

ECE171A: Linear Control System Theory

Lecture 10: Nyquist Stability

Nikolay Atanasov

natanasov@ucsd.edu

UC San Diego

JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

Outline

Nyquist's Idea

Principle of the Argument

Nyquist's Stability Criterion

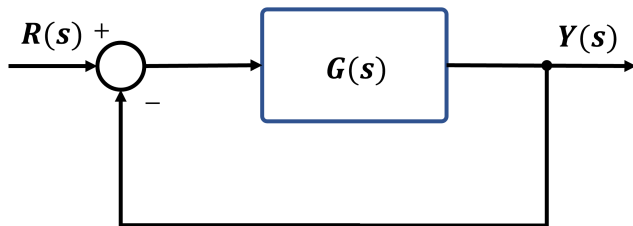
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Stability of Feedback Systems



- ▶ Consider a feedback control system with open-loop transfer function $G(s)$ (controller and plant) and closed-loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

- ▶ Testing BIBO stability using the poles of $T(s)$ requires knowledge of $G(s)$ and gives little guidance for control design, i.e., how should the controller be modified to make an unstable system stable?
- ▶ Given a Bode plot of $G(s)$, we aim to understand the stability of $T(s)$

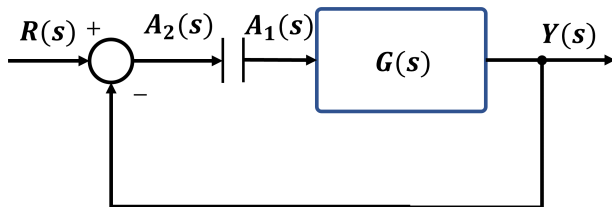
Nyquist's Idea

- ▶ **Harry Nyquist** made important contributions to control theory (stability of feedback systems), electronics (thermal noise), and communication theory (telegraph)
- ▶ Nyquist proposed an idea to determine the stability of a closed-loop system by investigating how sinusoidal signal propagate around the feedback loop
- ▶ Similar to return difference, break the feedback loop and ask whether a signal injected at $G(s)$ has larger or smaller magnitude when it returns to $G(s)$
- ▶ Nyquist's idea allows reasoning about **closed-loop stability** based on the **frequency response of the open-loop transfer function**



H. Nyquist

Nyquist's Idea

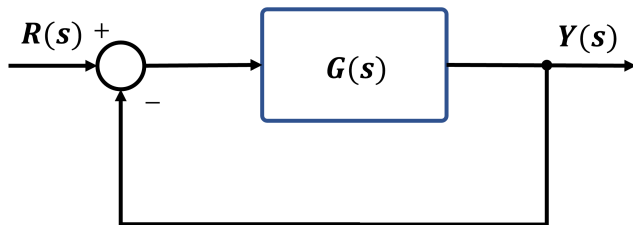


- ▶ Suppose that a sinusoid of frequency ω is injected at $A_1(s)$. In steady state, the signal at $A_2(s)$ will be a sinusoid with the same frequency ω , magnitude $|G(j\omega)|$, and phase $180^\circ + \angle G(j\omega)$
- ▶ **Critical point:** the signals at $A_1(s)$ and $A_2(s)$ are identical if:

$$|G(j\omega)| = 1 \quad \text{and} \quad \angle G(j\omega) = -180^\circ \quad \Leftrightarrow \quad G(j\omega) = -1$$

- ▶ **Nyquist's idea:** Let ω_p be such that $\angle G(j\omega_p) = -180^\circ$. A feedback control system is stable if $|G(j\omega_p)| < 1$ since the signal at $A_2(s)$ will have smaller amplitude than the injected signal at $A_1(s)$.

Open-Loop Poles vs Closed-Loop Poles



- ▶ Open-loop transfer function: $G(s) = \frac{b(s)}{a(s)}$
- ▶ Closed-loop transfer function: $T(s) = \frac{G(s)}{1 + G(s)} = \frac{b(s)}{a(s) + b(s)}$
- ▶ Let $\Delta(s) = 1 + G(s)$
 - ▶ The closed-loop poles are the zeros of $\Delta(s)$
 - ▶ The open-loop poles are the poles of $\Delta(s)$:

$$\Delta(s) = 1 + G(s) = 1 + \frac{b(s)}{a(s)} = \frac{a(s) + b(s)}{a(s)}$$

Outline

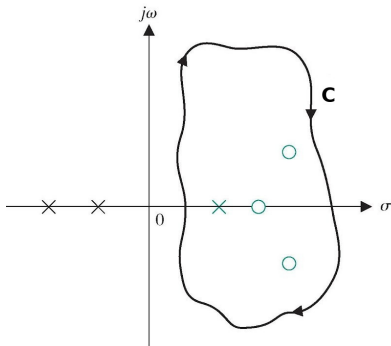
Nyquist's Idea

Principle of the Argument

Nyquist's Stability Criterion

Contours in the Complex Plane

- ▶ Nyquist's stability criterion utilizes a contour C in the complex plane to relate the locations of the **open-loop poles** and the **closed-loop poles**
- ▶ A **contour** C is a piecewise smooth path in the complex plane
- ▶ A contour C is **closed** if it starts and ends at the same point
- ▶ A contour C is **simple** if it does not cross itself at any point
- ▶ A parameterization $z(\theta) \in \mathbb{C}$ of a contour has direction indicated by increasing the parameter $\theta \in \mathbb{R}$
- ▶ **Cauchy's Principle of the Argument**: relates the arguments (phases) of the zeros and poles of $G(s)$ inside a contour C to the shape of $G(C)$
- ▶ $G(C)$ is a new closed contour obtained by evaluating $G(s)$ at all s on C



Open-loop Transfer Function

- ▶ Consider a control system with open-loop transfer function:

$$G(s) = \kappa \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- ▶ At each s , $G(s)$ is a complex number with magnitude and phase:

