

ECE171A: Linear Control System Theory

Lecture 13: PID Control

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Outline

PID Control

PID Tuning and Implementation

Inverted Pendulum Example

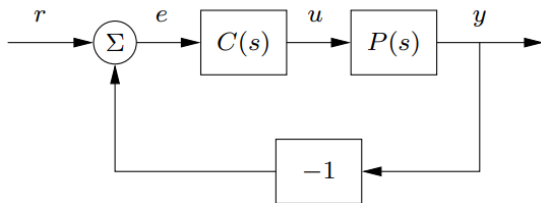
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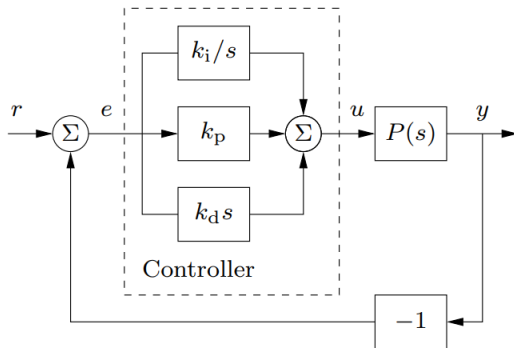
Feedback Control System



Signals	t domain	s domain
Input	$u(t)$	$U(s)$
Output	$y(t)$	$Y(s)$
Reference	$r(t)$	$R(s)$
Error	$e(t) = r(t) - y(t)$	$E(s) = R(s) - Y(s)$

Components	Transfer function
Plant	$P(s) = \frac{Y(s)}{U(s)}$
Controller	$C(s) = \frac{U(s)}{E(s)}$

Proportional Integral Derivative Control



Proportional Integral Derivative (PID) Controller

Uses proportional gain k_p , integral gain k_i , derivative gain k_d :

t domain

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \frac{de(t)}{dt}$$

s domain

$$\frac{U(s)}{E(s)} = C(s) = k_p + \frac{k_i}{s} + k_d s$$

PID Control

- ▶ PID control is the most common approach for utilizing feedback in engineering systems:
 - ▶ Survey of 100+ boiler-turbine controllers: 94.4% PI, 3.7% PID, 1.9% other
- ▶ PID control appears in both simple and complex systems: as a stand-alone controller, as an element of hierarchical or distributed systems, etc.
- ▶ PID control appears in biological systems, where proportional, integral, and derivative action is generated by subsystems with dynamic behavior
 - ▶ Example: Eye pupil opening regulates the amount of light entering the eye

Roles of PID Terms

- ▶ PID control terms:
 - ▶ **Proportional (P) term:** responds to present error
 - ▶ **Integral (I) term:** accumulates past error
 - ▶ **Derivative (D) term:** anticipates future error
- ▶ PID time constants:

$$u(t) = k_p \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right)$$

- ▶ **Integral time constant:** $T_i = k_p/k_i$
- ▶ **Derivative time constant:** $T_d = k_d/k_p$

Role of P Term

▶ **Proportional term:** $u(t) = k_p e(t)$

▶ Transfer function: $T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{k_p P(s)}{1 + k_p P(s)}$

▶ Error: $E(s) = R(s) - Y(s) = (1 - T(s))R(s)$

▶ Steady-state error of stable system for step reference $R(s) = 1/s$:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \frac{1}{1 + k_p P(0)}$$

▶ **Increasing k_p decreases steady-state error but also stability margins**

▶ **Feedforward term:** used to reduce steady-state error in early controllers:

$$u(t) = k_p e(t) + u_{ff}$$

▶ For step reference, if the DC gain is known, choose $u_{ff} = 1/P(0)$:

$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow \infty} s \left(\frac{1}{1 + k_p P(s)} R(s) - \frac{P(s)}{1 + k_p P(s)} \frac{u_{ff}}{s} \right) = \frac{1 - u_{ff} P(0)}{1 + k_p P(0)}$$

Role of I Term

- ▶ **Integral term:** feedforward term that **guarantees zero steady-state error:**

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau \quad U(s) = \left(k_p + \frac{k_i}{s} \right) E(s)$$

- ▶ Transfer function: $T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}$

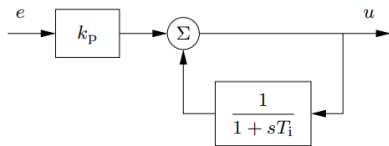
- ▶ Steady-state error of stable system for step reference $R(s) = 1/s$:

$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s(1 - T(s)) R(s) = \lim_{s \rightarrow 0} \frac{1}{1 + C(s)P(s)} \underbrace{=}_{{C(s) \rightarrow \infty}} 0$$

