ECE171A: Linear Control System Theory
Lecture 11: Performance Measures

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Outline

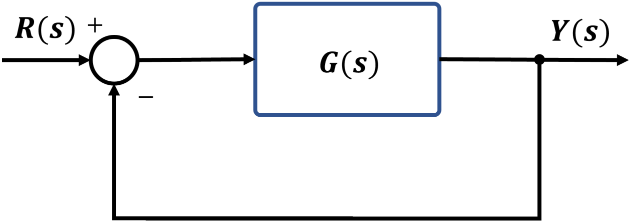
Outline

Stability Margins

Frequency Domain Performance Specifications

Closed-Loop Control from Open-Loop Frequency Response

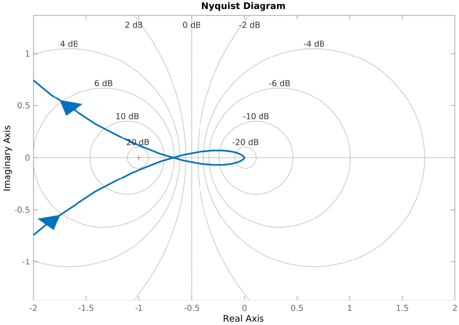
Stability Margins from a Nyquist Plot



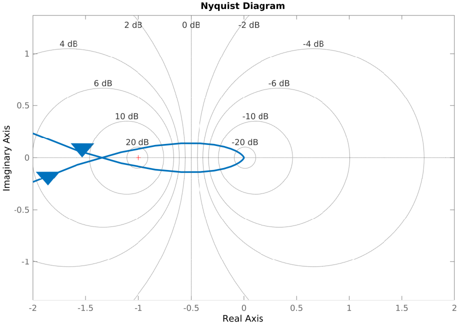
- Consider an open-loop transfer function: $G(s) = k\dfrac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$
- Increasing k increases the magnitude of all points on the Nyquist plot of $G(s)$, i.e, pushes the contour $G(C)$ further away from the origin

Stability Margins from a Nyquist Plot: Example

- Nyquist plot of $G(s) = \frac{k}{s(s+1)(s+10)}$



(a) $k = 75$

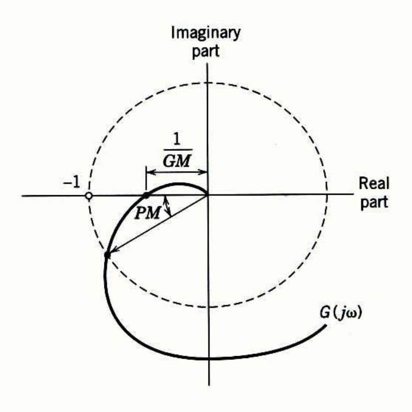


(b) $k = 150$

- The closed-loop system is stable for small k and unstable for large k
- In practice, it is not enough that the system is stable. There must also be a stability margin allowing robustness to disturbances.
- **Stability margin**: quantifies how far the Nyquist plot $G(C)$ is from the critical point -1

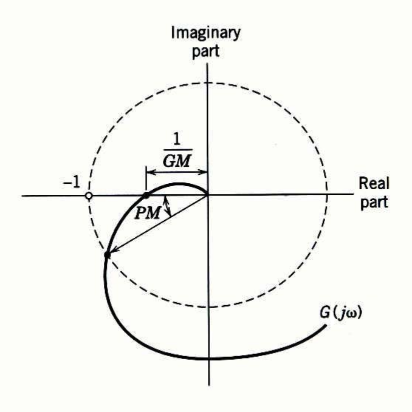
Gain Margin

- **Gain Margin** (GM):
  - the factor by which the open-loop gain can be increased before a stable closed-loop system becomes unstable
  - the factor by which the open-loop gain should be decreased until an unstable system becomes stable
- Nyquist plot: GM is the inverse of the distance from the origin to the first point where $G(C)$ crosses the real axis



Phase Margin

- **Phase Margin** (PM):
  - the amount by which the open-loop phase can be decreased before a stable closed-loop system becomes unstable
  - the amount by which the open-loop phase should be increased before an unstable system becomes stable
- Nyquist plot: PM is the smallest angle on the unit circle between -1 and $G(C)$



Algebraic Definitions of Gain Margin and Phase Margin

- **Phase-Crossover Frequency**: ω_p at which $G(j\omega)$ crosses the real axis:

$$\angle G(j\omega_p) = -180^\circ$$

- **Gain Margin**: the inverse of the open-loop gain at ω_p:

$$GM = 20\log\frac{1}{|G(j\omega_p)|} = -20\log|G(j\omega_p)| \text{ dB}$$

- **Gain-Crossover Frequency**: ω_g at which $G(j\omega)$ crosses the unit circle:

$$20\log|G(j\omega_g)| = 0 \text{ dB}$$

- **Phase Margin**: amount by which the open-loop phase at ω_g exceeds -180°:

$$PM = \angle G(j\omega_g) + 180^\circ$$

Gain Margin and Phase Margin

- For a stable minimum-phase system both GM and PM are positive. Larger gains mean larger relative stability.