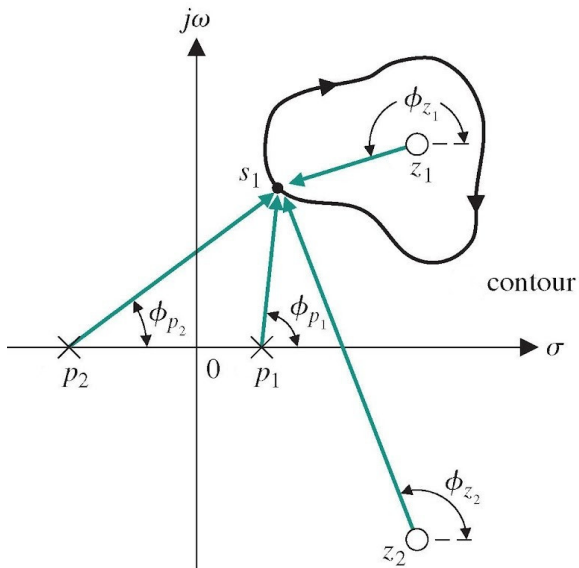
$$|G(s)| = |\kappa| \frac{\prod_{i=1}^m |s - z_i|}{\prod_{i=1}^n |s - p_i|} \quad \angle G(s) = \angle \kappa + \sum_{i=1}^m \angle (s - z_i) - \sum_{i=1}^n \angle (s - p_i)$$

- ▶ Graphical evaluation of the magnitude and phase:
 - ▶ $|s - z_i|$ is the length of the vector from z_i to s
 - ▶ $|s - p_i|$ is the length of the vector from p_i to s
 - ▶ $\angle (s - z_i)$ is the angle from the real axis to the vector from z_i to s
 - ▶ $\angle (s - p_i)$ is the angle from the real axis to the vector from p_i to s

Evaluating $G(s)$ along a Contour

- ▶ Let C be a simple closed clockwise contour C in the complex plane
- ▶ Evaluating $G(s)$ at all points on C produces a new closed contour $G(C)$
- ▶ **Assumption:** C does not pass through the origin or any of the poles or zeros of $G(s)$ (otherwise $\angle G(s)$ is undefined)
- ▶ A zero z_i outside the contour C :
 - ▶ As s moves around the contour C , the vector $s - z_i$ swings up and down but not all the way around
 - ▶ The net change in $\angle(s - z_i)$ is 0
- ▶ A zero z_i inside the contour C :
 - ▶ As s moves around the contour C , the vector $s - z_i$ turns all the way around
 - ▶ The net change in $\angle(s - z_i)$ is -360°
- ▶ A pole p_i outside the contour C : the net change in $\angle(s - p_i)$ is 0
- ▶ A pole p_i inside the contour C : the net change in $\angle(s - p_i)$ is -360°

Evaluating $G(s)$ along a Contour



Principle of the Argument

- ▶ Let Z and P be the number of zeros and poles of $G(s)$ inside C
- ▶ As s moves around C , $\angle G(s)$ undergoes a net change of $-(Z - P)360^\circ$
- ▶ A net change of -360° means that the vector from 0 to $G(s)$ swings clockwise around the origin one full rotation
- ▶ A net change of $-(Z - P)360^\circ$ means that the vector from 0 to $G(s)$ must encircle the origin in clockwise direction $(Z - P)$ times

Cauchy's Principle of the Argument

Consider a transfer function $G(s)$ and a simple closed clockwise contour C . Let Z and P be the number of zeros and poles of $G(s)$ inside C . Then, the contour generated by evaluating $G(s)$ along C will encircle the origin in a clockwise direction $Z - P$ times.

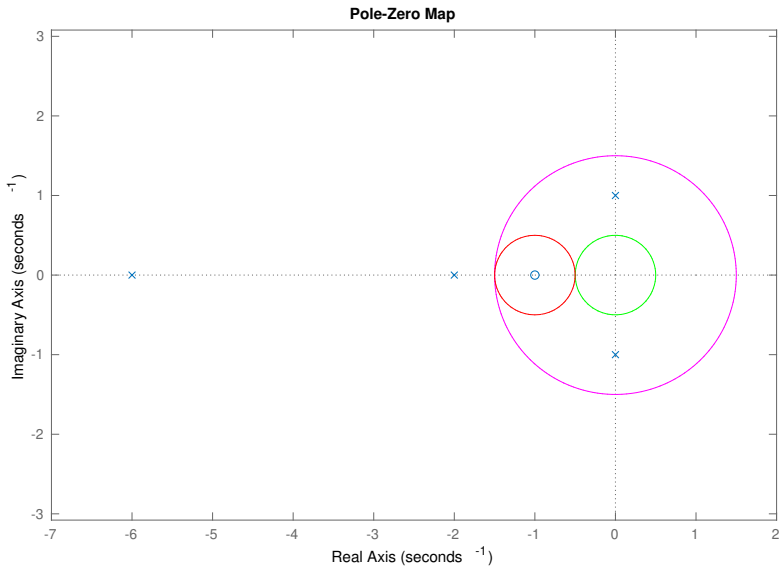
Winding Number

- ▶ To determine the number of encirclements of a point s by a contour Γ :
 1. Fix a pin at s pointing out of the page
 2. Attach a string from the pin to the contour Γ
 3. Let the end of the string attached to Γ traverse the contour
- ▶ The **winding number** $n(\Gamma, s)$ of Γ about s is equal to the number of times the string winds up on the pin when Γ is traversed:

$$n(\Gamma, s) = \frac{1}{2\pi j} \oint_{\Gamma} \frac{1}{s - z} dz$$

Principle of the Argument: Example

- ▶ Pole-zero map for $G(s) = \frac{10(s+1)}{(s+2)(s^2+1)(s+6)}$



Principle of the Argument: Example

- ▶ A circle contour C centered at the origin with radius 0.5 (green)
- ▶ The contour may be parameterized by $z(\theta) = 0.5e^{-j\theta}$ for $\theta \in [0, 2\pi]$
- ▶ The contour C is mapped by $G(s)$ to a new contour (from blue to red), e.g., parameterized by $G(z(\theta))$ for $\theta \in [0, 2\pi]$