- ▶ **Magic of integral action:** if a steady state exists, the error will be zero

- ▶ The PI term is implemented using a low-pass filter $H_{pi}(s) = \frac{1}{1+sT_i}$:

$$\frac{U(s)}{E(s)} = k_p \frac{1 + sT_i}{sT_i} = k_p + \frac{k_p}{sT_i}$$



(a) Integral action (automatic reset)

Role of D Term

- ▶ **Derivative term:** provides predictive action:

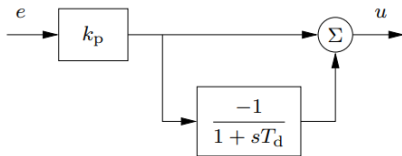
$$u(t) = k_p e(t) + k_d \frac{de(t)}{dt} = k_p \left(e(t) + T_d \frac{de(t)}{dt} \right) =: k_p e_p(t)$$

- ▶ **Prediction error** e_p : linear extrapolation of the error to time $t + T_d$
- ▶ In practice the error signal $e(t)$ is measured and contains high-frequency noise which should not be differentiated
- ▶ The D term is implemented using a low-pass filter $H_d(s) = \frac{1}{1+sT_d}$

- ▶ **Filtered derivative:** difference between a signal and its low-pass filtered version:

$$\frac{U_d(s)}{E(s)} = k_p \left(1 - \frac{1}{1+sT_d} \right) = \frac{k_d s}{1+sT_d}$$

- ▶ Acts as **differentiator** for low-frequency signals and as **constant gain** k_p for high-frequency signals



(b) Derivative action

Numerical Experiments

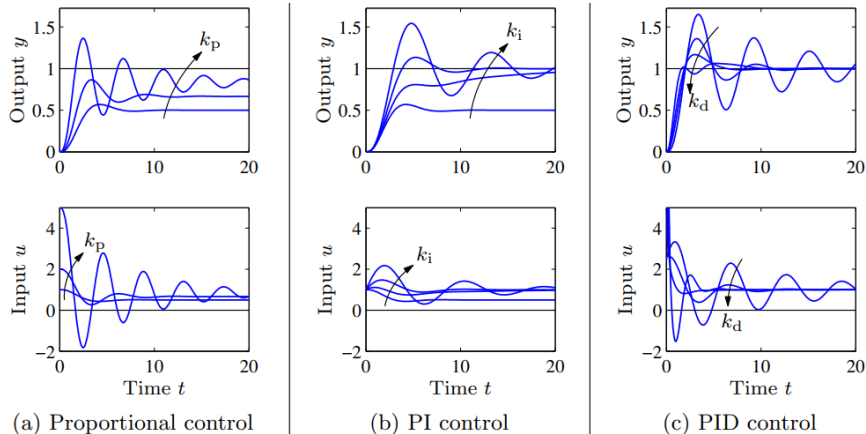
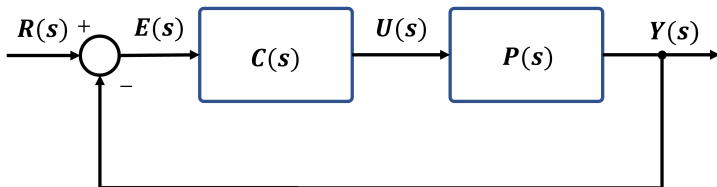


Figure 11.2: Responses to step changes in the reference value for a system with a proportional controller (a), PI controller (b), and PID controller (c). The process has the transfer function $P(s) = 1/(s+1)^3$, the proportional controller has parameters $k_p = 1, 2,$ and 5 , the PI controller has parameters $k_p = 1, k_i = 0, 0.2, 0.5,$ and 1 , and the PID controller has parameters $k_p = 2.5, k_i = 1.5,$ and $k_d = 0, 1, 2,$ and 4 .

Model Reduction

- ▶ Practical systems are complex
- ▶ While a high-order model may describe the system behavior accurately, a low-order model may simplify the system analysis and control design
- ▶ **Model reduction:** simplification of a system model that captures the essential properties needed for control design
- ▶ Various model reduction techniques are available:
 - ▶ **Dominant pole-zero approximation:** cancel pole-zero pairs or eliminate states that have little effect on the model response
 - ▶ **Mode selection:** eliminate poles and zeros that fall outside a specific frequency range of interest
- ▶ Low-order models can be obtained from first principles:
 - ▶ A system can be modeled as zeroth-order if its inputs are sufficiently slow
 - ▶ A system can be modeled as first-order if the change of its mass, momentum, or energy can be captured by a single variable (e.g., velocity)
 - ▶ A system can be modeled as second-order if the change of its mass, momentum, or energy can be captured by two variables (e.g., position and velocity)

Second-Order System Control Design



- ▶ Consider a feedback control system with a second-order plant:

$$P(s) = \frac{b_0}{s^2 + a_1s + a_0}$$

- ▶ How should the controller $C(s)$ be designed to ensure that the closed-loop system is **stable** and its **step response has zero steady-state error**?