- When $\omega_g = \omega_p = \omega_*$, there are closed-loop poles on the imaginary axis and instability starts to occur:

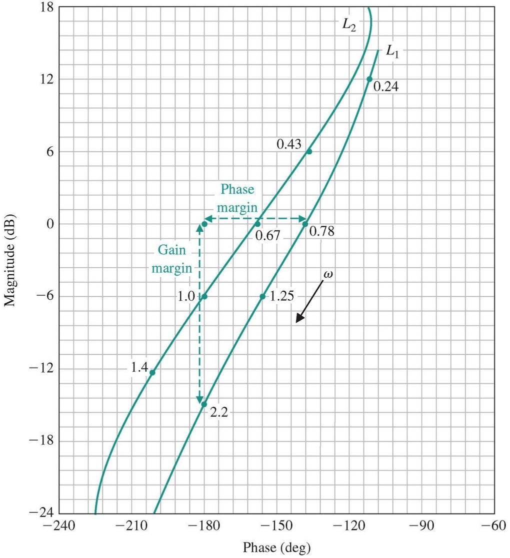
$$|G(j\omega_*)| = 1, \qquad \angle G(j\omega_*) = -180^\circ \qquad \Rightarrow \qquad 1 + G(j\omega_*) = 0$$

- **Bode plot** and **magnitude-phase** plot provide $|G(j\omega)|$ and $\angle G(j\omega)$ and hence ω_p, ω_g, GM, and PM can all be seen

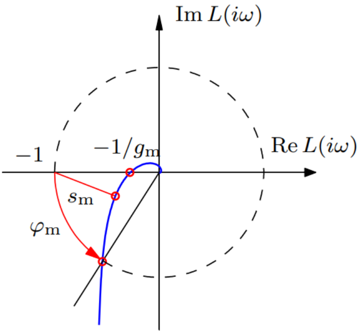
- **Caution**: the Bode plot or magnitude-phase plot interpretation of GM and PM to determine stability can be incorrect if the system is non-minimum phase or has delays. Only the Nyquist stability criterion should be used to determine stability.

Gain Margin and Phase Margin on a Magnitude-Phase Plot

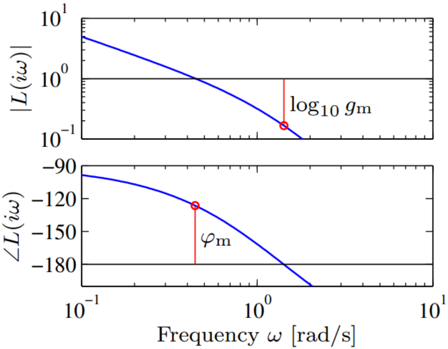
- Magnitude-phase plot of $G_1(s) = \frac{1}{s(s+1)(s/5+1)}$ and $G_2(s) = \frac{1}{s(s+1)^2}$



Gain Margin and Phase Margin on a Bode Plot



(a) Nyquist plot

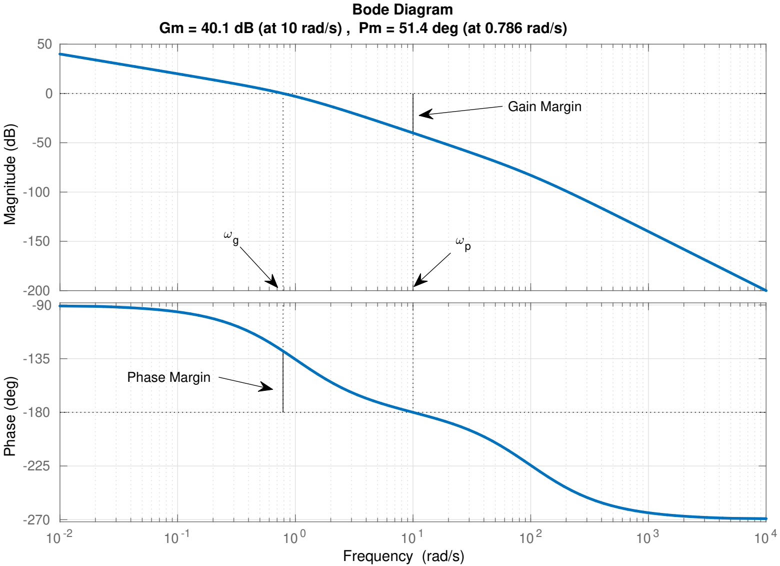


(b) Bode plot

- **Stability margin**: shortest distance s_m from Nyquist plot $G(C)$ to -1
- **Gain margin**: inverse gain g_m at phase-crossover ω_p
- **Phase margin**: phase distance φ_m from -180° at gain-crossover ω_g

Gain Margin and Phase Margin on a Bode Plot

- Bode plot of $G(s) = \frac{k}{s(s+1)(s/100+1)}$ with $k = 1$



Gain Margin and Phase Margin on a Bode Plot

- If $k > 0$, it has no effect on the phase and shifts the magnitude up or down by $20 \log k$. This changes the gain-crossover frequency ω_g but not the phase-crossover frequency ω_p.
- Some closed-loop poles lie on the imaginary axis when $\omega_g = \omega_p$
- Choose $k \approx 100$ to shift the magnitude up by ~ 40 dB, making $\omega_g \approx \omega_p$
- The imaginary axis crossing can be determined from the Bode plot but we do not know if we are going from stability to instability or vice versa
- Assuming that the system is stable initially (can only be verified by Nyquist or Routh-Hurwitz stability criteria), we expect the region of stability to be $0 < K < 100$