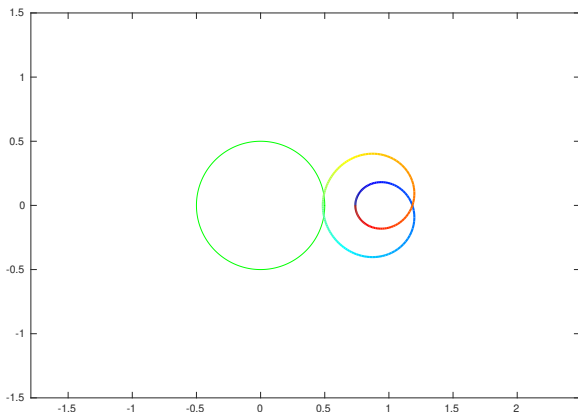


Figure: The origin is encircled 0 times clockwise

Principle of the Argument: Example

- ▶ A circle contour C centered at $(-1, 0)$ with radius 1 (red)
- ▶ The contour C is mapped by $G(s)$ to a new contour (from blue to red)

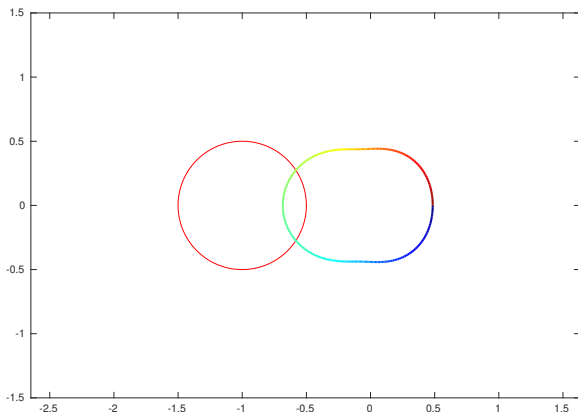


Figure: The origin is encircled 1 time clockwise

Principle of the Argument: Example

- ▶ A circle contour C centered at the origin with radius 1.5 (magenta)
- ▶ The contour C is mapped by $G(s)$ to a new contour (from blue to red)

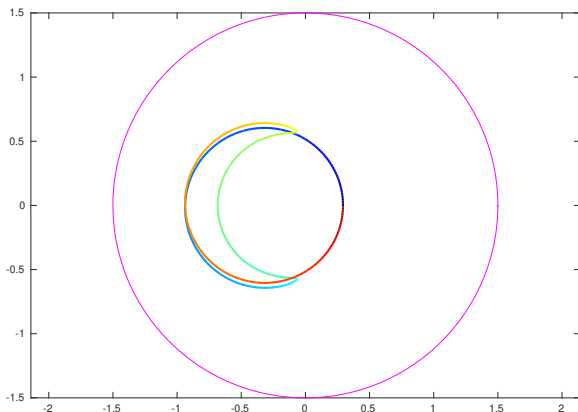


Figure: The origin is encircled 1 time counterclockwise

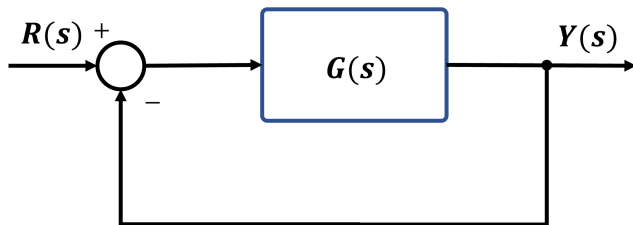
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Open-Loop Poles vs Closed-Loop Poles

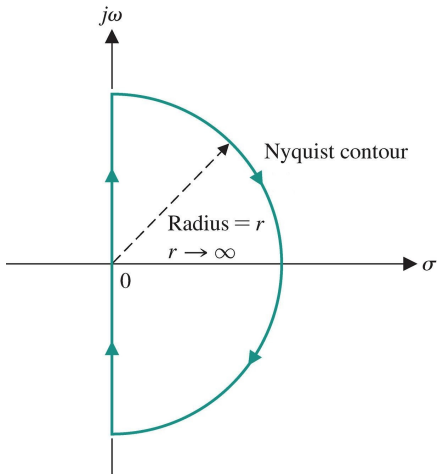


- ▶ Open-loop transfer function: $G(s) = \frac{b(s)}{a(s)}$
- ▶ Closed-loop transfer function: $T(s) = \frac{G(s)}{1 + G(s)} = \frac{b(s)}{a(s) + b(s)}$
- ▶ Let $\Delta(s) = 1 + G(s)$
 - ▶ The closed-loop poles are the zeros of $\Delta(s)$
 - ▶ The open-loop poles are the poles of $\Delta(s)$:

$$\Delta(s) = 1 + G(s) = 1 + \frac{b(s)}{a(s)} = \frac{a(s) + b(s)}{a(s)}$$

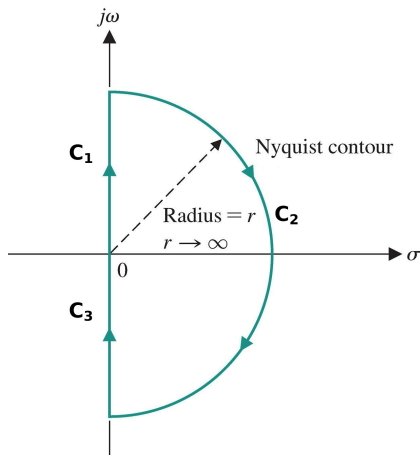
Nyquist Contour

- ▶ To determine how many closed-loop poles lie in the closed right half-plane, we apply the Principle of the Argument to $\Delta(s)$
- ▶ Define a clockwise contour C that covers the closed right half-plane