P Control for Second-Order System

- ▶ **P controller:**

$$u(t) = k_p e(t) \quad \Leftrightarrow \quad \frac{U(s)}{E(s)} = C(s) = k_p$$

- ▶ Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{k_p b_0}{s^2 + a_1 s + (a_0 + k_p b_0)}$$

- ▶ P control can accelerate the response of a second-order system by changing the natural frequency $\omega_n^2 = (a_0 + k_p b_0)$
- ▶ To ensure stability, we need $a_1 > 0$ and $a_0 + K_p b_0 > 0$
- ▶ P control can stabilize only some systems because it adjusts one coefficient of the characteristic equation

For $a_0 \neq 0$, $C(s)P(s)$ has 0 poles at the origin (type 0 system) and the closed-loop step response has a **constant finite steady-state error**:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} (1 - T(s)) = \frac{a_0}{a_0 + k_p b_0}.$$

PI Control for Second-Order System

- ▶ To achieve zero steady-state step error, we need to add a pole at the origin in $C(s)P(s)$ to obtain a type 1 system
- ▶ **PI controller:**

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau \quad \Leftrightarrow \quad \frac{U(s)}{E(s)} = C(s) = k_p + \frac{k_i}{s}$$

- ▶ Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{b_0(k_p s + k_i)}{s^3 + a_1 s^2 + (a_0 + k_p b_0)s + k_i b_0}$$

PI control achieves **zero steady-state error**:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} (1 - T(s)) = 1 - T(0) = 0$$

but the closed-loop system may be unstable if $a_1 < 0$.

PID Control for Second-Order System

- ▶ PID controller:

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \frac{de(t)}{dt} \quad \Leftrightarrow \quad C(s) = k_p + \frac{k_i}{s} + k_d s$$

- ▶ Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{b_0(k_p s + k_i + k_d s^2)}{s^3 + (a_1 + k_d b_0)s^2 + (a_0 + k_p b_0)s + k_i b_0}$$

- ▶ The coefficients of the characteristic polynomial can be set **arbitrarily** via an appropriate choice of k_p , k_i , k_d

For a second-order plant, PID control can guarantee **stability**, **good transient behavior**, and **zero steady-state step error**.

PID Control Example

- ▶ Consider the plant $P(s) = \frac{1}{s^2 - 3s - 1}$
- ▶ Design a PID controller $C(s)$ to achieve step response with zero steady-state error and place the closed-loop system poles at $-5, -6, -7$
- ▶ PID controller: $C(s) = k_p + \frac{k_i}{s} + k_d s$
- ▶ Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{k_d s^2 + k_p s + k_i}{s^3 + (k_d - 3)s^2 + (k_p - 1)s + k_i}$$

- ▶ Match coefficients with:

$$\Delta(s) = (s + 5)(s + 6)(s + 7) = s^3 + 18s^2 + 107s + 210$$

- ▶ PID control gains:

$$k_d = 21 \quad k_p = 108 \quad k_i = 210$$

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PID Control Gain Tuning

- ▶ **PID control gain tuning:** the process of determining satisfactory PID control gains
 - ▶ Manual tuning
 - ▶ Ziegler-Nichols method
 - ▶ First-order and time-delay (FOTD) method
 - ▶ Automatic tuning via relay feedback

Manual PID Control Gain Tuning

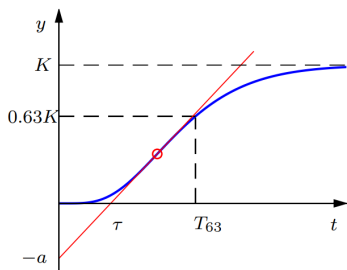
- ▶ Set $k_i = k_d = 0$
- ▶ Increase k_p slowly until the output of the closed-loop system oscillates on the verge of instability
- ▶ Reduce k_p to achieve **quarter amplitude decay** of the closed-loop response, i.e., the amplitude should be one-fourth of the maximum value during the oscillatory period
- ▶ Increase k_i and k_d to achieve the desired response

Table 7.4 Effect of Increasing the PID Gains K_p , K_D , and K_I on the Step Response

