ECE171A: Linear Control System Theory
Lecture 9: Frequency Response

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Outline

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Test Input Signals

- The transient and steady-state response of a system are often studied for specific test input signals

Test Signal	$u(t)$	$U(s)$
Impulse	$u(t) = \delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0 \end{cases}$	$U(s) = 1$
Step	$u(t) = H(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$	$U(s) = \frac{1}{s}$
Ramp	$u(t) = tH(t) = \begin{cases} t, & t \geq 0, \\ 0, & t < 0 \end{cases}$	$U(s) = \frac{1}{s^2}$
Parabola	$u(t) = \frac{t^2}{2} H(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0, \\ 0, & t < 0 \end{cases}$	$U(s) = \frac{1}{s^3}$
Sine	$u(t) = \begin{cases} \sin(\omega t), & t \geq 0, \\ 0, & t < 0 \end{cases}$	$U(s) = \frac{\omega}{s^2 + \omega^2}$
Cosine	$u(t) = \begin{cases} \cos(\omega t), & t \geq 0, \\ 0, & t < 0 \end{cases}$	$U(s) = \frac{s}{s^2 + \omega^2}$

Impulse Response

- LTI ODE System:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- **Impulse response**: response to an impulse input $u(t) = \delta(t)$:

$$\begin{aligned} y(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)d\tau + \mathbf{D}\delta(t) \\ &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t) \end{aligned}$$

- The impulse response with zero initial conditions reveals the transfer function:

$$Y(s) = G(s)\cancelto{1}{U(s)} \quad \Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = g(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t)$$

- By superposition, the forced response to any input $u(t)$ is the convolution of the input with the impulse response:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)}_{\text{natural response}} + \underbrace{\int_0^t g(t-\tau)u(\tau)d\tau}_{\text{forced response}}$$

Exponential Response

- LTI ODE System:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- **Exponential response**: response to exponential input $u(t) = e^{st}$ for $t \geq 0$ such that $s \in \mathbb{C}$ is not an eigenvalue of $\mathbf{A}$:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}(0) - (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right)}_{\text{transient response}} + \underbrace{\left(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)e^{st}}_{\text{steady-state response}}$$

- The transfer function $G(s)$ is a complex number:

$$G(s) = |G(s)|e^{j\angle G(s)}$$

- Steady-state exponential response:

$$y_{ss}(t) = |G(s)|e^{j\angle G(s)}e^{st} = |G(s)|e^{st + j\angle G(s)}$$

Step Response

- LTI ODE system:

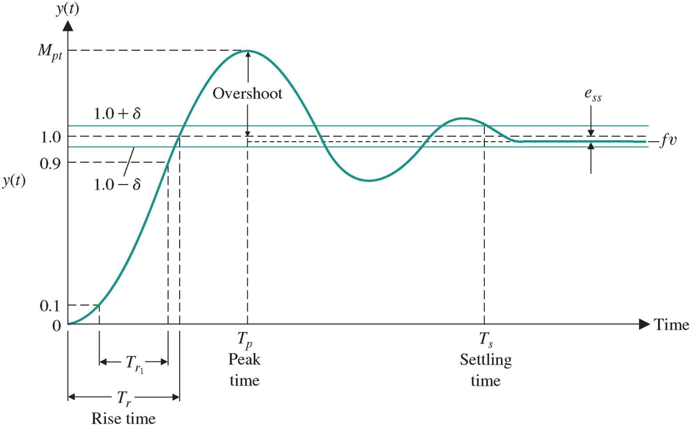
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- **Step response**: response to a step input $u(t) = 1$ for $t \geq 0$, which is a special case of $u(t) = e^{st}$ with $s = 0$:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}(\mathbf{x}(0) + \mathbf{A}^{-1}\mathbf{B})}_{\text{transient response}} + \underbrace{G(0)}_{\text{steady-state response}}$$

Step Response Performance Measures



- **Rise time**: from 10% to 90% of steady-state value: $t_r \approx \frac{2.16\zeta+0.6}{\omega_n}$
- **Peak time**: time at which the response is maximum: $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
- **Overshoot**: overshoot as percent of steady-state: $\text{p.o.} = 100\exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)\%$
- **Settling time**: response settles within 2% of steady-state: $t_s \approx \frac{4}{\zeta\omega_n}$
- **Steady-state error**: $e_{ss} = 1 - \lim_{t\to\infty} y(t) = 1 - G(0)$

Frequency Response

- LTI ODE System:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- **Frequency response**: response to a sinusoidal input $u(t) = \sin(\omega t + \phi)$

Frequency Response

The steady-state response of a system with transfer function $G(s)$ to a sinusoidal input $u(t) = \sin(\omega t + \phi)$ is a sinusoid of the **same frequency** with **amplitude scaled by** $|G(j\omega)|$ and **phase shifted by** $\angle G(j\omega)$:

$$y_{ss}(t) = |G(j\omega)| \sin(\omega t + \phi + \angle G(j\omega))$$

- The **magnitude** $|G(j\omega)|$ is determined from the ratio of the amplitudes of the output versus the input sinusoids
- The **phase** $\angle G(j\omega)$ is determined from the ratio of the time of the output versus the input zero crossings

Frequency Response Proof

- **Euler's Formula**: $\sin(\omega t + \phi) = \mathrm{Im}(e^{j(\omega t+\phi)}) = \dfrac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}$
- **Complex conjugate of** $G(s)$: $G^*(s) = |G(s)|e^{-j\angle G(s)}$
- **Conjugate symmetry of** $G(s)$:

$$G^*(s) = \left(\int_0^\infty g(t)e^{-st}dt\right)^* = \int_0^\infty g^*(t)e^{-s^*t}dt$$

$$\overset{g(t)\text{ is real}}{=\!=\!=\!=\!=} \int_0^\infty g(t)e^{-s^*t}dt = G(s^*)$$

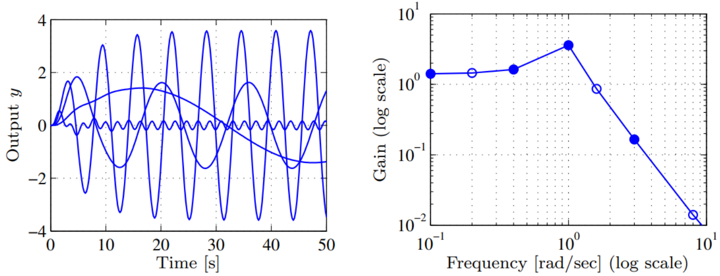
- **Proof**: by superposition the steady-state response to $u(t) = \sin(\omega t + \phi)$ is:

$$\begin{aligned} y_{ss}(t) &= \frac{1}{2j}G(j\omega)e^{j(\omega t+\phi)} - \frac{1}{2j}G(-j\omega)e^{-j(\omega t+\phi)} \\ &= \frac{1}{2j}|G(j\omega)|e^{j\angle G(j\omega)}e^{j(\omega t+\phi)} - \frac{1}{2j}|G(j\omega)|e^{-j\angle G(j\omega)}e^{-j(\omega t+\phi)} \\ &= |G(j\omega)|\sin(\omega t + \phi + \angle G(j\omega)) \end{aligned}$$

Empirical Transfer Function Determination

- The frequency response can be obtained empirically by applying a sinusoidal test signal at various frequencies and recording the magnitude and phase of the response. This can be used to identify the system's transfer function.