Nyquist Contour

- ▶ The Nyquist contour is made up of three parts:
 - ▶ **Contour C_1** : points $s = j\omega$ on the positive imaginary axis, as ω ranges from 0 to ∞
 - ▶ **Contour C_2** : points $s = re^{j\theta}$ on a semi-circle as $r \rightarrow \infty$ and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
 - ▶ **Contour C_3** : points $s = j\omega$ on the negative imaginary axis, as ω ranges from $-\infty$ to 0



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Nyquist Plot

- ▶ A **Nyquist plot** evaluates $\Delta(s) = 1 + G(s)$ over the Nyquist contour C
- ▶ Contour $\Delta(C)$ is obtained by shifting contour $G(C)$ by one unit to the right

Nyquist contour $C \Rightarrow$ Nyquist plot $G(C)$

- ▶ The contour $G(C)$ is obtained by combining $G(C_1)$, $G(C_2)$, and $G(C_3)$:
 - ▶ **Contour C_1 :**
 - ▶ plot $G(j\omega)$ for $\omega \in (0, \infty)$ in the complex plane
 - ▶ equivalent to a **polar plot** for $G(s)$
 - ▶ **Contour C_2 :**
 - ▶ plot $G(re^{j\theta})$ for $r \rightarrow \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
 - ▶ as $r \rightarrow \infty$, $s = re^{j\theta}$ dominates every factor it appears in
 - ▶ if $G(s)$ is strictly proper, then $G(re^{j\theta}) \rightarrow 0$
 - ▶ if $G(s)$ is not strictly proper, then $G(re^{j\theta}) \rightarrow \text{const}$
 - ▶ **Contour C_3 :**
 - ▶ plot $G(j\omega)$ for $\omega \in (-\infty, 0)$ in the complex plane
 - ▶ $G(-j\omega)$ is the complex conjugate of $G(j\omega)$
 - ▶ $G(-j\omega)$ and $G(j\omega)$ have the same magnitude but opposite phases
 - ▶ $G(C_3)$ is a reflected version of $G(C_1)$ about the real axis

Nyquist Plot: Example 1

- ▶ Draw a Nyquist plot of $G(s) = \frac{s+1}{s+10}$
- ▶ **Contour** C_1 : $s = j\omega$ with $\omega \in (0, \infty)$:
 - ▶ $\omega = 0$ and $\omega \rightarrow \infty$:

$$G(j0) = \frac{1}{10} \angle 0^\circ \qquad G(j\infty) = 1 \angle 0^\circ$$

- ▶ for $0 < \omega < \infty$:

$$|G(j\omega)| = \frac{1}{10} \frac{\sqrt{1+\omega^2}}{\sqrt{1+(\omega/10)^2}} \qquad \angle G(j\omega) = \tan^{-1}(\omega) - \tan^{-1}(\omega/10)$$

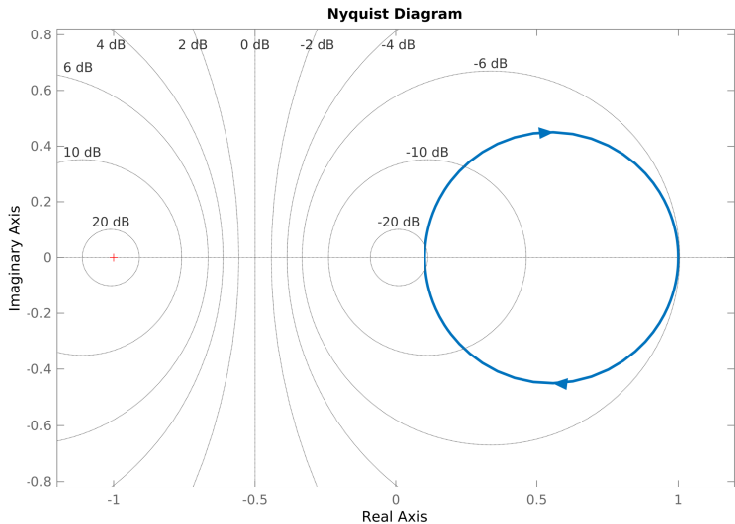
- ▶ **Contour** C_2 : $s = re^{j\theta}$ with $r \rightarrow \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r \rightarrow \infty} G(re^{j\theta}) = \lim_{r \rightarrow \infty} \frac{re^{j\theta} + 1}{re^{j\theta} + 10} = 1 \angle 0^\circ$$

- ▶ **Contour** C_3 : $s = j\omega$ with $\omega \in (-\infty, 0)$:
 - ▶ $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

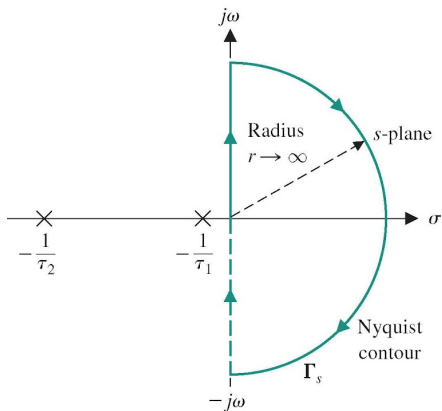
Nyquist Plot: Example 1

- ▶ Nyquist plot of $G(s) = \frac{s+1}{s+10}$
- ▶ Type 0 system as on Slide 51 of Lecture 9 with $\lim_{r \rightarrow \infty} G(re^{j\theta}) = 1$

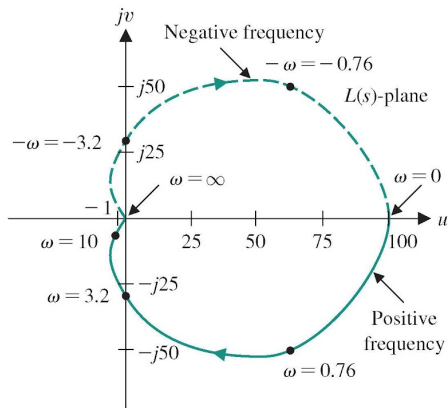


Nyquist Plot: Example 2

- ▶ Draw a Nyquist plot of $G(s) = \frac{\kappa}{(1+\tau_1 s)(1+\tau_2 s)} = \frac{100}{(1+s)(1+s/10)}$
- ▶ **Contour** C_1 : $G(j0) = \kappa \angle 0^\circ$, $G(j\infty) = 0 \angle -180^\circ$
- ▶ **Contour** C_2 : $\lim_{r \rightarrow \infty} G(re^{j\theta}) = 0$



