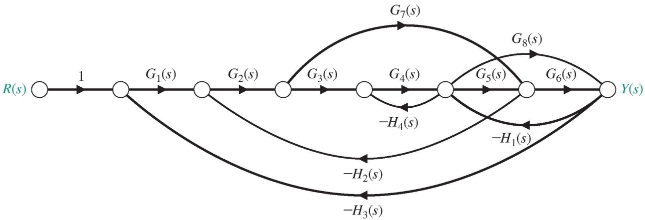


(b)

- The transfer function can also be obtained using block diagram transformations:

$$
\begin{aligned}
T(s) &= G_1(s)\left(\frac{G_2(s)}{1-G_2(s)H_2(s)}\right)\left(\frac{G_3(s)}{1-G_3(s)H_3(s)}\right)G_4(s) \\
&\quad + G_5(s)\left(\frac{G_6(s)}{1-G_6(s)H_6(s)}\right)\left(\frac{G_7(s)}{1-G_7(s)H_7(s)}\right)G_8(s) \\
&= G_1(s)G_2(s)G_3(s)G_4(s)\frac{\Delta_1(s)}{\Delta(s)} + G_5(s)G_6(s)G_7(s)G_8(s)\frac{\Delta_2(s)}{\Delta(s)}
\end{aligned}
$$

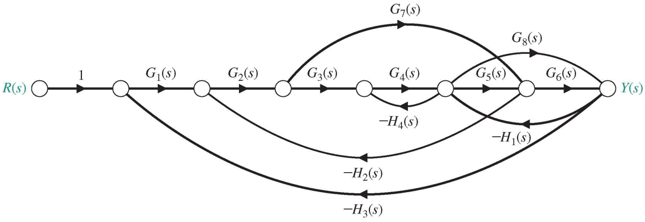
Mason's Gain Formula Example 2



- Determine the transfer function $\frac{Y(s)}{R(s)}$ using Mason's gain formula
- Forward paths from $R(s)$ to $Y(s)$:

$$
\begin{aligned}
P_1(s) &= G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_6(s) \\
P_2(s) &= G_1(s)G_2(s)G_7(s)G_6(s) \\
P_3(s) &= G_1(s)G_2(s)G_3(s)G_4(s)G_8(s)
\end{aligned}
$$

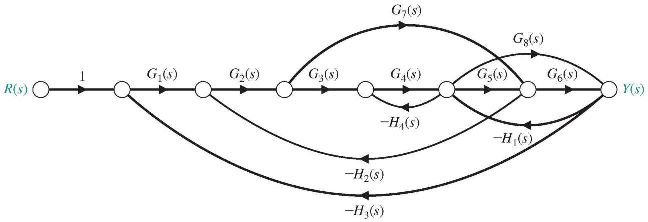
Mason's Gain Formula Example 2



- Loop gains:

$$L_1(s) = -G_2(s)G_3(s)G_4(s)G_5(s)H_2(s), \qquad L_2(s) = -G_5(s)G_6(s)H_1(s),$$
$$L_3(s) = -G_8(s)H_1(s), \qquad L_4(s) = -G_7(s)H_2(s)G_2(s)$$
$$L_5(s) = -G_4(s)H_4(s), \qquad L_6(s) = -G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)H_3(s)$$
$$L_7(s) = -G_1(s)G_2(s)G_7(s)G_6(s)H_3(s), \qquad L_8(s) = -G_1(s)G_2(s)G_3(s)G_4(s)G_8(s)H_3(s)$$

Mason's Gain Formula Example 2



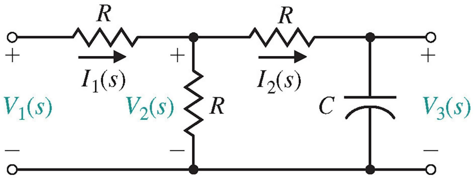
- Cofactors: $\Delta_1(s) = \Delta_3(s) = 1$ and $\Delta_2(s) = 1 - L_5(s)$
- Determinant: L_5 does not touch L_4 or L_7 and L_3 does not touch L_4:

$$\begin{aligned}\Delta(s) = 1 - (&L_1(s) + L_2(s) + L_3(s) + L_4(s) + L_5(s) + L_6(s) + L_7(s) + L_8(s)) \\ &+ (L_5(s)L_4(s) + L_5(s)L_7(s) + L_3(s)L_4(s))\end{aligned}$$