  1. Apply a sinusoidal signal at a fixed frequency ω
  2. Measure response amplitude ratio and phase lag at steady state
  3. Repeat as ω varies from 0 to ∞



(a) Time domain simulations (b) Frequency response

Figure: Gain computed by measuring system response to individual sinusoid inputs

Frequency Domain Plots

- Plotting the magnitude and phase of the transfer function $G(j\omega)$ versus the input frequency ω provides insight about the behavior of a linear control system
- The following frequency-domain plots of the transfer function are used:
  - **Bode plot**: plot of magnitude $20\log_{10}|G(j\omega)|$ in decibels (dB) and phase $\angle G(j\omega)$ in degrees versus $\log_{10}\omega$ as ω varies from 0 to ∞
  - **Polar plot**: plot of $\text{Im}(G(j\omega))$ versus $\text{Re}(G(j\omega))$ as ω varies from 0 to ∞
  - **Magnitude-phase plot**: plot of magnitude $20\log_{10}|G(j\omega)|$ in decibels (dB) versus phase $\angle G(j\omega)$ in degrees as ω varies from 0 to ∞

Decibel Units

- **Bel**: relative measurement unit of log-ratio of measured power P to reference power P_0

$$\text{Log-power ratio} = \log_{10}\left(\frac{P}{P_0}\right) \text{ Bels}$$

- **Decibel**: ten Bels:

$$\text{Log-power ratio} = 10\log_{10}\left(\frac{P}{P_0}\right) \text{ dB}$$

- The **power spectral density** of $y(t)$ is the Fourier transform $S_{yy}(j\omega)$ of the autocorrelation function

- The input-output power spectral density relationship for an LTI system with input $U(s)$, transfer function $G(s)$, and output $Y(s)$ is:

$$S_{yy}(j\omega) = |Y(j\omega)|^2 = |G(j\omega)|^2|U(j\omega)|^2 = |G(j\omega)|^2 S_{uu}(j\omega)$$

- The log-power ratio at ω in dB is:

$$10\log_{10}\left(\frac{S_{yy}(j\omega)}{S_{uu}(j\omega)}\right) = 10\log_{10}|G(j\omega)|^2 = 20\log_{10}|G(j\omega)|$$

Outline

System Response to Test Input Signals

Bode Plot

Non-minimum Phase Systems

Polar Plot

Magnitude-Phase Plot

Bode Plot

- **Hendrik Bode**: a pioneer of modern control theory and electronic telecommunications
- **Bode plot**: represents the frequency response of a linear system with transfer function $G(s)$ by two plots:
  - Plot of **magnitude** $20\log_{10}|G(j\omega)|$ in dB versus $\log_{10}\omega$
  - Plot of **phase** $\angle G(j\omega)$ in degrees versus $\log_{10}\omega$

H. Bode

- Logarithmic scale is used for the input frequency ω to capture the system behavior over a wide frequency range
- The log-scale intervals are known as decades (base 10) or octaves (base 2):
  - The number of **decades** between ω_1 and ω_2 is $\log_{10}\frac{\omega_2}{\omega_1}$
  - The number of **octaves** between ω_1 and ω_2 is $\log_2\frac{\omega_2}{\omega_1}$
  - There are $\log_2(10) \approx 3.32$ octaves in one decade
  - A slope of 20 dB/decade is the same as $\frac{20 \text{ dB/decade}}{\log_2(10) \text{ octave/decade}} \approx 6$ dB/octave

Transfer Function Magnitude and Phase

- The magnitude and phase of $G(s)$ are needed to draw a Bode plot

- Consider a transfer function $G(s) = \dfrac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$

- **Magnitude** of $G(s)$ in log-scale is the sum/difference of magnitudes corresponding to terms in the numerator/denominator:

$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|$$

- **Phase** of $G(s)$ is the sum/difference of phases corresponding to terms in the numerator/denominator:

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s).$$

Transfer Function in Bode Form

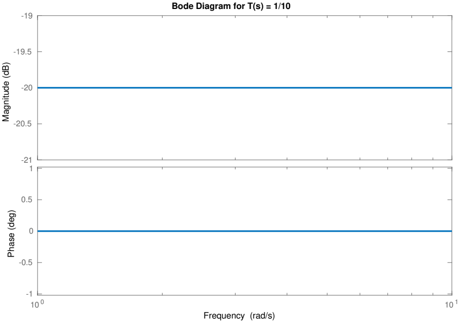
- Instead of computing the magnitude and phase of $G(s)$ directly, it is preferable to obtain rules for drawing Bode plots of individual terms
- **Transfer function in Bode form**: a transfer function with m_1 real zeros, m_2 complex conjugate zero pairs, n_0 poles at the origin, n_1 real poles, and n_2 complex conjugate pole pairs:

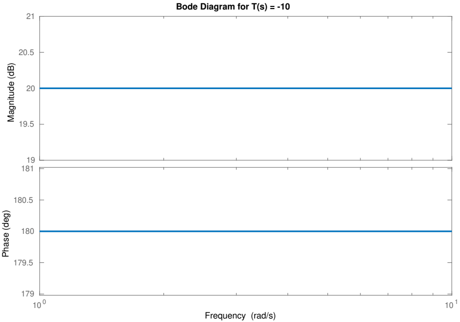
$$G(s) = \kappa \frac{\prod_{i=1}^{m_1}\left(\frac{s}{z_i}+1\right)\prod_{l=1}^{m_2}\left(\left(\frac{s}{\omega_{n_l}}\right)^2 + 2\zeta_l\left(\frac{s}{\omega_{n_l}}\right)+1\right)}{s^{n_0}\prod_{i=1}^{n_1}\left(\frac{s}{p_i}+1\right)\prod_{k=1}^{n_2}\left(\left(\frac{s}{\omega_{n_k}}\right)^2 + 2\zeta_k\left(\frac{s}{\omega_{n_k}}\right)+1\right)}$$

- A transfer function may contain only four kinds of factors:
  - Constant term: κ
  - Poles s^{-q} or zeros s^q at the origin
  - Real poles $\left(\frac{s}{p}+1\right)^{-1}$ or zeros $\left(\frac{s}{z}+1\right)$
  - Complex conjugate poles or zeros: $\left(\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right)+1\right)^{\pm 1}$
- If we determine the magnitude and phase plots for these four factors, we can add them together graphically to obtain a Bode plot for any transfer function

Bode Plot for a Constant Term κ

- **Magnitude**: $20 \log |\kappa|$
- **Phase**: $\angle\kappa = \begin{cases} 0^\circ & \text{if } \kappa > 0 \\ 180^\circ & \text{if } \kappa < 0 \end{cases}$
- Example: Bode plot for $G(s) = \frac{1}{10}$ and $G(s) = -10$





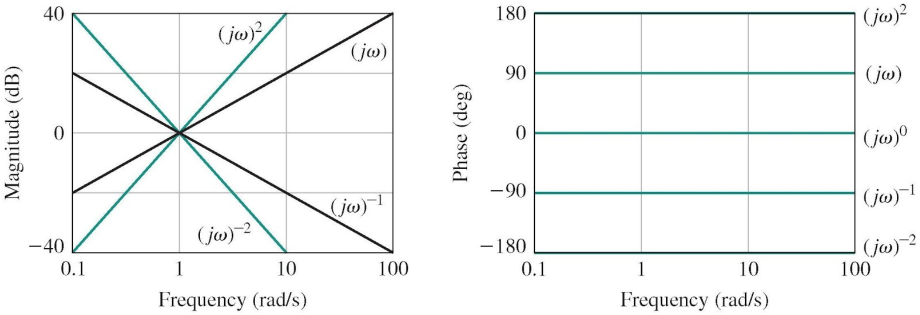
Bode Plot for Pole or Zero at the Origin: s^q

- **Magnitude**: straight line (log scale) through the origin with slope $20q$:

$$20 \log |(j\omega)^q| = 20q \log |\omega|$$

- **Phase**: a horizontal line at $q90^\circ$:

$$\underline{/(j\omega)^q} = q\underline{/(j\omega)} = q90^\circ$$



Bode Plot for Real Zero $\left(\frac{s}{z}+1\right)$

- **Magnitude**: $20\log\left|j\frac{\omega}{z}+1\right| = 20\log\sqrt{1+\left(\frac{\omega}{z}\right)^2}$
- **Phase**: $\underline{/\left(j\frac{\omega}{z}+1\right)} = \tan^{-1}\frac{\omega}{z}$
- Extreme ω values:
  - **Case 1**: $\omega \ll z$: horizontal line at 0:

$$20\log\left|j\frac{\omega}{z}+1\right| \approx 0 \qquad \underline{/\left(j\frac{\omega}{z}+1\right)} \approx 0^\circ$$