PID Gain	Percent Overshoot	Settling Time	Steady-State Error
Increasing K_p	Increases	Minimal impact	Decreases
Increasing K_I	Increases	Increases	Zero steady-state error
Increasing K_D	Decreases	Decreases	No impact

Ziegler-Nichols Method

- ▶ Developed by John Ziegler and Nathaniel Nichols in the 1940s
- ▶ Perform a simple experiment on the system to extract features from its time domain or frequency domain response
- ▶ **Time-domain method**
 - ▶ Apply a unit step input to the **open-loop** system
 - ▶ Record the x-intercept τ and y-intercept $-a$ with the coordinate axes of the steepest tangent to the step response
 - ▶ Use τ and a to choose the PID control gains



(a) Step response method

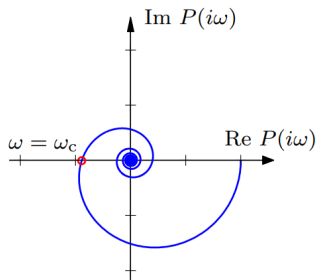
Type	k_p	T_i	T_d
P	$1/a$		
PI	$0.9/a$	$\tau/0.3$	
PID	$1.2/a$	$\tau/0.5$	0.5τ

(a) Step response method

Ziegler-Nichols Method

► Frequency-domain method

- Connect a PID controller to the plant with $k_i = k_d = 0$
- Increase k_p until the closed-loop response oscillates on the verge of instability
- Record the critical proportional gain k_c and the period of oscillation T_c
- Nyquist contour of $k_c P(s)$ passes through -1 at frequency $\omega_c = 2\pi/T_c$
- Use k_c and T_c to choose the PID control gains



(b) Frequency response method

Type	k_p	T_i	T_d
P	$0.5k_c$		
PI	$0.45k_c$	$T_c/1.2$	
PID	$0.6k_c$	$T_c/2$	$T_c/8$

(b) Frequency response method

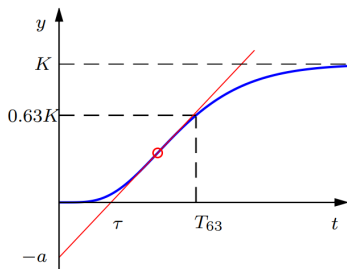
FOTD method

- ▶ Ziegler–Nichols methods use 2 parameters to determine the PID control gains
- ▶ **First-order and time-delay (FOTD) method**: uses plant model with more parameters:

$$P(s) = \frac{K}{1 + sT} e^{-\tau s}$$

- ▶ Apply unit-step input to **open-loop** system
- ▶ Record **time delay** τ (x-intercept of steepest tangent), **steady-state value** K , and $T = T_{63} - \tau$, where T_{63} is the time when the output reaches 63% of K
- ▶ Use τ , K , and T to choose the PI gains:

$$k_p = \frac{0.15\tau + 0.35T}{K\tau} \quad k_i = \frac{0.46\tau + 0.02T}{K\tau^2}$$



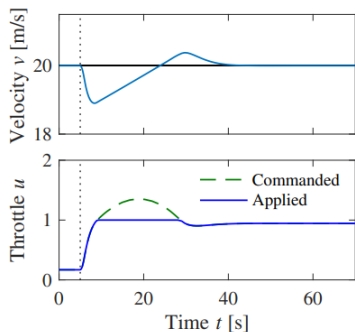
(a) Step response method

Integral Windup

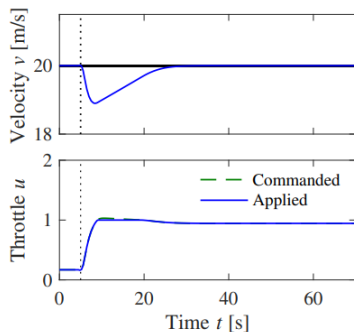
- ▶ **Integral windup**: accumulation of integral error due to input saturation
- ▶ Physical actuators have limits, e.g., a motor has maximum speed, a valve cannot be more than fully opened
- ▶ When actuator limits are reached, the input remains at its limit (**input saturation**) and the system runs in open-loop
- ▶ The integral error $\int_0^t e(\tau)d\tau$ accumulates while the input is saturated
- ▶ Once the input leaves the saturation range the accumulated integral error induces **large transient response**

Example: Cruise control

- ▶ When a car encounters a steep hill (e.g., 6°), the throttle saturates
- ▶ The resulting integral windup leads to velocity overshoot



(a) Windup



(b) Anti-windup

Figure 11.10: Simulation of PI cruise control with windup (a) and anti-windup (b). The figure shows the speed v and the throttle u for a car that encounters a slope that is so steep that the throttle saturates. The controller output is a dashed line. The controller parameters are $k_p = 0.5$, $k_i = 0.1$ and $k_{aw} = 2.0$. The anti-windup compensator eliminates the overshoot by preventing the error from building up in the integral term of the controller.