Gain Margin and Phase Margin on a Bode Plot

- Use Routh-Hurwitz to verify the region of stability for:

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{k}{s(s+1)(s/100+1)+k} = \frac{100k}{s^3+101s^2+100s+100k}$$

- Characteristic polynomial $a(s) = s^3 + 101s^2 + 100s + 100k$
- The Routh table is:

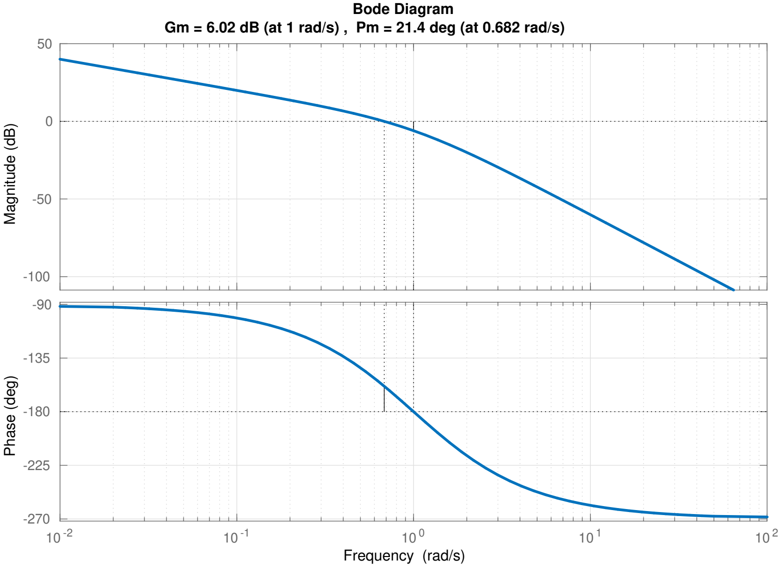
s^3	1	100
s^2	101	$100k$
s^1	$100 - \frac{100k}{101}$	0
s^0	$100k$	0

- Stability region: $0 < k < 101$
- Auxiliary polynomial roots for $k = 101$:

$$A(s) = 101(s^2+100) \qquad \Rightarrow \qquad s = \pm j10$$

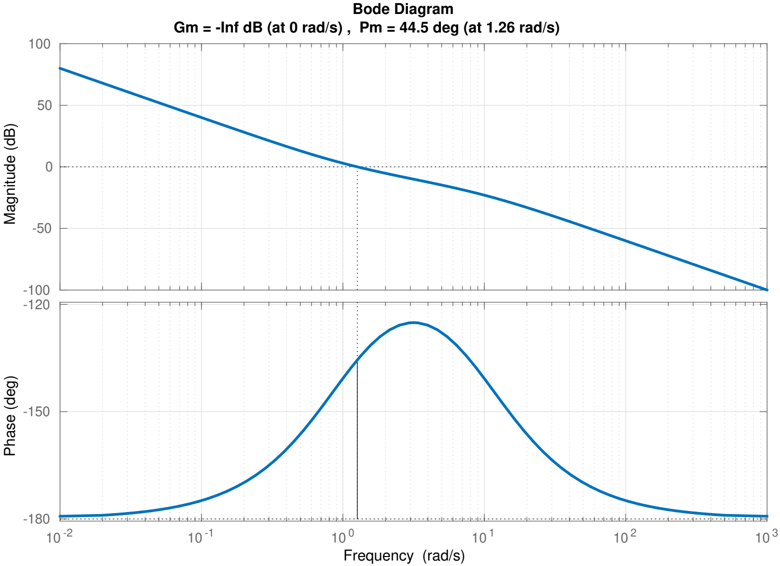
Stability Margins: Example 1

- What are the gain margin and phase margin of $G(s) = \frac{1}{s(s+1)^2}$?



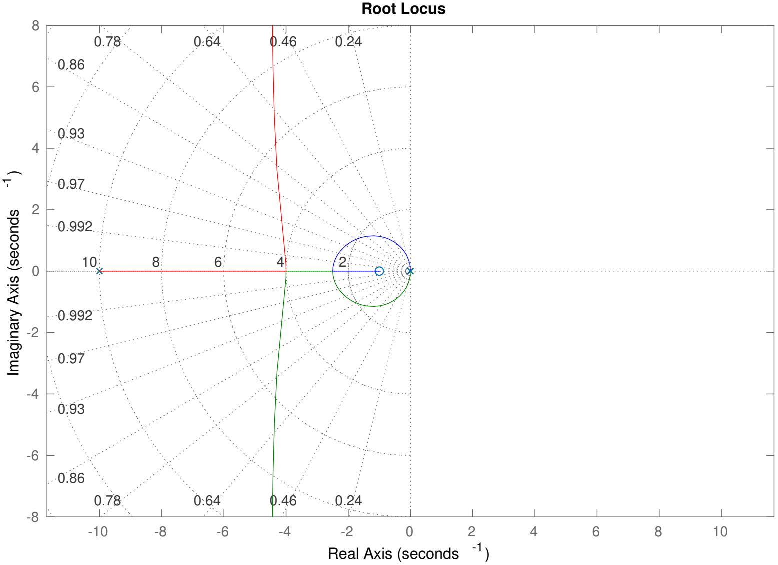
Stability Margins: Example 2

- What are the gain margin and phase margin of $G(s) = \frac{(s+1)}{s^2(s/10+1)}$?