  - **Case 2**: $\omega \gg z$: log-scale line of slope 20 going through 0 when $\omega = z$:

$$20\log\left|j\frac{\omega}{z}+1\right| \approx 20\log\frac{1}{z} + 20\log\omega \qquad \underline{/\left(j\frac{\omega}{z}+1\right)} \approx 90^\circ$$

  - **Case 3**: $\omega = z$ (**corner frequency**):

$$20\log\left|j\frac{\omega}{z}+1\right| \approx 3dB \qquad \underline{/\left(j\frac{\omega}{z}+1\right)} = 45^\circ$$

Bode Plot for Real Pole $\left(\frac{s}{p}+1\right)^{-1}$

- **Magnitude**: $20\log\left|\left(j\frac{\omega}{p}+1\right)^{-1}\right| = -20\log\sqrt{1+\left(\frac{\omega}{p}\right)^2}$
- **Phase**: $\angle\left(j\frac{\omega}{p}+1\right)^{-1} = -\tan^{-1}\frac{\omega}{p}$
- Extreme ω values:
  - **Case 1**: $\omega \ll p$: horizontal line at 0:

$$-20\log\left|j\frac{\omega}{p}+1\right| \approx 0 \qquad \angle\left(j\frac{\omega}{p}+1\right)^{-1} \approx 0^\circ$$

  - **Case 2**: $\omega \gg p$: log-scale line of slope -20 going through 0 when $\omega = p$:

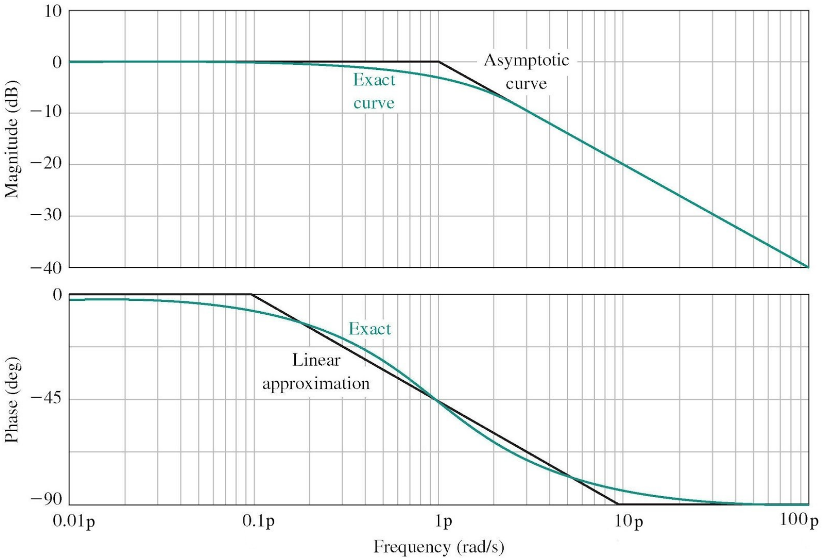
$$-20\log\left|j\frac{\omega}{p}+1\right| \approx -20\log\frac{1}{p} - 20\log\omega \qquad \angle\left(j\frac{\omega}{p}+1\right)^{-1} \approx -90^\circ$$

  - **Case 3**: $\omega = p$ (**corner frequency**):

$$-20\log\left|j\frac{\omega}{p}+1\right| \approx -3dB \qquad \angle\left(j\frac{\omega}{p}+1\right)^{-1} \approx -45^\circ$$

Bode Plot for Real Pole $\left(\frac{s}{p}+1\right)^{-1}$

- A real pole behaves like a constant at low frequencies and like an integrator at high frequencies



Bode Plot Example 1

- Draw a Bode plot for $G(s) = 10\frac{s+10}{(s+1)(s+100)} = \frac{(s/10+1)}{(s+1)(s/100+1)}$

Step 1 : Find frequency break points (poles and zeros): 1, 10, 100

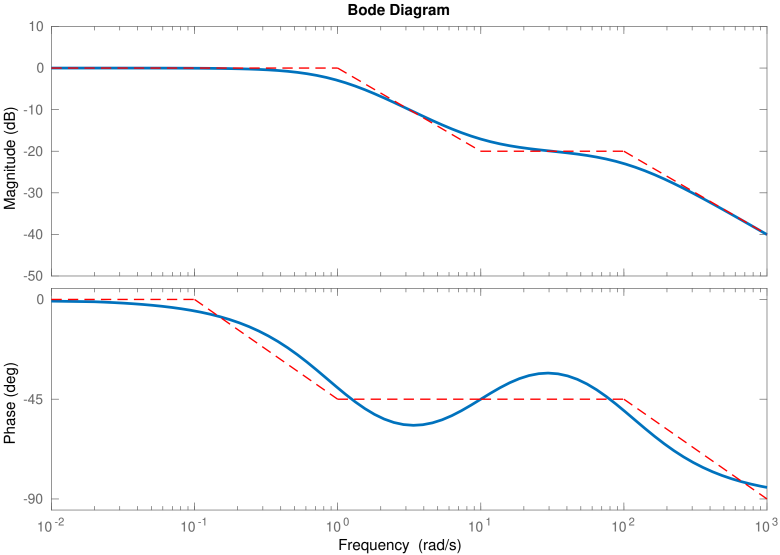
Step 2 : Calculate $|G(0)|$ and $\angle G(0)$ to determine the starting points

Step 3 : Sketch the Bode plot by the rules:

- **Magnitude increases with a zero**: the slope is $+20$ dB/decade for a real zero
- **Magnitude decreases with a pole**: the slope is -20 dB/decade for a real pole
- **Phases increases with a zero**: by $+90°$ starting from $z/10$ and ending at $10z$
- **Phases decreases with a pole**: by $-90°$ starting from $p/10$ and ending at $10p$

Bode Plot Example 1

- Draw a Bode plot for $G(s) = 10\frac{s+10}{(s+1)(s+100)} = \frac{(s/10+1)}{(s+1)(s/100+1)}$



Bode Plot for Complex Conjugate Zeros

- Consider $G(s) = \left(\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right)$
- **Magnitude**:

$$|G(j\omega)| = \left|-\frac{\omega^2}{\omega_n^2} + 2\zeta\frac{\omega}{\omega_n}j + 1\right| = \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}$$

- **Phase**:

$$\angle G(j\omega) = \angle -\frac{\omega^2}{\omega_n^2} + 2\zeta\frac{\omega}{\omega_n}j + 1 = \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Bode Plot for Complex Conjugate Zeros

$$|G(j\omega)| = \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} \qquad \angle G(j\omega) = \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

- Extreme ω values:
  - **Case 1**: $\omega \ll \omega_n$: horizontal line at 0:

$$20\log|G(j\omega)| \approx 0 \qquad \angle G(j\omega) \approx 0^\circ$$

  - **Case 2**: $\omega \gg \omega_n$: log-scale line of slope 40 going through 0 when $\omega = \omega_n$:

$$20\log|G(j\omega)| \approx 20\log\sqrt{\left(\frac{\omega}{\omega_n}\right)^4} = 40\log\omega - 40\log\omega_n \qquad \angle G(j\omega) \approx 180^\circ$$

  - **Case 3**: $\omega = \omega_n$:

$$20\log|G(j\omega)| = 20\log(2\zeta) \qquad \angle G(j\omega) = 90^\circ$$

Bode Plot for Complex Conjugate Poles

- Consider $G(s) = \left(\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right)^{-1}$

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} \qquad \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