Avoiding Integral Windup

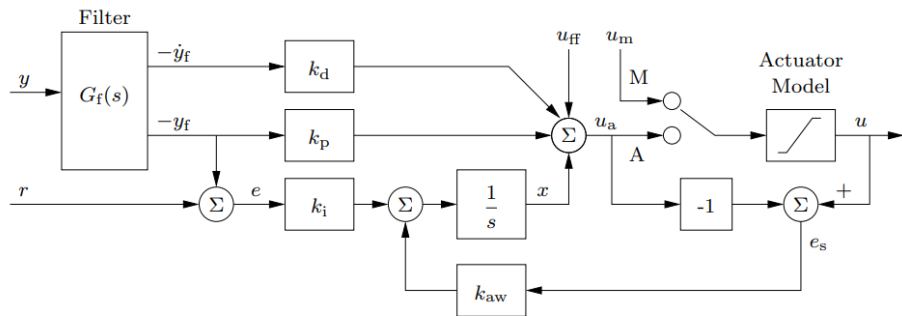


Figure: Anti-windup PID controller with output filtering, feedforward input u_{ff} , and input saturation error e_s

- ▶ The controller has an extra feedback path from the saturating actuator to measure saturation error $e_s = u - u_a$
- ▶ When the actuator saturates, the saturation error e_s is fed back to the integrator to reduce the integral error

Avoiding Derivative Noise

- ▶ Derivative control requires differentiation of the error signal:

$$\dot{e}(t) \approx \frac{e(t) - e(t - \tau)}{\tau}$$

- ▶ In practice, the error signal is measured and contains high-frequency noise, which should not be differentiated
- ▶ The derivative term $k_d s$ is implemented using a low-pass filter $H_d(s) = \frac{1}{\tau_f s + 1}$ with a small filter time constant τ_f
- ▶ PID control with high-frequency noise attenuation:

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \dot{e}_f(t) \quad C(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{\tau_f s + 1}$$
$$\tau_f \dot{e}_f(t) = -e_f(t) + e(t)$$

Discrete-time PID Control Implementation

- ▶ sampling interval: τ_s
- ▶ filter time constant: τ_f
- ▶ sampled error: $e[k] = e(k\tau_s)$
- ▶ filtered error: $e_f[k] = \frac{\tau_s}{\tau_f} e[k] + \left(1 - \frac{\tau_s}{\tau_f}\right) e_f[k - 1]$
- ▶ derivative error: $e_d[k] = \frac{e_f[k] - e_f[k-1]}{\tau_s}$
- ▶ integral error: $e_i[k] = e_i[k - 1] + \tau_s e[k - 1]$
- ▶ control: $u[k] = k_p e[k] + k_i e_i[k] + k_d e_d[k]$

Outline

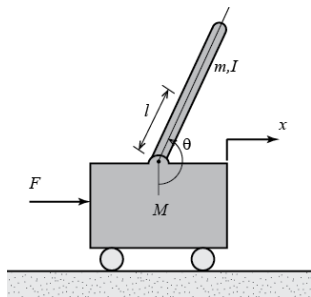
PID Control

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Inverted Pendulum Example

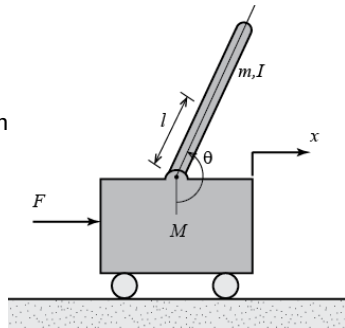
Inverted Pendulum Example

- ▶ Consider an inverted pendulum mounted on a motorized cart
- ▶ **Objective:** control the cart force to balance the inverted pendulum in an upright position
- ▶ Popular example in control theory and reinforcement learning
- ▶ Nonlinear system that is unstable without control



Inverted Pendulum: Parameters

- ▶ Cart mass: $M = 0.5$ kg
- ▶ Pendulum mass: $m = 0.2$ kg
- ▶ Cart friction coefficient: $b = 0.1$ N/m/sec
- ▶ Length to pendulum center of mass: $\ell = 0.3$ m
- ▶ Pendulum moment of inertia:
 $I = 0.006$ kg m²
- ▶ Cart input force: F
- ▶ Cart position: x
- ▶ Pendulum angle: θ