Stability Margins: Example 2

- Root locus of $G(s) = \frac{(s+1)}{s^2(s/10+1)}$



Stability Margins: Example 2

- What are the gain margin and phase margin of $G(s) = \frac{k(s+1)}{s^2(s/10+1)}$?
- The gain margin is ∞ since the phase hits -180° at $\omega_p = \infty$
- As $k \to \infty$, the gain-crossover frequency ω_g moves to the right and the **phase margin decreases**
- As $k \to \infty$, a pair of closed-loop poles moves vertically on the root locus and the **damping ratio ζ decreases**
- There is a relationship between **phase margin** PM and **damping ratio** ζ
- We will analyze a second-order system to determine this and establish a relationship between **frequency response** and **transient step response**

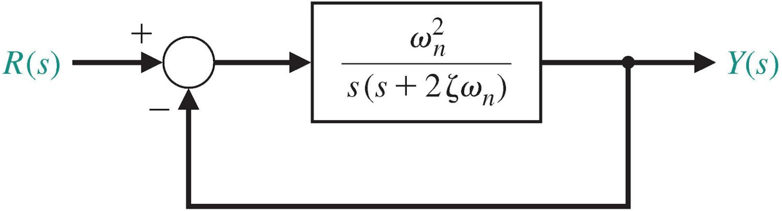
Outline

Stability Margins

Frequency Domain Performance Specifications

Closed-Loop Control from Open-Loop Frequency Response

Frequency Domain Performance Specifications



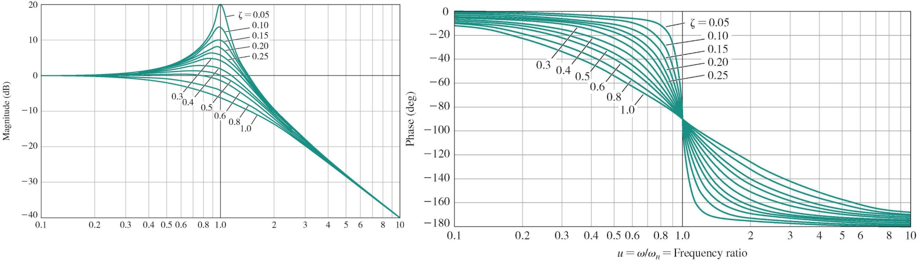
- Consider a second-order system:

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1}$$

- How does the closed-loop **frequency response** $T(j\omega)$ relate to the **transient step response** (rise time, overshoot, settling time)?

Frequency Response of a Second-order System

- Bode plot of $T(s) = \frac{1}{\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1}$



- The **damping ratio** ζ is related to the **resonant peak** $\max_\omega |T(j\omega)|$

- The **natural frequency** ω_n and **rise time** t_r are related to the **bandwidth** ω_b (frequency range $(0, \omega_b)$ over which the system tracks an input signal well)

Frequency Domain Performance Specifications

- **Low-frequency (DC) gain**: the magnitude of the transfer function $|T(j\omega)|$ for low frequencies $\omega \to 0$ is equal to the steady-state step response
- **Bandwidth**: the frequency ω_b at which the transfer function magnitude drops 3 dB below the DC gain:

$$|T(j\omega_b)| = \frac{1}{\sqrt{2}}|T(0)|$$

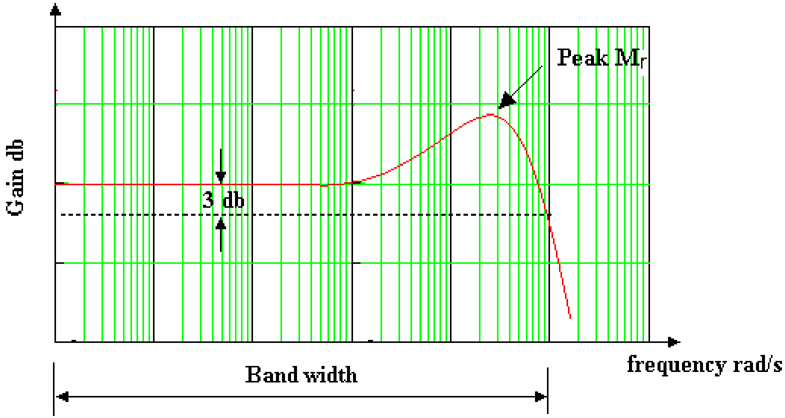
- **Resonant frequency**: ω_r where the transfer function magnitude is maximized:

$$\omega_r = \arg\max_{\omega} |T(j\omega)|$$

- **Resonant peak**: the maximum value of the transfer function magnitude:

$$M_r = |T(j\omega_r)|$$

Frequency Domain Performance Specifications



Frequency Response of a Second-order System

- Consider a second-order system:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1}$$