- Transfer function:

$$T(s) = \frac{P_1(s) + P_2(s)\Delta_2(s) + P_3(s)}{\Delta(s)}$$

Mason's Gain Formula Example 3



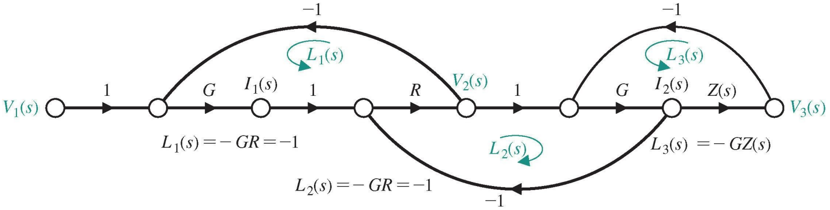
(a)

- Consider a ladder circuit with one energy storage element
- Determine the transfer function from $V_1(s)$ to $V_3(s)$
- The current and voltage equations are:

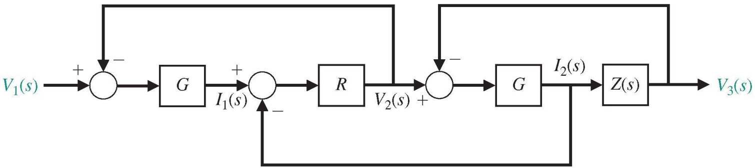
$$I_1(s) = \frac{1}{R}(V_1(s) - V_2(s)) \qquad I_2(s) = \frac{1}{R}(V_2(s) - V_3(s))$$

$$V_2(s) = R(I_1(s) - I_2(s)) \qquad V_3(s) = \frac{1}{Cs}I_2(s)$$

Mason's Gain Formula Example 3



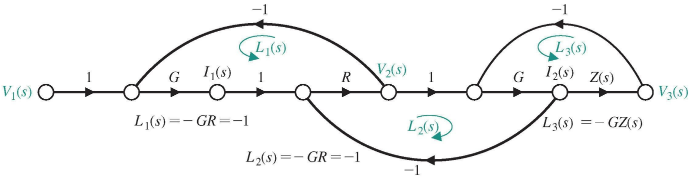
(b)



(c)

- Admittance: $G = \frac{1}{R}$
- Impedence: $Z(s) = \frac{1}{Cs}$

Mason's Gain Formula Example 3



(b)

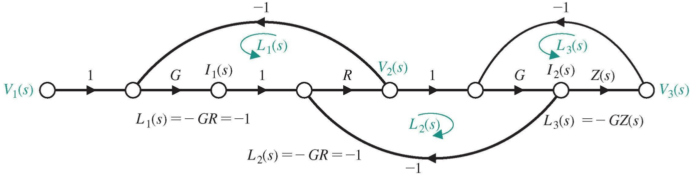
- Forward path: $P_1(s) = GRGZ(s) = GZ(s) = \frac{1}{RCs}$
- Loops: $L_1(s) = -GR = -1$, $L_2(s) = -GR = -1$, $L_3(s) = -GZ(s)$
- Cofactor: all loops touch the forward path: $\Delta_1(s) = 1$
- Determinant: loops $L_1(s)$ and $L_3(s)$ are non-touching:

$$\Delta(s) = 1 - (L_1(s) + L_2(s) + L_3(s)) + L_1(s)L_3(s) = 3 + 2GZ(s)$$

- Transfer function:

$$T(s) = \frac{V_3(s)}{V_1(s)} = \frac{P_1(s)}{\Delta(s)} = \frac{GZ(s)}{3 + 2GZ(s)} = \frac{1/(3RC)}{s + 2/(3RC)}$$

Mason's Gain Formula Example 3



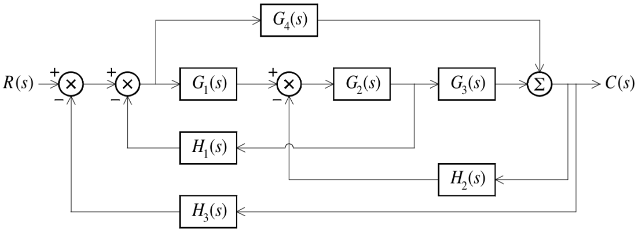
(b)