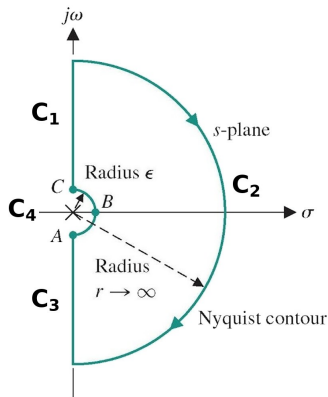
(a)



(b)

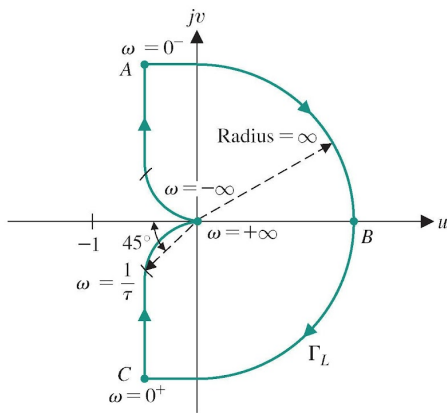
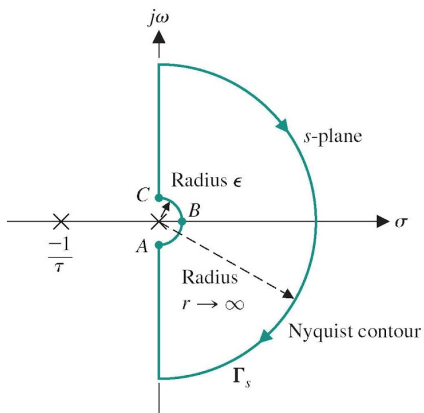
Nyquist Plot: Poles on the Imaginary Axis

- ▶ The Principle of the Argument assumes C does not pass through zeros or poles of $\Delta(s)$
- ▶ There might be poles of $G(s)$ on the imaginary axis, which are poles of $\Delta(s)$
- ▶ The Nyquist contour needs to be modified to take a small detour around poles of $G(s)$ on the imaginary axis
- ▶ **Contour C_4 :** avoid poles of $G(s)$ at origin:
 - ▶ plot $G(\epsilon e^{j\theta})$ for $\epsilon \rightarrow 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- ▶ If $G(s)$ has other poles p on the imaginary axis, more contours need to be introduced. Substitute $s = p + \epsilon e^{j\theta}$ into $G(s)$ and examine what happens as $\epsilon \rightarrow 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.



Nyquist Plot: Example 3

- ▶ Draw a Nyquist plot of a type 1 system: $G(s) = \frac{\kappa}{s(1+\tau s)}$
- ▶ Since there is a pole at the origin, we need to use a modified Nyquist contour



Nyquist Plot: Example 3

- ▶ **Contour** C_1 : $s = j\omega$ with $\omega \in (0, \infty)$: polar plot as on Slide 58 of Lecture 9:

$$G(j0^+) = \infty \underline{-90^\circ}$$

$$G(j\infty) = \lim_{\omega \rightarrow \infty} \frac{\kappa}{j\omega(1 + j\omega T)} = \lim_{\omega \rightarrow \infty} \left| \frac{\kappa}{\tau\omega^2} \right| \underline{-90^\circ - \tan^{-1}(\omega T)} = 0 \underline{-180^\circ}$$

- ▶ **Contour** C_2 : $s = re^{j\theta}$ with $r \rightarrow \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r \rightarrow \infty} G(re^{j\theta}) = \lim_{r \rightarrow \infty} \left| \frac{\kappa}{\tau r^2} \right| e^{-2j\theta} = 0 \underline{-2\theta}$$

- ▶ The phase of $G(s)$ changes from -180° at $\omega = \infty$ to 180° at $\omega = -\infty$

- ▶ **Contour** C_3 : $s = j\omega$ with $\omega \in (-\infty, 0)$:

- ▶ $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

- ▶ **Contour** C_4 : $s = \epsilon e^{j\theta}$ with $\epsilon \rightarrow 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon \rightarrow 0} G(\epsilon e^{j\theta}) = \lim_{\epsilon \rightarrow 0} \frac{\kappa}{\epsilon e^{j\theta}(1 + \tau \epsilon e^{j\theta})} \stackrel{\frac{1}{1+\epsilon} \approx 1-\epsilon}{=} -\kappa T + \lim_{\epsilon \rightarrow 0} \frac{\kappa}{\epsilon} e^{-j\theta} = \infty \underline{-\theta}$$

- ▶ $G(\epsilon e^{j\theta})$ approaches an asymptote at $-\kappa T$ as $\epsilon \rightarrow 0$

- ▶ The phase of $G(s)$ changes from 90° at $\omega = 0^-$ to -90° at $\omega = 0^+$

Nyquist Plot: Example 4

▶ Draw a Nyquist plot of a type 1 system: $G(s) = \frac{\kappa}{s(1+\tau_1s)(1+\tau_2s)}$

▶ **Contour** C_4 : $s = \epsilon e^{j\theta}$ with $\epsilon \rightarrow 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

▶ C_4 maps into a semicircle with infinite radius as in Example 3:

$$G(j0) = \infty \angle -\theta$$

▶ **Contour** C_2 : $s = re^{j\theta}$ with $r \rightarrow \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