- **Transfer function magnitude** at $s = j\omega$:

$$|T(j\omega)| = \frac{1}{|-\frac{\omega^2}{\omega_n^2} + 2\zeta\frac{\omega}{\omega_n}j + 1|} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

- **Transfer function phase** at $s = j\omega$:

$$\angle T(j\omega) = \angle \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)j + 1} = -\arctan\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Resonant Frequency of a Second-order System

- **Transfer function magnitude** at $s = j\omega$:

$$|T(j\omega)| = \frac{1}{|-\frac{\omega^2}{\omega_n^2} + 2\zeta\frac{\omega}{\omega_n}j + 1|} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

- **Resonant frequency**:

$$\frac{d|T(j\omega)|}{d\omega} = 0 \qquad \Rightarrow \qquad \omega_r = 0 \quad \text{or} \quad \omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

- **Resonant peak**:
  - Case 1: $\zeta \leq \frac{1}{\sqrt{2}}$:

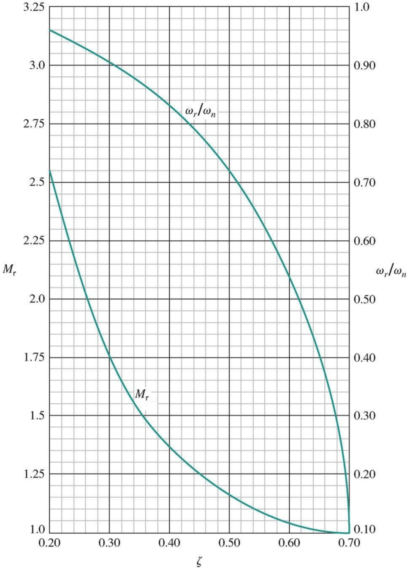
$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2} \qquad\qquad M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

  - Case 2: $\zeta > \frac{1}{\sqrt{2}}$:

$$\omega_r = 0 \qquad\qquad M_r = 1$$

Resonant Frequency of a Second-order System

- Plot of $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ and $\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$ as a function of ζ



- The **resonant peak** M_r is related to the **percent overshoot** via ζ
- Example:
  - The resonant peak of the closed-loop system should be less than 1.75 ($\approx$ 5 dB)
  - Equivalent to ζ should be greater than 0.3
  - Equivalent to p.o. should be less than 37%

Bandwidth of a Second-order System

- **Bandwidth**: the low frequency range $(0, \omega_b)$ over which the closed-loop system tracks an input signal well

$$|T(j\omega_b)| = \frac{1}{\sqrt{2}}|T(0)|$$

- **Relationship between ω_b, ω_n, and ζ**: with $u = \omega_b/\omega_n$:

$$u^4 + 2(\zeta^2 - 1)u^2 + 1 = 2 \quad \Rightarrow \quad u^2 = (1 - 2\zeta^2) \pm \sqrt{4\zeta^4 - 4\zeta^2 + 2}$$

$$\omega_b = \omega_n\sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

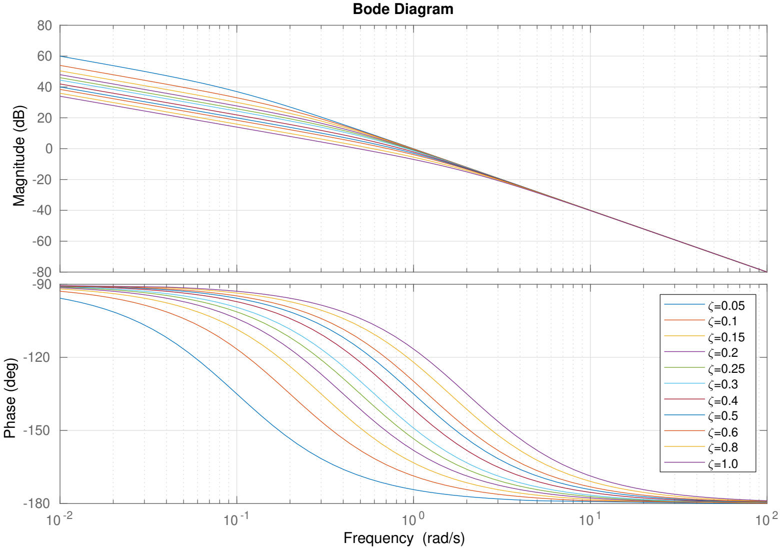
- Bandwidth ω_b and rise time $t_r \approx \frac{2.16\zeta + 0.6}{\omega_n}$ are inversely proportional:
  - If $\omega_n \uparrow$, then $\omega_b \uparrow$ and $t_r \downarrow$
  - If $\zeta \uparrow$, then $\omega_b \downarrow$ and $t_r \uparrow$

- Adding a zero to $G(s)$ increases ω_b of the closed-loop transfer function $T(s)$

- Adding a pole to $G(s)$ decreases ω_b of the closed-loop transfer function $T(s)$