- Determine the transfer function from $I_1(s)$ to $I_2(s)$
- Instead of re-drawing the signal flow graph, we can use:

$$\frac{I_2(s)}{I_1(s)} = \frac{I_2(s)/V_1(s)}{I_1(s)/V_1(s)} = \frac{G}{G(2+GZ(s))} = \frac{1}{2+GZ(s)} = \frac{s}{2s+1/(RC)}$$

- One forward path from $V_1(s)$ to $I_2(s)$ with gain $GRG = G$ and cofactor 1
- One forward path from $V_1(s)$ to $I_1(s)$ with gain G and cofactor $1 - (L_2(s) + L_3(s)) = 2 + GZ(s)$

Mason's Gain Formula Example 4



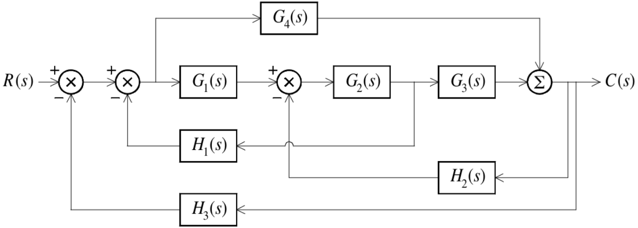
- Determine the transfer function from $R(s)$ to $C(s)$
- Forward paths:

$$P_1(s) = G_1(s)G_2(s)G_3(s) \qquad P_2(s) = G_4(s)$$

- Loops:

$$L_1(s) = -G_1(s)G_2(s)H_1(s) \qquad L_2(s) = -G_2(s)G_3(s)H_2(s)$$
$$L_3(s) = -G_1(s)G_2(s)G_3(s)H_3(s) \qquad L_4(s) = -G_4(s)H_3(s)$$
$$L_5(s) = G_2(s)H_1(s)G_4(s)H_2(s)$$

Mason's Gain Formula Example 4



- Cofactors: both forward paths touch all loops: $\Delta_1(s) = \Delta_2(s) = 1$
- Determinant: all loop pairs are touching:

$$\Delta(s) = 1 - (L_1(s) + L_2(s) + L_3(s) + L_4(s) + L_5(s))$$

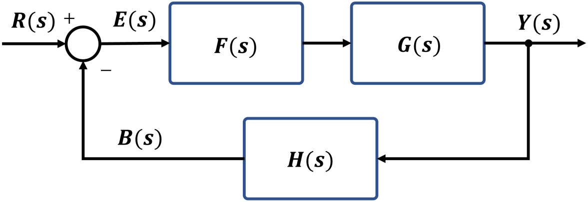
- Transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{P_1(s) + P_2(s)}{\Delta(s)} = \frac{G_1(s)G_2(s)G_3(s) + G_4(s)}{\Delta(s)}$$

Outline

Parameter Sensitivity

- Feedback control is useful for reducing sensitivity to parameter variations in the plant $G(s)$



- Transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)F(s)}{1 + G(s)F(s)H(s)}$$

- Suppose that $G(s)$ undergoes a change $\Delta G(s)$ so that the true plant model is $G(s) + \Delta G(s)$

- What is the change $\Delta T(s)$ in the overall transfer function $T(s)$?

Parameter Sensitivity

- Since $T(s)$ and $G(s)$ might have different units, parameter sensitivity is defined as a percentage change in $T(s)$ over percentage change in $G(s)$
- **Parameter sensitivity**: ratio of the incremental change in the overall system transfer function to the incremental change in the transfer function of one component:

$$\boxed{S_G^T(s) = \frac{dT(s)}{dG(s)} \frac{G(s)}{T(s)}} \approx \frac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)}$$

- Parameter sensitivity should be small to allow robustness to changes in $G(s)$
- Conversely, the transfer function of elements with high sensitivity should be estimated well because minor mismatch might have a significant effect on the overall system transfer function. These are the system elements we should really be careful about.