Inverted Pendulum: System Model

- ▶ Horizontal direction force balance for the cart:

$$M\ddot{x} + b\dot{x} + N = F$$

- ▶ Horizontal direction force balance for the pendulum:

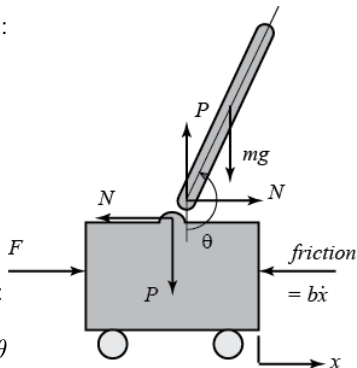
$$N = m\ddot{x} + ml\ddot{\theta} \cos \theta - ml\dot{\theta}^2 \sin \theta$$

- ▶ Force balance perpendicular to the pendulum:

$$P \sin \theta + N \cos \theta - mg \sin \theta = ml\ddot{\theta} + m\ddot{x} \cos \theta$$

- ▶ Torque balance about the pendulum centroid:

$$-P\ell \sin \theta - N\ell \cos \theta = I\ddot{\theta}$$



Inverted Pendulum: System Model

- ▶ Eliminating reaction force N and normal force P and denoting the input force F by u , we get the cart-pole equations of motion:

$$(M + m)\ddot{x} + b\dot{x} + m\ell\ddot{\theta} \cos \theta - m\ell\dot{\theta}^2 \sin \theta = u$$

$$(I + m\ell^2)\ddot{\theta} + mgl \sin \theta = -m\ell\ddot{x} \cos \theta$$

- ▶ Since our control techniques apply to linear time-invariant systems only, we need to linearize the equations of motion
- ▶ Linearize about the upright pendulum position $\theta_e = \pi$ and assume that the pendulum remains within a small neighborhood: $\phi = \theta - \pi$
- ▶ Small angle approximation:

$$\cos \theta = \cos(\pi + \phi) \approx -1 \quad \sin \theta = \sin(\pi + \phi) \approx -\phi \quad \dot{\theta}^2 = \dot{\phi}^2 \approx 0$$

- ▶ Linearized equations of motion:

$$(M + m)\ddot{x} + b\dot{x} - m\ell\ddot{\phi} = u$$

$$(I + m\ell^2)\ddot{\phi} - mgl\phi = m\ell\ddot{x}$$

Inverted Pendulum: Transfer Function

- ▶ Laplace transform of the equations of motion with zero initial conditions:

$$(M + m)s^2X(s) + bsX(s) - mls^2\Phi(s) = U(s)$$

$$(I + ml^2)s^2\Phi(s) - mgl\Phi(s) = mls^2X(s)$$

- ▶ Eliminating $X(s)$ leads to:

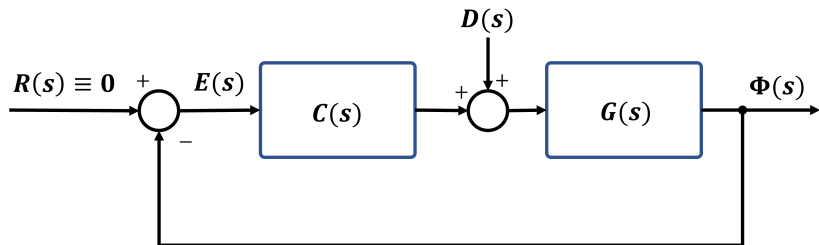
$$(M + m) \left(\frac{I + ml^2}{ml} - \frac{g}{s^2} \right) s^2\Phi(s) + b \left(\frac{I + ml^2}{ml} - \frac{g}{s^2} \right) s\Phi(s) - mls^2\Phi(s) = U(s)$$

- ▶ Pendulum transfer function with $q = (M + m)(I + ml^2) - (ml)^2$:

$$G(s) = \frac{\Phi(s)}{U(s)} = \frac{mls^2}{qs^4 + b(I + ml^2)s^3 - (M + m)mgl s^2 - bmgls}$$

Inverted Pendulum: PID Control

- ▶ Design a controller $C(s)$ to maintain the pendulum vertically upward when the cart input F is subjected to a 1-Nsec impulse disturbance $D(s)$
- ▶ Design specifications:
 - ▶ Settling time of less than 5 seconds
 - ▶ Maximum pendulum deviation from the vertical position of 0.05 rad