- Extreme ω values:
  - **Case 1**: $\omega \ll \omega_n$: horizontal line at 0:

$$20\log|G(j\omega)| \approx 0 \qquad \angle G(j\omega) \approx 0^\circ$$

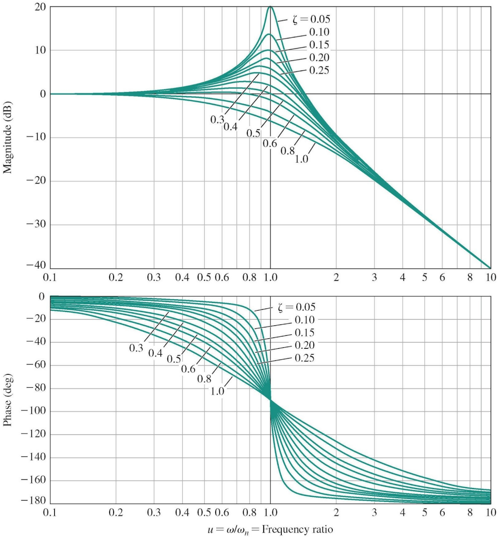
  - **Case 2**: $\omega \gg \omega_n$: log-scale line of slope -40 going through 0 when $\omega = \omega_n$

$$20\log|G(j\omega)| \approx -20\log\sqrt{\left(\frac{\omega}{\omega_n}\right)^4} = -40\log\omega + 40\log\omega_n \qquad \angle G(j\omega) \approx -180^\circ$$

  - **Case 3**: $\omega = \omega_n$:

$$20\log|G(j\omega)| = -20\log(2\zeta) \qquad \angle G(j\omega) = -90^\circ$$

Bode Plot for Complex Conjugate Poles



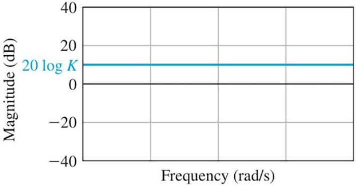
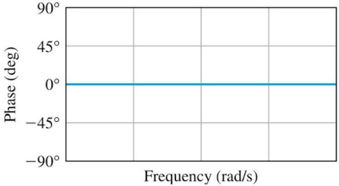
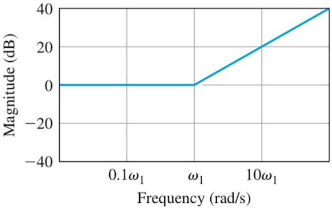
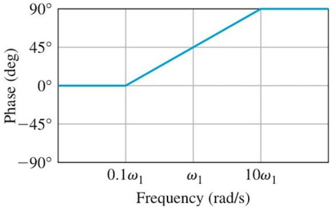
- $G(s) = \dfrac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$

- **Resonant frequency**: the largest gain $\max_\omega |G(j\omega)| \approx \frac{1}{2\zeta}$ occurs at $\omega \approx \omega_n$

- The asymptotic approximation is poor near $\omega = \omega_n$ and the magnitude and phase depend on ζ

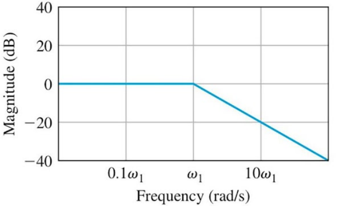
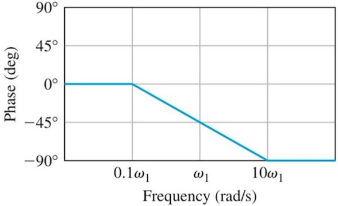
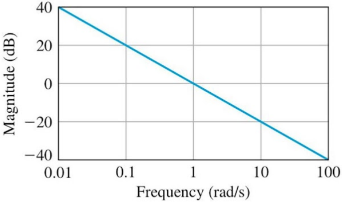
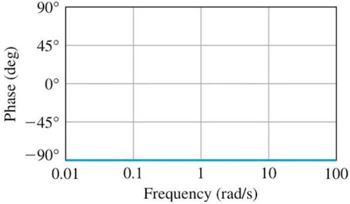
Bode Plot Approximations for Basic Transfer Function Terms

Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude $20 \log_{10}\lvert G(j\omega)\rvert$	Phase $\phi(\omega)$
1. Gain, $G(j\omega) = K$		
2. Zero, $G(j\omega) = 1 + j\omega/\omega_1$		

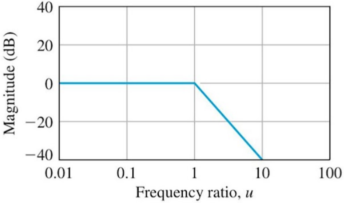
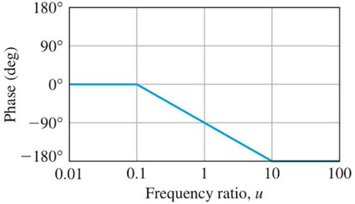
Bode Plot Approximations for Basic Transfer Function Terms

Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude $20 \log_{10}\lvert G(j\omega)\rvert$	Phase $\phi(\omega)$
3. Pole, $G(j\omega) = (1 + j\omega/\omega_1)^{-1}$		
4. Pole at the origin, $G(j\omega) = 1/j\omega$		

Bode Plot Approximations for Basic Transfer Function Terms

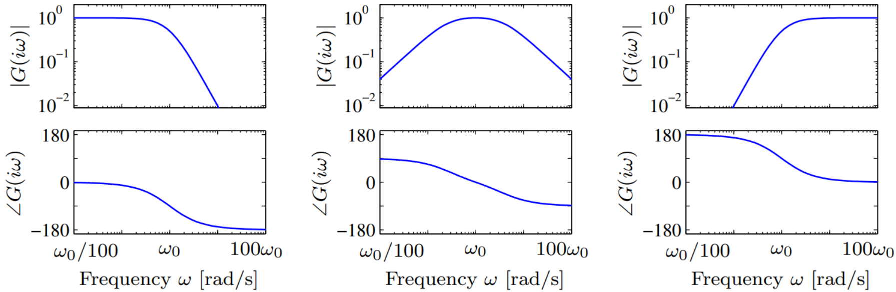
Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude $20 \log_{10}\lvert G(j\omega)\rvert$	Phase $\phi(\omega)$
5. Two complex poles, $0.1 < \zeta < 1$, $G(j\omega) = (1 + j2\zeta u - u^2)^{-1}$ $u = \omega/\omega_n$		

LTI Systems as Filters

- A Bode plot allows viewing a stable linear system as a filter that changes input signals depending on the frequency range
- **Low-pass filter**:
$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$
- **Band-pass filter**:
$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$
- **High-pass filter**:
$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

LTI Systems as Filters



$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(a) Low-pass filter

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(b) Band-pass filter

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(c) High-pass filter

Figure: Bode plots for low-pass, band-pass, and high-pass filters. Each system passes frequencies in a specific range and attenuates the frequencies outside of that range.

Bode Plot Example 2

- Draw a Bode plot for $G(s) = \dfrac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}$ with $a \ll b \ll \omega_0$

- **Magnitude plot:**
  - Begin with $G(0) = \frac{kb}{a\omega_0^2}$
  - At $\omega = a$, the effect of the real pole begins and the gain decreases with slope -20 dB/decade
  - At $\omega = b$, the real zero increases the slope by 20 dB/decade, leaving a net slope of 0 dB/decade
  - This slope is used until the second-order pole affects it at $\omega = \omega_0$ by -40 dB/decade

- **Phase plot:**
  - The approximation process is similar but effect of the poles and zeros on the phase begin one decade earlier and terminate one decade later.

Bode Plot Example 2

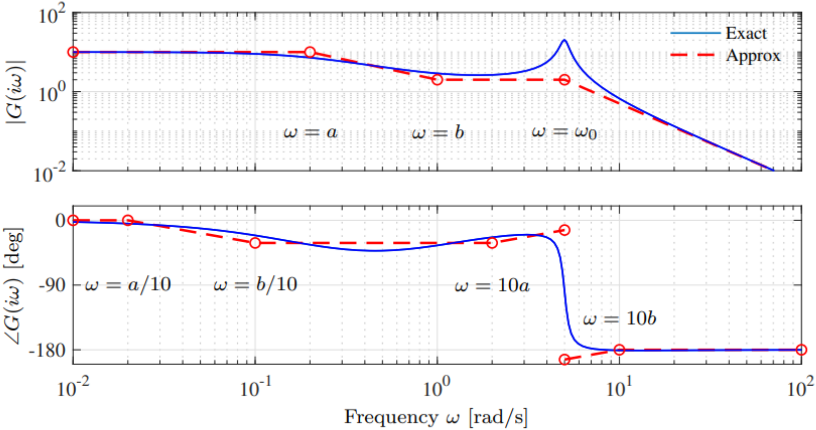


Figure 9.15: Asymptotic approximation to a Bode plot. The solid curve is the Bode plot for the transfer function $G(s) = k(s+b)/(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)$, where $a \ll b \ll \omega_0$. Each segment in the gain and phase curves represents a separate portion of the approximation, where either a pole or a zero begins to have effect. Each segment of the approximation is a straight line between these points at a slope given by the rules for computing the effects of poles and zeros.