Stability Margins of a Second-order System

- Bode plot of $G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$



Stability Margins of a Second-order System

- The phase plot of $G(s)$ shows that the **phase-crossover frequency** is:

$$\omega_p = \infty$$

- The **gain margin** is:

$$GM = \infty$$

- Set $|G(j\omega)|$ to 1 to obtain the gain-crossover frequency ω_g:

$$1 = |G(j\omega_g)| = \frac{\omega_n^2}{|j\omega_g||j\omega_g + 2\zeta\omega_n|} = \frac{\omega_n^2}{\omega_g\sqrt{4\zeta^2\omega_n^2 + \omega_g^2}}$$

- The **gain-crossover frequency** is:

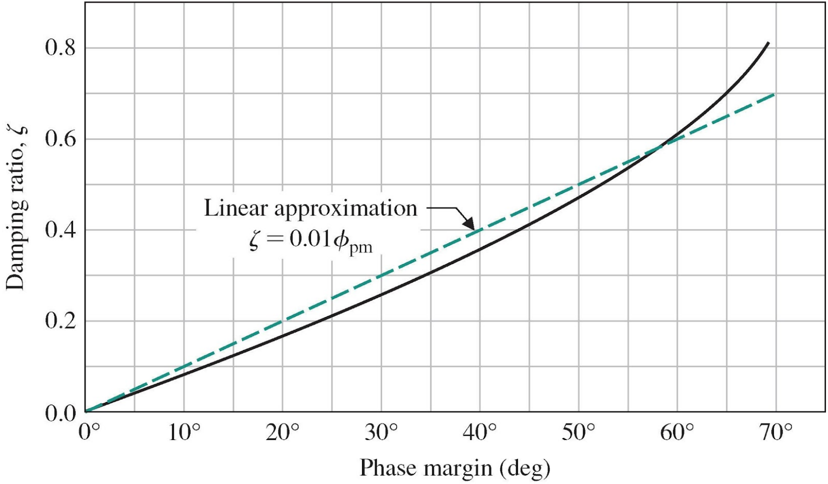
$$\omega_g = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

- The **phase margin** is:

$$\text{PM} = \angle G(j\omega_g) + \pi = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right)$$

Phase Margin of a Second-order System

- The phase margin of a second-order system is a function of ζ but not ω_n
- The relationship between PM and ζ can be approximated well by a straight line for small values of ζ



Phase Margin of a Second-order System

- For $0 \leq \zeta \leq 0.7$, the phase margin PM (in degrees) and the damping ratio ζ of a second-order system are related by:

$$PM \approx 100\zeta$$

- The relationship between ζ and PM can be used to design control systems in the frequency domain meeting time-domain specifications
- Poles that are ignored in a dominant-pole-pair approximation contribute phase lag so it is important to keep a large phase margin
- For $0.2 \leq \zeta \leq 0.8$, the gain-crossover frequency ω_g of $G(s)$ is related to the closed-loop system bandwidth ω_b:

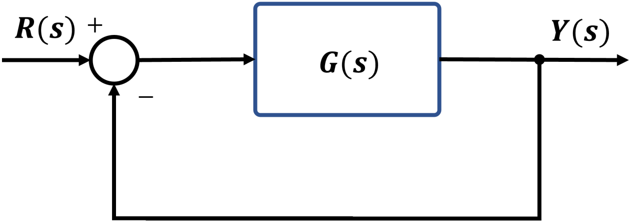
$$\omega_b \approx 1.8\omega_g$$

Frequency Domain Control Design

- Consider proportional control design with gain k
- To obtain low steady-state error, we want large gain k
- To obtain fast transient response we want large ω_g since $\omega_b \uparrow$, $t_r \downarrow$
- Increasing k, increases ω_g but decreases the phase margin and the system becomes less stable and might exhibit oscillatory behavior
- More complicated control design may be needed to simultaneously provide good phase margin, good gain-crossover frequency, and good steady state tracking

Outline

Frequency Domain Performance Specifications



- Feedback control system with control gain k and open-loop transfer function:

$$G(s) = k\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

- How can the closed-loop frequency-domain performance specifications (resonant peak M_r, resonant frequency ω_r, bandwidth ω_b) be related to the open-loop frequency response ($G(j\omega)$)?

- How can the gain k be adjusted to meet frequency-domain performance specifications?

Closed-Loop Transfer Function Magnitude

- Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

- Closed-loop transfer function magnitude:

$$M(s) = |T(s)| = \frac{|G(s)|}{|1 + G(s)|}$$

- Obtain $M(s)$ as a function of the real and imaginary parts of $G(s) = x(s) + jy(s)$:

$$M = \frac{\sqrt{x^2 + y^2}}{\sqrt{(1 + x)^2 + y^2}}$$

- This equation turns out to be a circle on a Nyquist plot

Constant Magnitude Circles

- Relationship between the magnitude of the closed-loop transfer function M and the real part x and imaginary part y of the open-loop transfer function:

$$M^2(1+x)^2 + M^2y^2 = x^2 + y^2$$
$$M^2 = (1-M^2)x^2 - 2M^2x + (1-M^2)y^2$$

- Assume $M \neq 1$ and divide both sides by $(1-M^2)$:

$$x^2 - 2\frac{M^2}{1-M^2}x + y^2 = \frac{M^2}{1-M^2}$$

- Add $M^4/(1-M^2)^2$ to both sides to complete the square for x:

$$\left(x - \frac{M^2}{1-M^2}\right)^2 + y^2 = \frac{M^2}{(1-M^2)^2}$$

Constant Magnitude Circles

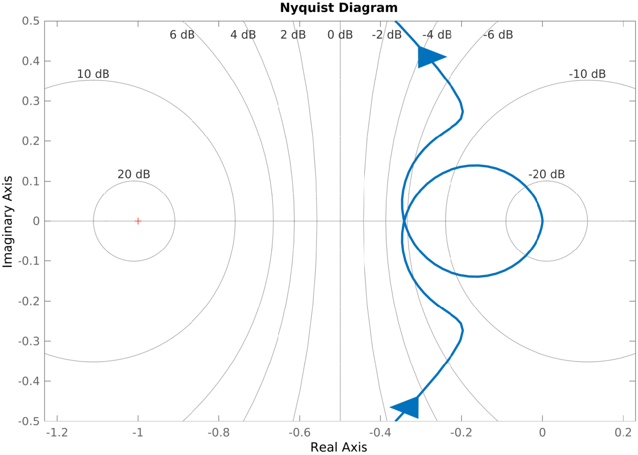
- M **circle**: a circle of constant closed-loop transfer function magnitude on a polar/Nyquist plot:

$$\left(x - \frac{M^2}{1-M^2}\right)^2 + y^2 = \frac{M^2}{(1-M^2)^2}$$

- An M circle is centered at $\left(\frac{M^2}{1-M^2}, 0\right)$ with radius $\frac{M}{|(1-M^2)|}$
- As $M \to \infty$, the M circle is centered at $(-1, 0)$ with radius 0
- For $1 < M < \infty$, the M circle center moves to the left of $(-1, 0)$, while the radius increases
- As $M \to 0$, the M circle is centered at $(0, 0)$ with radius 0
- For $0 < M < 1$, the M circle center moves to the right of $(0, 0)$, while the radius increases
- At $M = 1$, we get a degenerate circle at $(\pm\infty, 0)$ with radius ∞

Constant Magnitude Circles on a Nyquist Plot

- Nyquist plot of $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$
- If the frequencies ω along the polar plot of $G(s)$ are available, we can construct a closed-loop Bode plot using the M circles

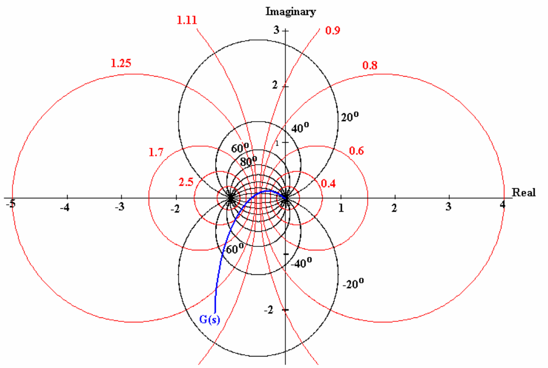


Constant Phase Circles

- N **circle**: a circle of constant $N = \tan \angle T(s)$ on a polar/Nyquist plot:

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4}\left(1 + \frac{1}{N^2}\right)$$

- An N circle is centered at $(-0.5, 0.5/N)$ with radius $0.5\sqrt{1 + 1/N^2}$
- N circles are orthogonal to M circles, i.e., intersect at 90°

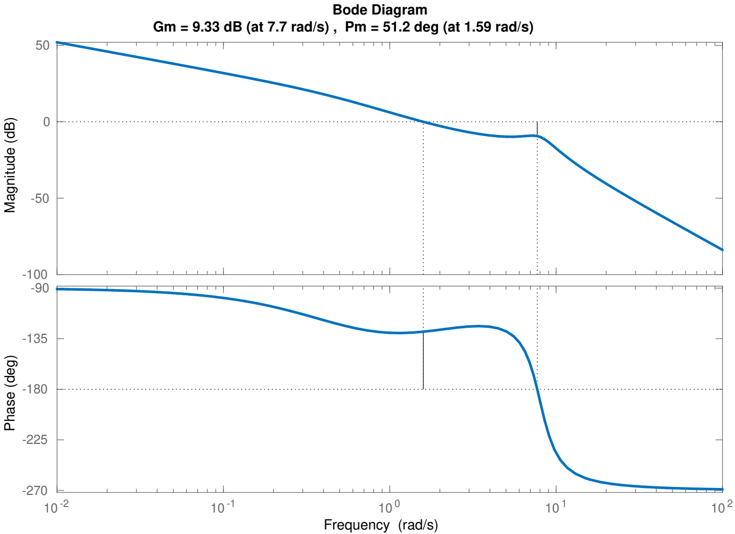


Frequency Domain Performance Specifications

- Given the frequency response of an open-loop transfer function $G(s)$, we can verify stability and frequency domain performance metrics
- **Stability**:
  - Determine using the Nyquist criterion
  - What if $k < 0$? Rotate the Nyquist plot clockwise by 180°.
- **Gain margin** GM **and phase margin** PM:
  - Can be obtained from a Nyquist plot, Bode plot, or magnitude-phase plot
- **Resonant peak** M_r, **resonant frequency** ω_r, and **bandwidth** ω_b:
  - Use the M circles on a Nyquist plot