▶ C_2 maps into a point at 0 with phase $\angle -3\theta$

▶ **Contour** C_1 : $s = j\omega$ with $\omega \in (0, \infty)$: polar plot as on Slide 59 of Lecture 9:

$$G(j\infty) = 0 \angle -270^\circ$$

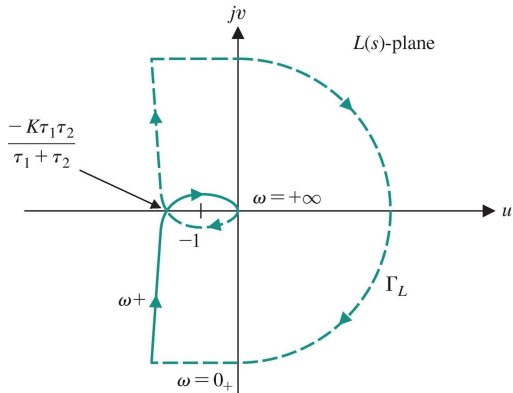
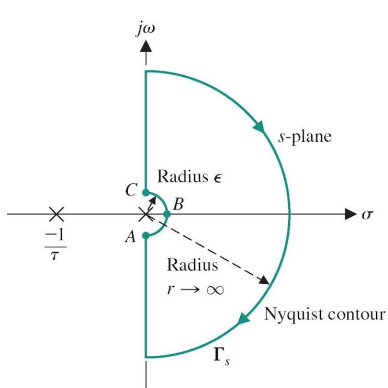
▶ **Contour** C_3 : $G(C_3)$ is a reflection of $G(C_1)$ about the real axis

Nyquist Plot: Example 4

► Contour C_1 with $\omega \in (0, \infty)$:

$$G(j\omega) = \frac{\kappa}{j\omega(1+j\omega\tau_1)(1+j\omega\tau_2)} = \frac{-\kappa(\tau_1 + \tau_2) - j\kappa(1 - \omega^2\tau_1\tau_2)\omega}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4\tau_1^2\tau_2^2}$$

$$= \frac{\kappa}{\sqrt{\omega^4(\tau_1 + \tau_2)^2 + \omega^2(1 - \omega^2\tau_1\tau_2)^2}} \angle -90^\circ - \tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)$$



Nyquist Plot: Example 5

- ▶ Draw a Nyquist plot of a type 2 system: $G(s) = \frac{\kappa}{s^2(1+\tau s)}$
- ▶ Two poles at the origin \Rightarrow need to use a modified Nyquist contour
- ▶ Magnitude and phase:

$$G(j\omega) = \frac{\kappa}{(j\omega)^2(1+j\omega\tau)} = \frac{|\kappa|}{\sqrt{\omega^4 + \omega^6\tau^2}} \underline{\underline{-180^\circ - \tan^{-1}(\omega\tau)}}$$

- ▶ **Contour** C_1 : $s = j\omega$ with $\omega \in (0, \infty)$:

$$G(j0^+) = \infty \underline{\underline{-180^\circ}}$$

$$\begin{aligned} G(j\infty) &= \lim_{\omega \rightarrow \infty} \frac{\kappa}{(j\omega)^2(1+j\omega\tau)} = \lim_{\omega \rightarrow \infty} \left| \frac{\kappa}{\tau\omega^3} \right| \underline{\underline{-180^\circ - \tan^{-1}(\omega\tau)}} \\ &= 0 \underline{\underline{-270^\circ}} \end{aligned}$$

- ▶ **Contour** C_3 : $s = j\omega$ with $\omega \in (-\infty, 0)$:

- ▶ $G(C_3)$ is a reflection (complex conjugate) of $G(C_1)$ about the real axis

Nyquist Plot: Example 5

- Magnitude and phase:

$$G(j\omega) = \frac{\kappa}{(j\omega)^2(1 + j\omega\tau)} = \frac{|\kappa|}{\sqrt{\omega^4 + \omega^6\tau^2}} \underline{-180^\circ - \tan^{-1}(\omega\tau)}$$

- **Contour** C_2 : $s = re^{j\theta}$ with $r \rightarrow \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r \rightarrow \infty} G(s) = \lim_{r \rightarrow \infty} \frac{\kappa}{\tau s^3} = \lim_{r \rightarrow \infty} \left| \frac{\kappa}{\tau r^3} \right| e^{-3j\theta} = 0 \underline{-3\theta}$$

- The phase of $G(s)$ changes from -270° at $\omega = \infty$ to 270° at $\omega = -\infty$

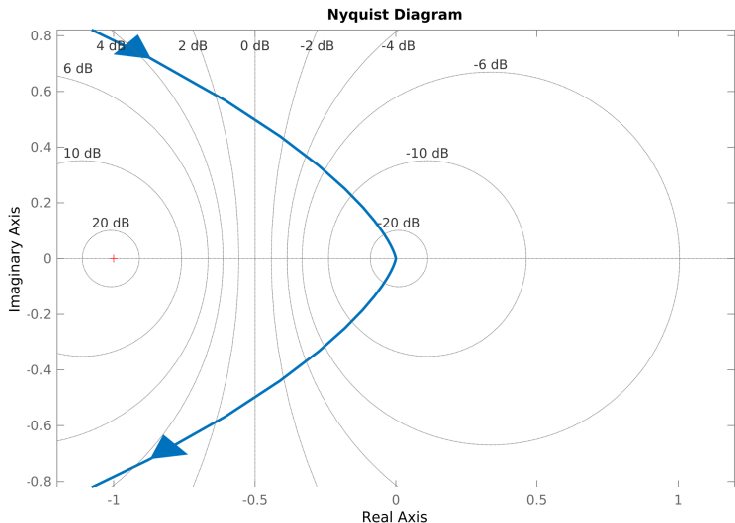
- **Contour** C_4 : $s = \epsilon e^{j\theta}$ with $\epsilon \rightarrow 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon \rightarrow 0} G(s) = \lim_{\epsilon \rightarrow 0} \frac{\kappa}{s^2} = \lim_{\epsilon \rightarrow 0} \frac{\kappa}{\epsilon^2} e^{-2j\theta} = \infty \underline{-2\theta}$$

- The phase of $G(s)$ changes from 180° at $\omega = 0^-$ to -180° at $\omega = 0^+$