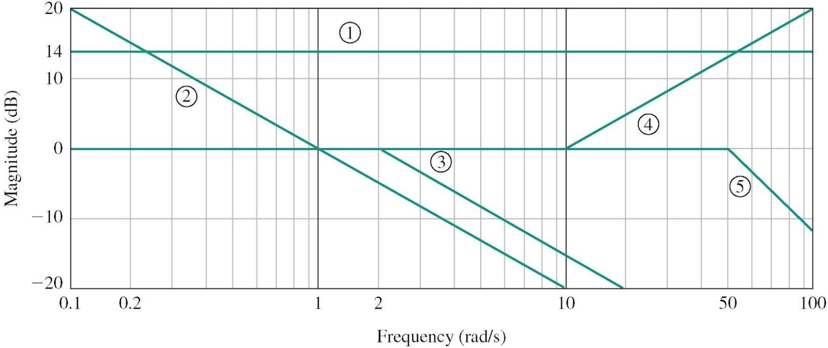
Bode Plot Example 3

- Draw a Bode plot for $G(s) = \dfrac{4(1 + 0.1s)}{s(1 + 0.5s)(1 + 0.6(s/50) + (s/50)^2)}$
- Factors in order of their occurrence as $s = j\omega$ increases:
  1. A constant gain $\kappa = 4$
  2. A pole at the origin
  3. A pole at $\omega = 2$
  4. A zero at $\omega = 10$
  5. A pair of complex poles at $\omega = \omega_n = 50$

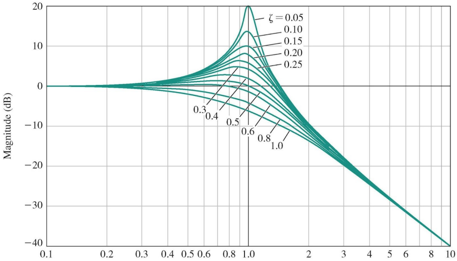
Bode Plot Example 3

- Consider the approximate magnitude plots:
  1. **Constant gain**: $20 \log |\kappa| = 14$ dB
  2. **Pole at the origin**: a line with slope -20 dB/decade through 0 when $\omega = 1$
  3. **Pole at** $\omega = 2$: horizontal line at 0 dB until the corner frequency at $\omega = 2$ and a line with slope -20 dB/decade after
  4. **Zero at** $\omega = 10$: horizontal line at 0 dB until the corner frequency at $\omega = 10$ and a line with slope 20 dB/decade after
  5. **Complex pole pair at** $\omega = \omega_n = 50$: horizontal line at 0 dB until the corner frequency at $\omega = 50$ and a line with slope -40 dB/decade after
- The approximations must be corrected at the corner frequencies:
  - Real zero/pole: $\pm 3dB$
  - Complex pair of zeros/poles: based on ζ

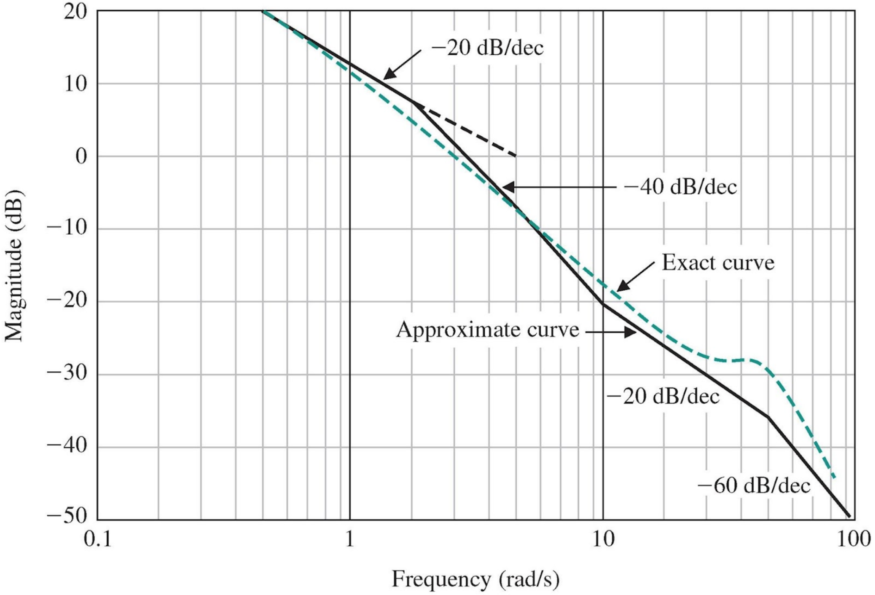
Bode Plot Example 3



- Complex pole pair correction:



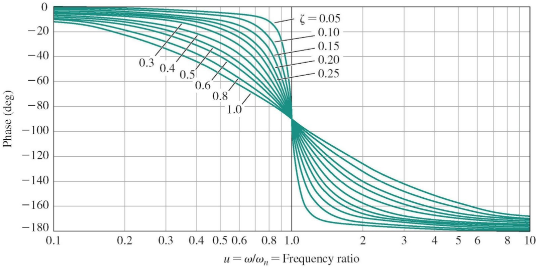
Bode Plot Example 3



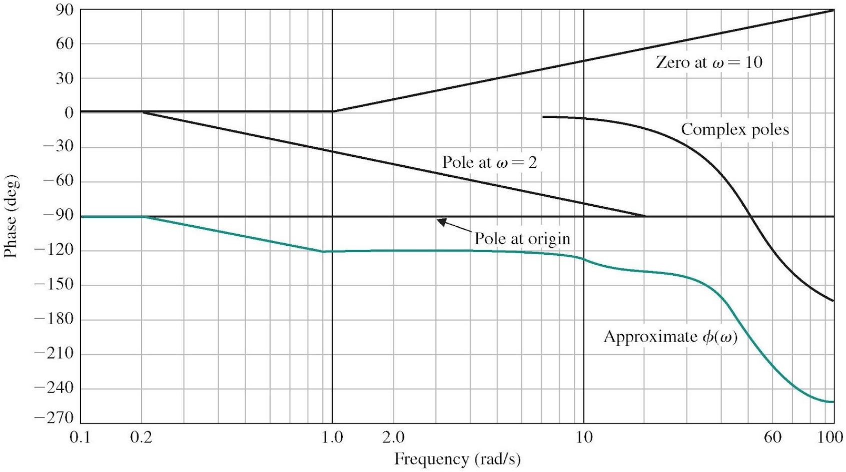
Bode Plot Example 3

- Consider the approximate phase plots:
  1. **Constant gain**: $\angle K = 0^\circ$
  2. **Pole at the origin**: -90°
  3. **Pole at** $\omega = 2$: a line with slope -45 deg/decade from $\omega = 0.2$ to $\omega = 20$
  4. **Zero at** $\omega = 10$: a line with slope 45 deg/decade from $\omega = 1$ to $\omega = 100$
  5. **Complex pole pair at** $\omega = \omega_n = 50$: phase shift of -90 deg/decade from $\omega = 5$ to $\omega = 500$

- The phase characteristic for the complex pole pair should be obtained from:



Bode Plot Example 3



- The exact phase shift can be evaluated at important frequencies:

$$\underline{/G(j\omega)} = \underline{/K} + \sum_{i=1}^{m_1} \tan^{-1}\left(\frac{\omega}{z_i}\right) + \sum_{l=1}^{m_2} \tan^{-1}\left(\frac{2\zeta_l \omega_{n_l}\omega}{\omega_{n_l}^2 - \omega^2}\right) - n_0\frac{\pi}{2} - \sum_{i=1}^{n_1} \tan^{-1}\left(\frac{\omega}{p_i}\right) - \sum_{k=1}^{n_2} \tan^{-1}\left(\frac{2\zeta_k \omega_{n_k}\omega}{\omega_{n_k}^2 - \omega^2}\right)$$