Return Difference

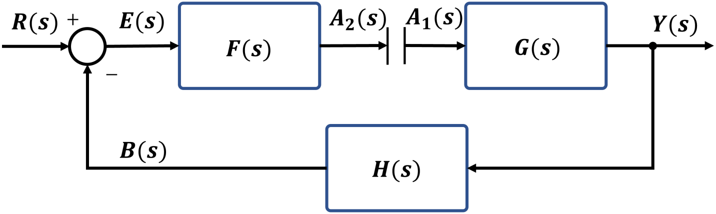
- Hendrik Bode was interested in measuring the effect of feedback on a specific element in a closed-loop control system

- Bode defined **return difference** as an impulse input $U(s) = 1$ at a system element minus the loop transfer function $L(s)$ back to the element:

$$\rho(s) = 1 - L(s)$$

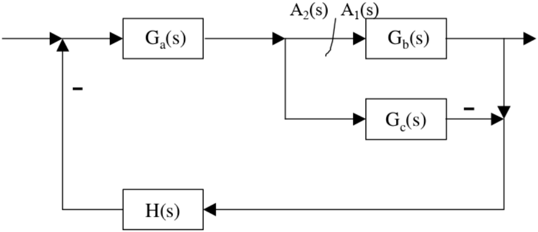
- Return difference computation:
  - open the feedback loop immediately prior to the element of interest
  - compute the transfer function $L(s) = \frac{A_2(s)}{A_1(s)}$ from the element input $(A_1(s))$ back to the cut connection $(A_2(s))$
  - the return difference is $\rho(s) = 1 - L(s)$

Return Difference Example 1



- Return difference with respect to $G(s)$
- Cut the loop immediately prior to $G(s)$
- Compute the loop gain: $L(s) = \frac{A_2(s)}{A_1(s)} = -G(s)H(s)F(s)$
- Return difference: $\rho_G(s) = 1 - L(s) = 1 + G(s)H(s)F(s)$

Return Difference Example 2



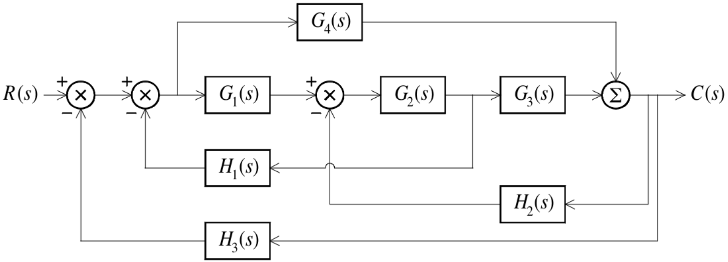
- Return difference with respect to $G_b(s)$
- Cut the loop immediately prior to $G_b(s)$
- Compute the loop gain via Mason's formula:

$$L(s) = \frac{G_1(s)\Delta_1(s)}{\Delta(s)} = \frac{-H(s)G_a(s)G_b(s)}{1 - H(s)G_a(s)G_c(s)}$$

- Return difference:

$$\rho_{G_b}(s) = 1 - L(s) = 1 + \frac{H(s)G_a(s)G_b(s)}{1 - H(s)G_a(s)G_c(s)} = \frac{1 + H(s)G_a(s)(G_b(s) - G_c(s))}{1 - H(s)G_a(s)G_c(s)}$$

Return Difference Example 3



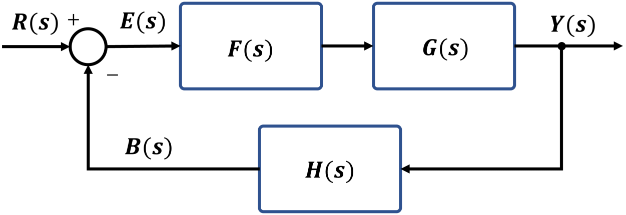
- Return difference with respect to $G_2(s)$
- Cut the loop immediately prior to $G_2(s)$
- Compute the loop gain via Mason's formula:

$$L(s) = \frac{-G_2(s)H_1(s)G_1(s) - G_2(s)G_3(s)H_2(s) - G_2(s)G_3(s)H_3(s)G_1(s) + G_2(s)H_1(s)G_4(s)H_2(s)}{1 + G_4(s)H_3(s)}$$

- Return difference: $\rho_{G_2}(s) = 1 - L(s)$

Parameter Sensitivity is Inverse Return Difference

- How is parameter sensitivity related to return difference?