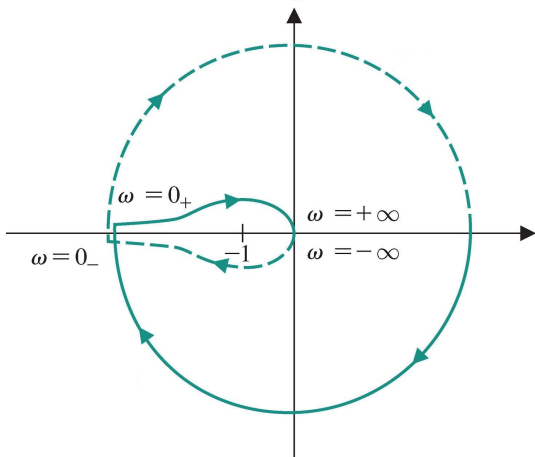
Nyquist Plot: Example 5

- ▶ Nyquist plot of a type 2 system: $G(s) = \frac{\kappa}{s^2(1+\tau s)} = \frac{1}{s^2(s+1)}$
- ▶ **Caution:** Matlab's *nyquistplot* does not generate $G(C_4)$



Nyquist Plot: Example 5

- ▶ Nyquist plot of a type 2 system: $G(s) = \frac{K}{s^2(1+\tau s)}$



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Nyquist Plot: Example 6

- ▶ Draw a Nyquist plot of $G(s) = \frac{1}{(s+a)^3}$

$$G(j\omega) = \frac{1}{(j\omega + a)^3} = \frac{(a - j\omega)^3}{(a^2 + \omega^2)^3} = \frac{a^3 - 3a\omega^2}{(a^2 + \omega^2)^3} + j\frac{\omega^3 - 3a^2\omega}{(a^2 + \omega^2)^3}$$

- ▶ **Contour** C_1 : $s = j\omega$ with $\omega \in (0, \infty)$:

$$G(j0) = \frac{1}{a^3} \angle 0^\circ, \quad G(j\infty) = 0 \angle -270^\circ$$

- ▶ **Contour** C_2 : $s = re^{j\theta}$ with $r \rightarrow \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$G(re^{j\theta}) = \frac{1}{(re^{j\theta} + a)^3} \rightarrow 0 \angle -3\theta$$

- ▶ **Contour** C_3 : a reflection (complex conjugate) of $G(C_1)$ about the real axis

Nyquist Plot: Example 6

- Draw a Nyquist plot of $G(s) = \frac{1}{(s+0.6)^3}$

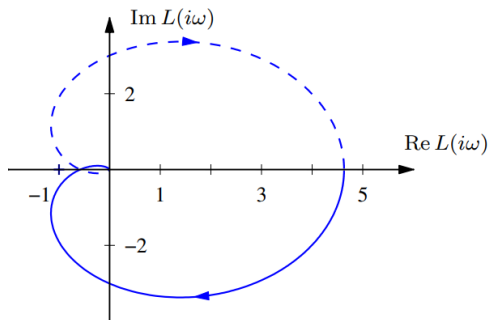
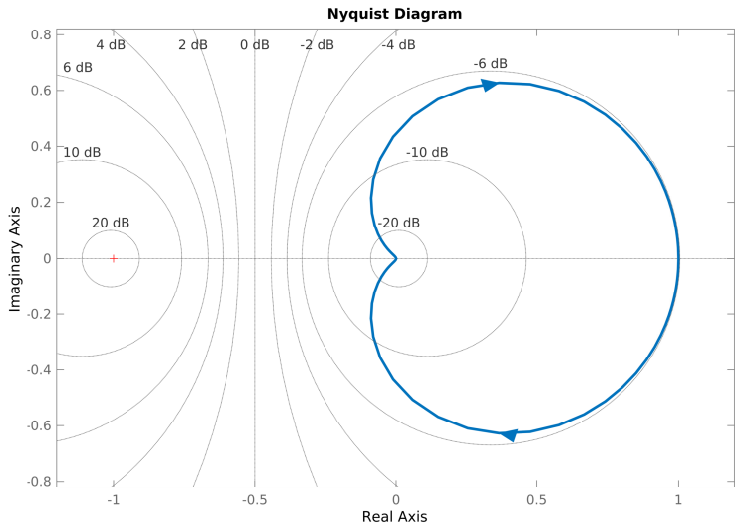


Figure 10.5: Nyquist plot for a third-order transfer function $L(s)$. The Nyquist plot consists of a trace of the loop transfer function $L(s) = 1/(s+a)^3$ with $a = 0.6$. The solid line represents the portion of the transfer function along the positive imaginary axis, and the dashed line the negative imaginary axis. The outer arc of the Nyquist contour Γ maps to the origin.

Nyquist Plot: Example 7

- ▶ Draw a Nyquist plot of $G(s) = \frac{s(s+1)}{(s+10)^2}$



Nyquist's Stability Criterion

- ▶ Consider the stability of the closed-loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{G(s)}{\Delta(s)}$$

- ▶ **Open-loop poles:** the poles of $\Delta(s)$ are the poles of $G(s)$
- ▶ **Closed-loop poles:** the zeros of $\Delta(s)$ are the poles of $T(s)$
- ▶ Principle of the Argument applied to $\Delta(s) = 1 + G(s)$:
 - ▶ Let C be a Nyquist contour
 - ▶ Let P be the number of poles of $\Delta(s)$ (open-loop poles) inside C
 - ▶ Let Z be the number of zeros of $\Delta(s)$ (closed-loop poles) inside C
 - ▶ Then, $\Delta(C)$ encircles the origin in clockwise direction $N = Z - P$ times

Nyquist's Stability Criterion

- ▶ From the Principle of the Argument applied to $\Delta(s)$, the number of closed-loop poles in the closed right half-plane is:

$$Z = N + P$$

where:

- ▶ N : the clockwise encirclements of the origin by $\Delta(C)$ correspond to the clockwise encirclements of $-1 + j0$ by $G(C)$ and can be determined from a Nyquist plot of $G(s)$
- ▶ P : the number of poles of $\Delta(s)$ inside C corresponds to the number of poles of $G(s)$ inside C and can be determined from $G(s)$ or its Bode plot

Nyquist's Stability Criterion

Consider a unity feedback control system with open-loop transfer function $G(s)$. Let C be a Nyquist contour. The system is stable if and only if the number of counterclockwise encirclements of $-1 + j0$ by $G(C)$ is equal to the number of poles of $G(s)$ inside C .