Bode Plot Example 4

- Draw a Bode plot for

$$G(s) = \frac{(s+1)(s^2+3s+100)}{s^2(s+10)(s+100)} = \frac{(s+1)((s/10)^2+2(0.15)(s/10)+1)}{10s^2(s/10+1)(s/100+1)}$$

- Magnitude and phase at $\omega = 0.1$:

$$20\log|G(j\omega)| \approx 20dB \qquad \angle G(j\omega) \approx -180^\circ$$

- Magnitude slope in dB/decade:

ω	Zero at -1	Zeros with $\omega_n = 10$	Double pole at 0	Pole at -10	Pole at -100
0.1 - 1	0	0	-40	0	0
1 - 10	20	0	-40	0	0
10 - 100	20	40	-40	-20	0
100 - 1000	20	40	-40	-20	-20

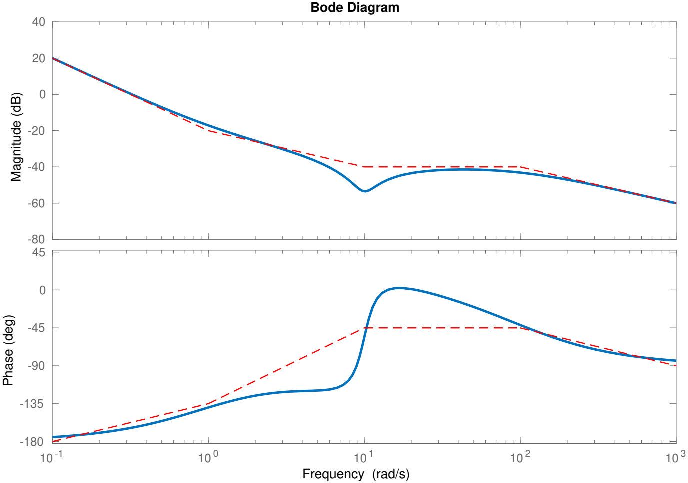
- Phase slope in degrees/decade:

ω	Zero at -1	Zeros with $\omega_n = 10$	Double pole at 0	Pole at -10	Pole at -100
0.1 - 1	45	0	0	0	0
1 - 10	45	90	0	-45	0
10 - 100	0	90	0	-45	-45
100 - 1000	0	0	0	0	-45

Bode Plot Example 4

- Draw a Bode plot for

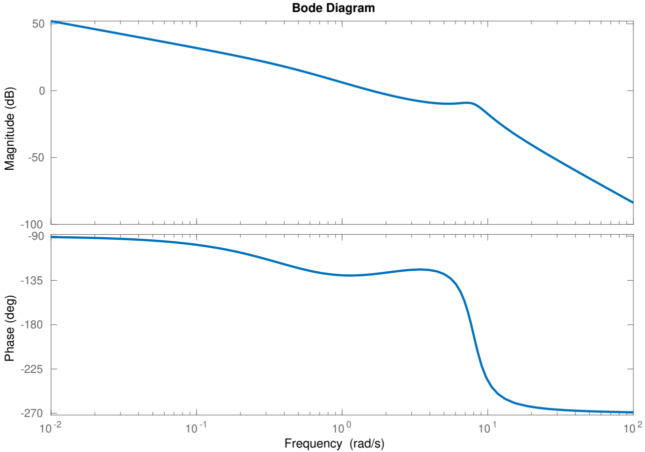
$$G(s) = \frac{(s+1)(s^2+3s+100)}{s^2(s+10)(s+100)} = \frac{(s+1)((s/10)^2+2(0.15)(s/10)+1)}{10s^2(s/10+1)(s/100+1)}$$



Bode Plot in Matlab

- Bode plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$

```
s = tf('s');
G = 4*(s/2+1)/s/(1+2*s)/(1+0.4*(s/8) + (s/8)^2);
bodeplot(G);
```



Outline

Non-minimum Phase Systems

- **Minimum phase system**: a system whose transfer function poles and zeros are in the closed left half-plane
- **Non-minimum phase system**: a system whose transfer function has zeros or poles in the right half-plane
- Bode plots can also be drawn for non-minimum phase systems
- The magnitude of a transfer function does not depend on whether the zeros and poles are in the left or right half-plane
- The phase contribution of a zero or pole in the right half-plane is always at least as large as the phase contribution of a zero or pole in the left half-plane

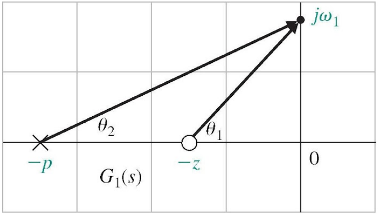
Non-minimum Phase Systems

- To understand the difference between minimum and non-minimum phase systems compare the transfer functions:

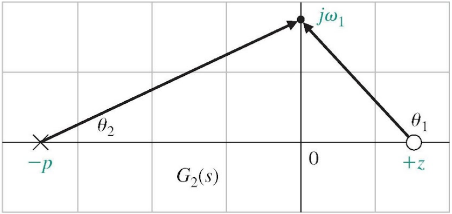
$$G_1(s) = \frac{s+z}{s+p} \qquad\qquad G_2(s) = \frac{s-z}{s+p}$$

- Magnitude: $|G_1(j\omega)| = |G_2(j\omega)| = \frac{\sqrt{\omega^2+z^2}}{\sqrt{\omega^2+p^2}}$

- Phase: $\angle G_1(j\omega_1)$ vs $\angle G_2(j\omega_1)$



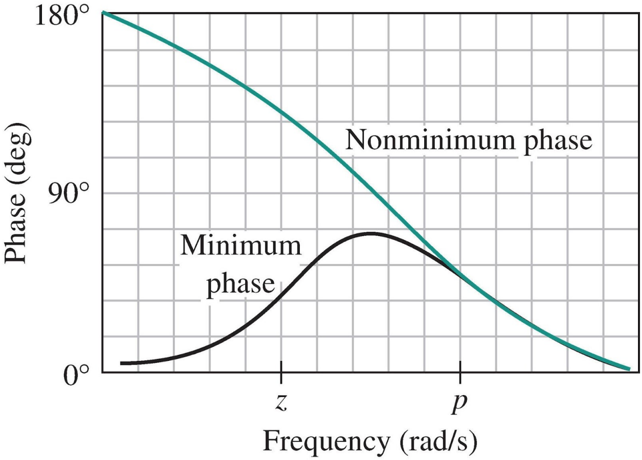
(a)



(b)

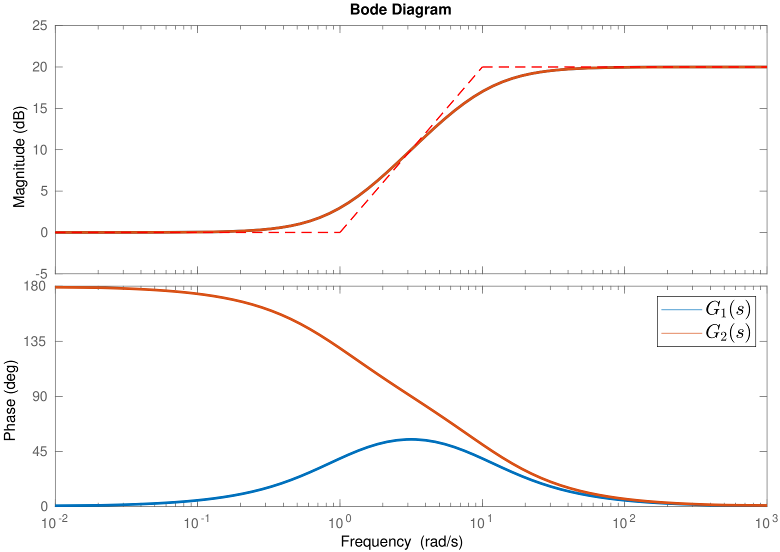
Non-minimum Phase Systems

- A minimum phase system has the smallest phase lag of all systems with the same magnitude curve