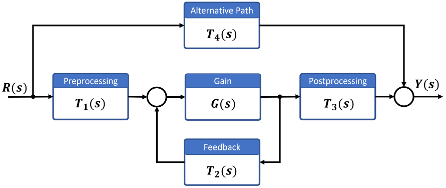
For a control system with a single feedback loop, parameter sensitivity $S_G(s)$ is equal to the inverse of the return difference $\rho_G(s)$.

$$
\begin{aligned}
S_G(s) &= \frac{dT(s)}{dG(s)}\frac{G(s)}{T(s)} = \frac{d}{dG(s)}\left(\frac{G(s)F(s)}{1+G(s)F(s)H(s)}\right)\frac{G(s)}{T(s)} \\
&= \frac{F(s)}{(1+G(s)F(s)H(s))^2}\frac{G(s)}{T(s)} = \frac{1}{1+G(s)F(s)H(s)} \\
&= \frac{1}{1-L(s)} = \frac{1}{\rho_G(s)}
\end{aligned}
$$

Canonical Feedback Control Architecture

- Transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = T_4(s) + \frac{T_1(s)G(s)T_3(s)}{1 - G(s)T_2(s)}$$



- Sensitivity of $T(s)$ with respect to $G(s)$:

$$\frac{dT}{dG} = T_1 T_3 \left(\frac{1}{1 - GT_2} + \frac{GT_2}{(1 - GT_2)^2} \right) = \frac{T_1 T_3}{(1 - GT_2)^2}$$

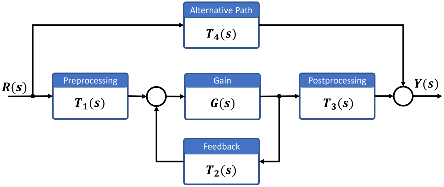
$$S_G^T = \frac{G}{T}\frac{dT}{dG} = \frac{G(1 - GT_2)}{T_4(1 - GT_2) + T_1 T_3 G} \frac{T_1 T_3}{(1 - GT_2)^2}$$

$$= \frac{GT_1 T_3}{T_4(1 - GT_2)^2 + T_1 T_3 G(1 - GT_2)}$$

Canonical Feedback Control Architecture

- Transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = T_4(s) + \frac{T_1(s)G(s)T_3(s)}{1 - G(s)T_2(s)}$$



- Sensitivity of $T(s)$ with respect to $G(s)$:

$$S_G^T(s) = \frac{G(s)T_1(s)T_3(s)}{T_4(s)(1 - G(s)T_2(s))^2 + T_1(s)T_3(s)G(s)(1 - G(s)T_2(s))}$$

- Note that $G(s)$ does not affect $T_4(s)$ in the transfer function. Consider only the portion that $G(s)$ affects:

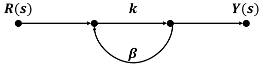
$$T'(s) = \frac{T_1(s)G(s)T_3(s)}{1 - G(s)T_2(s)}$$

- Letting $T_4(s) = 0$ in $S_G^T(s)$ shows that $S_G^{T'}(s)$ is the inverse of the return difference:

$$S_G^{T'}(s) = \frac{1}{1 - G(s)T_2(s)} = \frac{1}{\rho_G^{T'}(s)}$$

Example: Feedback OpAmp Sensitivity

- Feedback amplifier with input voltage $R(s)$, feedforward gain k, feedback gain β, and output voltage $Y(s)$



- Transfer function: $T(s) = \frac{Y(s)}{R(s)} = \frac{k}{1-k\beta}$
- Return difference: $\rho_k = 1 - k\beta$
- Sensitivity wrt k: $S_k^T = \frac{1}{1-k\beta}$
- Sensitivity wrt β: $S_\beta^T = \frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta(1-k\beta)}{k}\frac{k^2}{(1-\beta k)^2} = \frac{k\beta}{1-k\beta}$
- When $k \approx 10^3$ and $\beta \approx -0.1$, then $S_k^T \approx 0$ and $S_\beta^T \approx -1$.
- When designing an OpAmp, the forward gain k can be arbitrary but we need to be careful with the design of β because it affects the response almost one-to-one