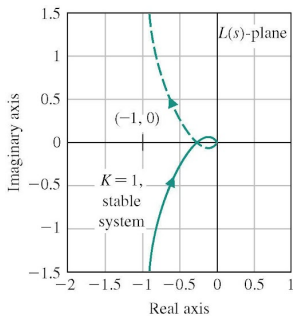
Nyquist Stability: Example 4

- ▶ Determine the closed-loop stability of $G(s) = \frac{\kappa}{s(1+\tau_1s)(1+\tau_2s)} = \frac{\kappa}{s(1+s)^2}$
- ▶ $G(C_1)$ crosses the real axis when:

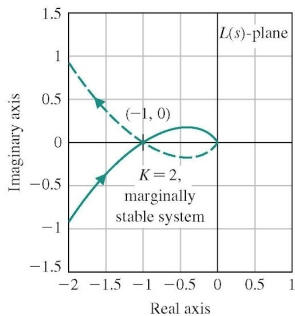
$$G(j\omega) = \frac{-\kappa(\tau_1 + \tau_2) - j\kappa(1 - \omega^2\tau_1\tau_2)\omega}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4\tau_1^2\tau_2^2} = \alpha + j0$$

$$\Rightarrow \omega = \frac{1}{\sqrt{\tau_1\tau_2}} \quad \alpha = -\frac{\kappa\tau_1\tau_2}{\tau_1 + \tau_2}$$

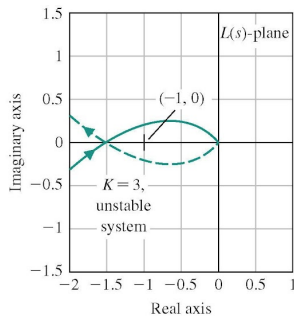
- ▶ The system is stable when $\alpha = -\frac{\kappa\tau_1\tau_2}{\tau_1 + \tau_2} \geq -1$



(a)



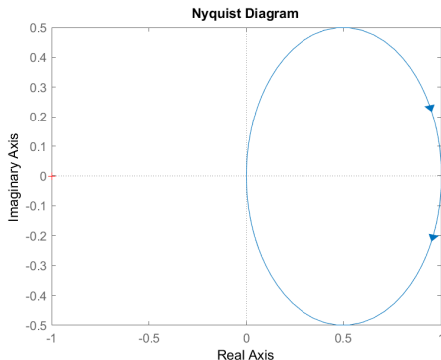
(b)



(c)

Nyquist Plot: Example 8

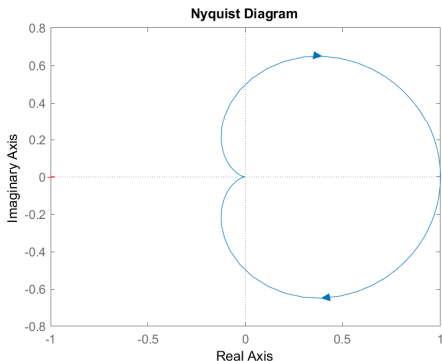
- ▶ Open-loop transfer function: $G(s) = \frac{1}{s+1}$
- ▶ Number of closed-loop poles in CRHP: $Z = N + P = 0$



- ▶ Closed-loop transfer function: $T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s+2}$

Nyquist Plot: Example 9

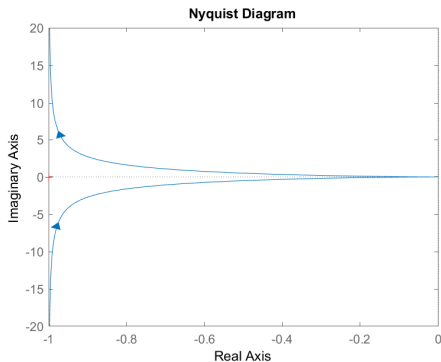
- ▶ Open-loop transfer function: $G(s) = \frac{1}{(s+1)^2}$
- ▶ Number of closed-loop poles in CRHP: $Z = N + P = 0$



- ▶ Closed-loop transfer function: $T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s^2+2s+2}$

Nyquist Plot: Example 10

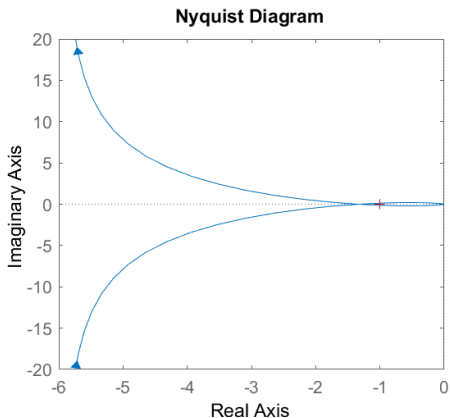
- ▶ Open-loop transfer function: $G(s) = \frac{1}{s(s+1)}$
- ▶ Number of closed-loop poles in CRHP: $Z = N + P = 0$



- ▶ Closed-loop transfer function: $T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s^2+s+1}$

Nyquist Plot: Example 11

- ▶ Open-loop transfer function: $G(s) = \frac{1}{s(s+1)(s+0.5)}$
- ▶ Number of closed-loop poles in CRHP: $Z = N + P = 2$



- ▶ Closed-loop transfer function: $T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s^3+1.5s^2+0.5s+1}$
- ▶ Closed-loop poles: $p_{1,2} = 0.0416 \pm j0.7937$ and $p_3 = -1.5832$