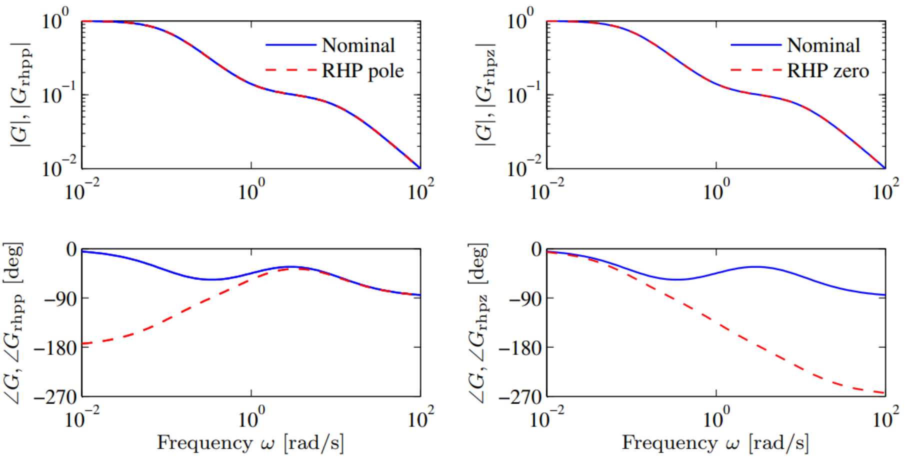
Non-minimum Phase Systems: Example 1

- Draw a Bode plot for $G_1(s) = 10\frac{s+1}{s+10}$ and $G_2(s) = 10\frac{s-1}{s+10}$



Non-minimum Phase Systems: Example 2

$$G(s) = \frac{s+1}{(s+0.1)(s+10)} \qquad G_{\mathrm{rhpp}}(s) = \frac{s+1}{(s-0.1)(s+10)} \qquad G_{\mathrm{rhpz}}(s) = \frac{-s+1}{(s+0.1)(s+10)}$$



(a) Right half-plane pole (b) Right half-plane zero

Non-minimum Phase Systems: Example 3

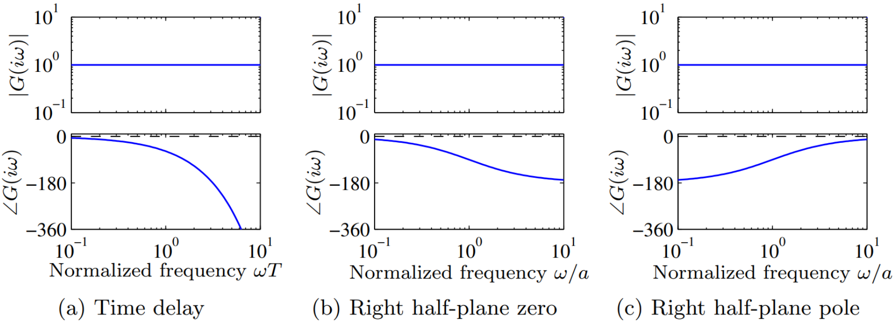


Figure: Bode plots of non-minimum phase systems: (a) Time delay $G(s) = e^{-sT}$, (b) system with right half-plane zero $G(s) = (a - s)/(a + s)$, (c) system with right half-plane pole $G(s) = (s + a)/(s - a)$. The corresponding minimum phase system has transfer function $G(s) = 1$ in all cases.

Non-minimum Phase System Control

- The presence of poles and zeros in the right half-plane imposes limitations on the achievable control performance
- The extra phase causes difficulty fot control because there is a delay between applying an input and seeing its effect
- **Zeros** depend on the relationship of inputs and outputs of a system. They can be changed by moving or adding sensors and actuators
- **Poles** are intrinsic to a system and do not depend on sensors or actuators

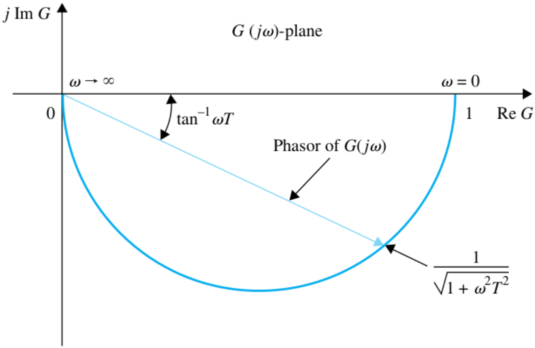
Outline

Polar Plot

- **Polar plot**: a plot of $\text{Im}(G(j\omega))$ versus $\text{Re}(G(j\omega))$ of a transfer function $G(j\omega)$ as ω varies from 0 to ∞
- A polar plot contains less information than a Bode plot because the frequency values ω are not captured
- The general shape of the polar plot can be determined from:
  - Magnitude $|G(j\omega)|$ and phase $\underline{/G(j\omega)}$ at $\omega = 0$ and $\omega = \infty$
  - Intersection of the polar plot with the real and imaginary axes

Polar Plot: Type 0 System

- Draw a polar plot for $G(s) = \frac{1}{1+Ts}$
- Magnitude: $|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2 T^2}}$
- Phase: $\angle G(j\omega) = -\tan^{-1}(\omega T)$
- Polar plot: $|G(j0)| = 1$, $\angle G(j0) = 0$; $|G(j\infty)| = 0$, $\angle G(j\infty) = -90°$

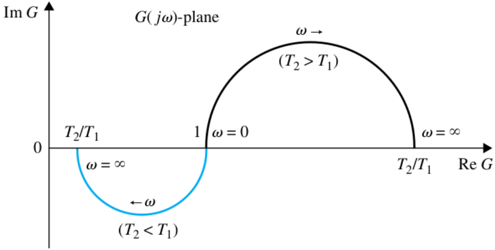


Polar Plot: Type 0 System

- Draw a polar plot for $G(s) = \frac{1+T_2 s}{1+T_1 s}$
- Magnitude: $|G(j\omega)| = \frac{\sqrt{1+\omega^2 T_2^2}}{\sqrt{1+\omega^2 T_1^2}}$
- Phase: $\angle G(j\omega) = \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_1)$
- The polar plot depends on the relative magnitudes of T_1 and T_2
- If $T_2 > T_1$:

  $|G(j\omega)| \geq 1 \qquad \angle G(j\omega) \geq 0$
- If $T_1 > T_2$:

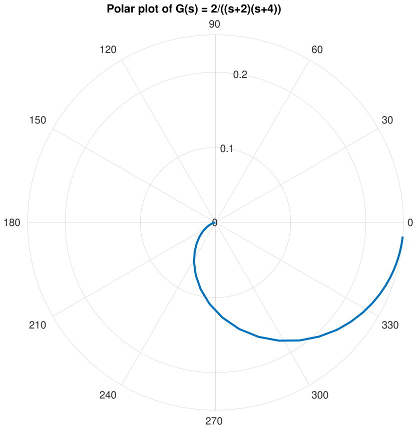
  $|G(j\omega)| \leq 1 \qquad \angle G(j\omega) \leq 0$



Polar Plot: Type 0 System

- Draw a polar plot for $G(s) = \frac{\kappa}{(1+T_1 s)(1+T_2 s)}$
- Magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$ at $\omega = 0$ and $\omega = \infty$:

$$G(j0) = \kappa\angle 0^\circ \qquad G(j\infty) = 0\angle -180^\circ$$



Polar Plot: Type 1 System

- Draw a polar plot for $G(s) = \frac{\kappa}{s(1+\tau s)}$
- Magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$:

$$|G(j\omega)| = \frac{\kappa}{\sqrt{\omega^2 + \omega^4 \tau^2}}$$

$$\angle G(j\omega) = -\frac{\pi}{2} - \tan^{-1}(\omega\tau)$$

- Values at $\omega = 0$, $\omega = 1/\tau$, $\omega = \infty$:

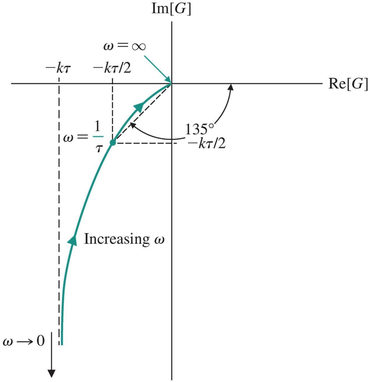
$$G(j0) = \infty \angle -90^\circ$$

$$G(j\frac{1}{\tau}) = \frac{\kappa\tau}{\sqrt{2}} \angle -135^\circ$$

$$G(j\infty) = 0 \angle -180^\circ$$

- Asymptote as $\omega \to 0$:

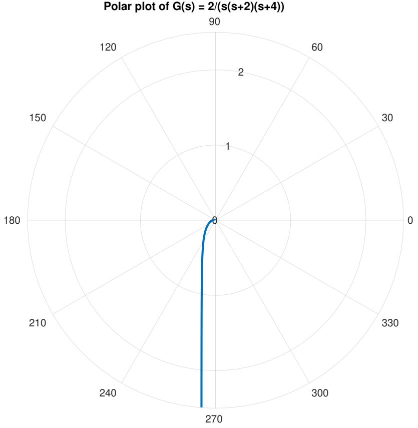
$$G(j\omega) = \frac{\kappa}{j\omega(1+\tau j\omega)} \overset{\text{small } \omega}{\approx} \frac{\kappa}{j\omega}(1 - j\tau\omega) = -\kappa\tau - j\frac{\kappa}{\omega}$$



Polar Plot: Type 1 System

- Draw a polar plot for $G(s) = \frac{K}{s(1+T_1 s)(1+T_2 s)}$
- Magnitude $|G(j\omega)|$ and phase $\underline{/G(j\omega)}$ at $\omega = 0$ and $\omega = \infty$:

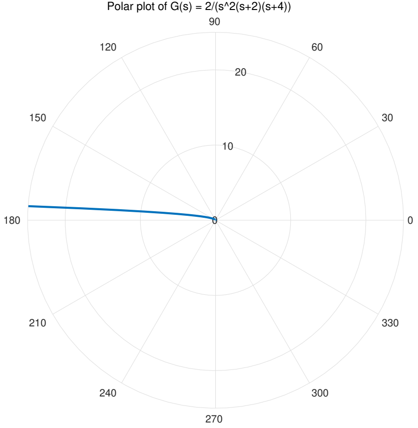
$$G(j0) = \infty\underline{/-90^\circ} \qquad G(j\infty) = 0\underline{/-270^\circ}$$



Polar Plot: Type 2 System

- Draw a polar plot for $G(s) = \frac{K}{s^2(1+T_1 s)(1+T_2 s)}$
- Magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$ at $\omega = 0$ and $\omega = \infty$:

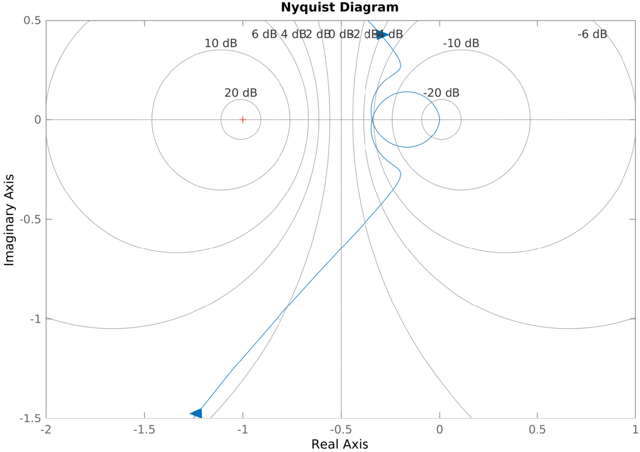
$$G(j0) = \infty\angle -180^\circ \qquad G(j\infty) = 0\angle -360^\circ$$



Polar Plot in Matlab

- Nyquist plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$

```
s = tf('s');
G = 4*(s/2+1)/s/(1+2*s)/(1+0.4*(s/8) + (s/8)^2);
nyquistplot(G);
```

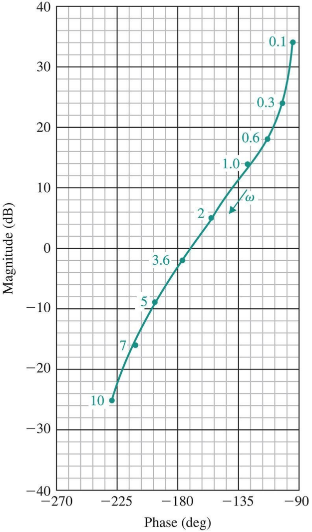


Outline

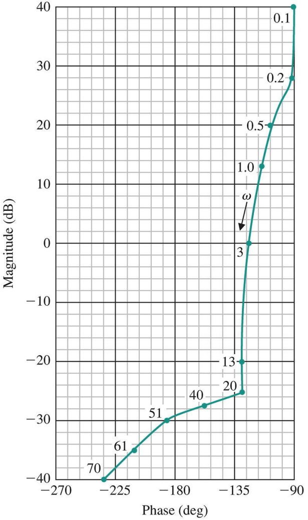
Magnitude-Phase Plot

- **Magnitude-phase plot**: a plot of the magnitude $20\log_{10}|G(j\omega)|$ in dB versus the phase $\angle G(j\omega)$ in degrees as ω varies from 0 to ∞
- A magnitude-phase plot can be obtained from the information on a Bode plot
- A magnitude-phase plot is shifted up or down when the gain factor κ varies
- The Bode plot property of adding plots of individual components does not carry over

Magnitude-Phase Plot



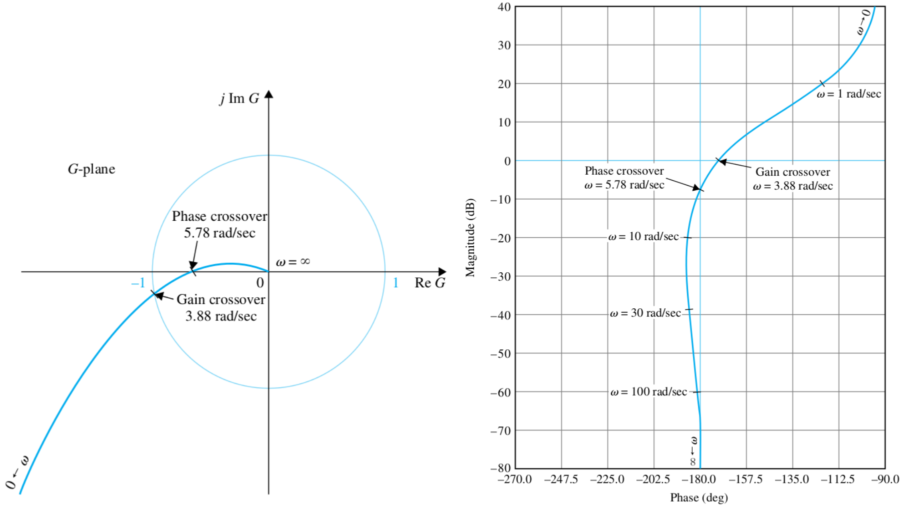
(a) $G_1(s) = \frac{5}{s(s/2+1)(s/6+2)}$



(b) $G_2(s) = \frac{5(s/10+1)}{s(s/2+1)(1+0.6(s/50)+(s/50)^2)}$

Magnitude-Phase Plot

- Draw a polar plot and a magnitude-phase plot for $G(s) = \frac{10(s+10)}{s(s+2)(s+5)}$



Magnitude-Phase Plot in Matlab

- Nichols plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$

```
s = tf('s');
G = 4*(s/2+1)/s/(1+2*s)/(1+0.4*(s/8) + (s/8)^2);
nicholsplot(G);
```