Inverted Pendulum: PID Control

- Pendulum transfer function with $q = (M + m)(l + ml^2) - (ml)^2$:

$$G(s) = \frac{\Phi(s)}{U(s)} = \frac{m l s^2}{q s^4 + b(l + ml^2)s^3 - (M + m)mg l s^2 - b m g l s}$$

```
M = 0.5; m = 0.2; b = 0.1; I = 0.006;  
g = 9.8; l = 0.3; q = (M+m)*(I+m*l^2)-(m*l)^2;  
s = tf('s');  
G = (m*l*s^2)/(q*s^4 + b*(I + m*l^2)*s^3 - (M + m)*m*g*l*s^2 - b*m*g*l*s);
```

- PID control design: $C(s) = k_p + k_i \frac{1}{s} + k_d s$

```
Kp = 100; Ki = 1; Kd = 1;  
C = pid(Kp,Ki,Kd);
```

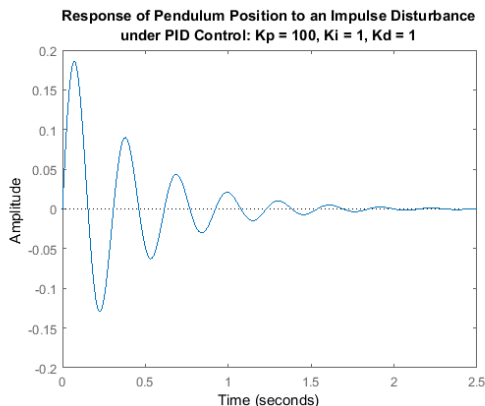
- Closed-loop transfer function from $D(s)$ to $\Phi(s)$:

$$T(s) = \frac{\Phi(s)}{D(s)} = \frac{G(s)}{1 + C(s)G(s)}$$

```
T = feedback(G,C);
```

Inverted Pendulum: PID Control

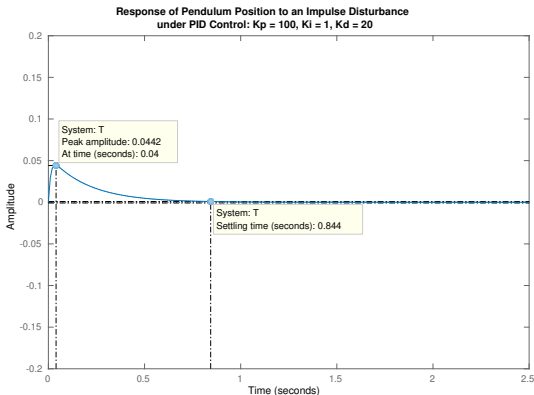
```
1 t=0:0.01:10;  
2 impulse(T,t)  
3 axis([0, 2.5, -0.2, 0.2]);  
title({'Response of Pendulum Position to an Impulse Disturbance'; 'under PID  
Control: Kp = 100, Ki = 1, Kd = 1'});
```



- ▶ **Settling time:** 1.64 sec meets the specifications (no additional integral control is needed)
- ▶ **Peak response:** 0.2 rad exceeds the requirement of 0.05 rad (the overshoot can be reduced by increasing the derivative control gain)

Inverted Pendulum: PID Control

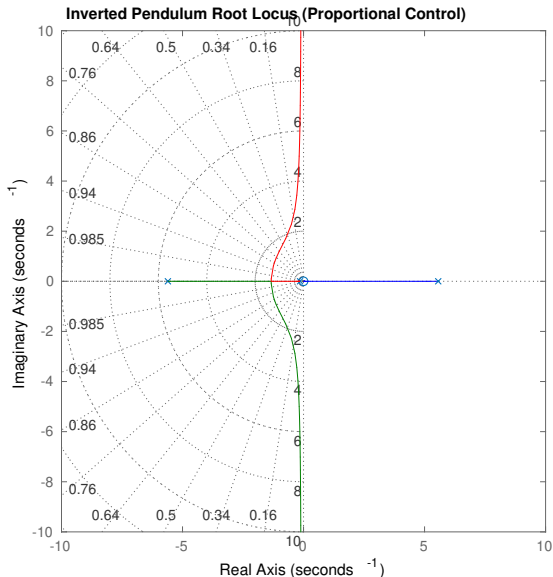
```
t=0:0.01:10;  
impulse(T,t)  
axis([0, 2.5, -0.2, 0.2]);  
title({'Response of Pendulum Position to an Impulse Disturbance';  
      'under PID Control: Kp = 100, Ki = 1, Kd = 20'});
```



- ▶ **Settling time:** 0.844 sec meets the specifications
- ▶ **Peak response:** 0.044 rad meets the specifications