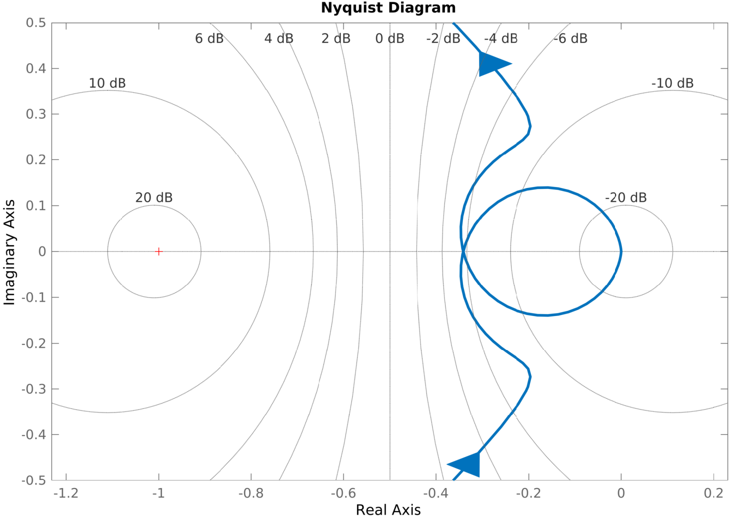
Open-Loop Bode Plot

- Open-loop Bode plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$



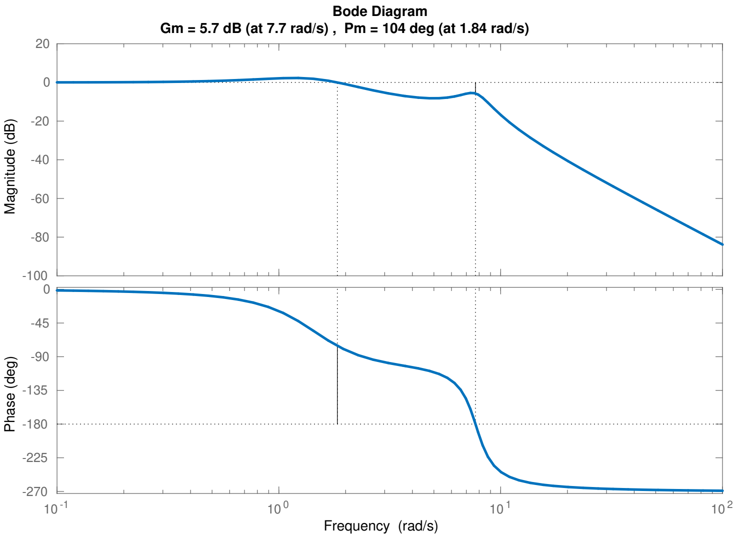
Nyquist Plot

- Nyquist plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$



Closed-Loop Bode Plot

- Closed-loop Bode plot for $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$



Frequency Domain Control Design

- How should k be adjusted to meet desired closed-loop frequency domain specifications?
  - It is difficult to determine how much to change k to meet a resonant peak specification on a Nyquist plot
  - It is difficult to tell where the Nyquist plot would become tangent to the desired M circle

- **Nathaniel Nichols** proposed to transform the M and N circles from a Nyquist plot to a magnitude-phase plot

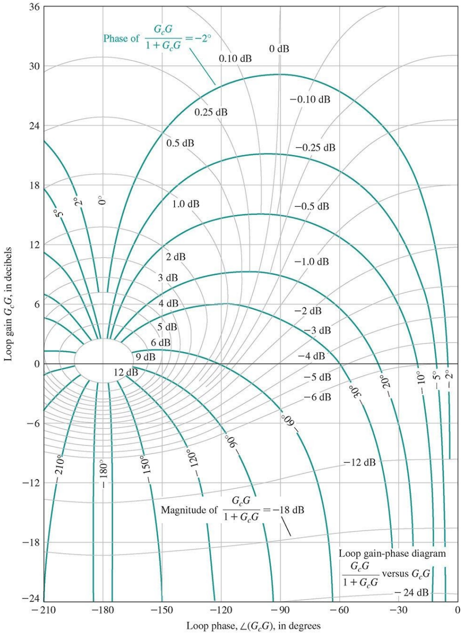
- On a magnitude-phase plot, the M and N contours are no longer circles

- If k changes, a magnitude-phase plot only moves up or down, which is much easier to interpret that the change of the shape on a Nyquist plot

N. Nichols

Nichols Plot

- **Nichols plot**: a magnitude-phase plot with overlaid M and N contours of constant closed-loop transfer-function magnitude and phase
- The gain margin and phase margin can be obtained
- The resonant peak M_r and bandwidth ω_b can be obtained
- A change in the gain k moves the response up or down and can be used to meet closed-loop frequency domain specifications



Nichols Plot

- Nichols plot of $G(s) = \frac{4(s/2+1)}{s(2s+1)(1+0.4(s/8)+(s/8)^2)}$