Inverted Pendulum: Root Locus with Proportional Control

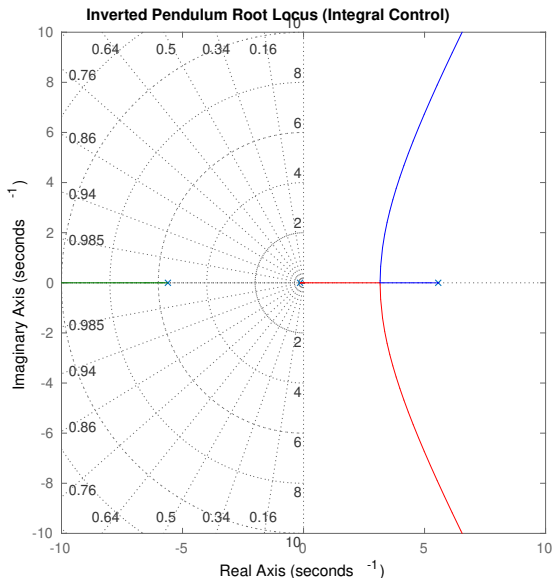
- ▶ Positive root locus for the inverted pendulum plant $G(s)$



- ▶ One branch entirely in the right half-plane
- ▶ Need to add a pole at the origin to cancel the plant zero at the origin
- ▶ This will produce two closed-loop poles in the right half-plane that we can then draw to the left-half plane to stabilize the closed-loop system

Inverted Pendulum: Root Locus with Integral Control

- ▶ Positive root locus for integral control of the inverted pendulum $\frac{1}{s}G(s)$



- ▶ We need to draw the two branches to the left-half plane to stabilize the closed-loop system
- ▶ Adding a zeros to the controller will pull the branches to the left

Inverted Pendulum: Root Locus Manipulation

- ▶ Poles and zeros of $\frac{1}{s}G(s) = \frac{m\ell s^2}{qs^5 + b(I+m\ell^2)s^4 - (M+m)mgl s^3 - bmgl s^2}$:

$$z_1 = z_2 = 0$$

$$p_1 = p_2 = 0, \quad p_3 = -0.143, \quad p_4 = -5.604 \quad p_5 = 5.565$$

- ▶ Suppose we introduce a zero to the controller: $\frac{(s-z_3)}{s}G(s)$

- ▶ There will be $5 - 3 = 2$ asymptotes with angles $\frac{\pi}{2}$, $\frac{3\pi}{2}$ and centroid:

$$\alpha = \frac{1}{2}(-5.604 + 5.565 - 0.143 - z_3) = -\frac{0.182 + z_3}{2}$$

- ▶ We cannot have z_3 in the right half-plane so the best we can do to pull the root locus branches is to have $z_3 \approx 0$ so that $\alpha \approx -0.1$.
- ▶ The real parts of the two poles $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ will approach $\alpha \approx -0.1$ as $K \rightarrow \infty$
- ▶ This design is insufficient to meet the settling time specification:

$$t_s \approx \frac{4}{\zeta\omega_n} \approx \frac{4}{0.1} = 40 \text{ s}$$

Inverted Pendulum: Root Locus Manipulation

- ▶ Adding a single zero to the controller is not sufficient to pull the root locus branches far enough to the left
- ▶ Add two zeros between $p_3 = -0.143$ and $p_4 = -5.604$ to pull the root locus branches towards them, leaving a single asymptote at $-\pi$
- ▶ Let $z_3 = -3$ and $z_4 = -4$ and consider the controller:

$$C(s) = \frac{(s+3)(s+4)}{s} = 7 + 12\frac{1}{s} + s$$

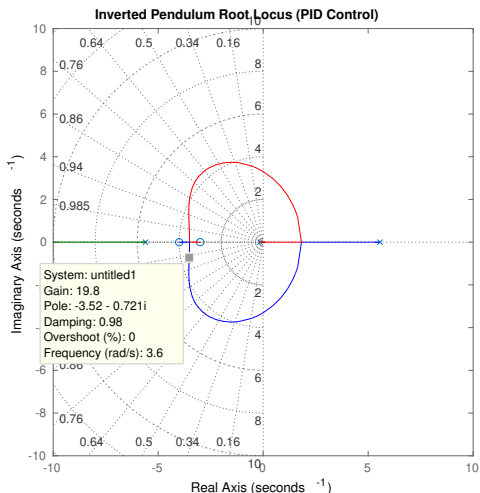
- ▶ Note that $kC(s)$ is a PID controller:

$$k_p = 7k \quad k_i = 12k \quad k_d = k$$

Inverted Pendulum: Root Locus with PID Control

- ▶ Positive root locus for PID control of the inverted pendulum:

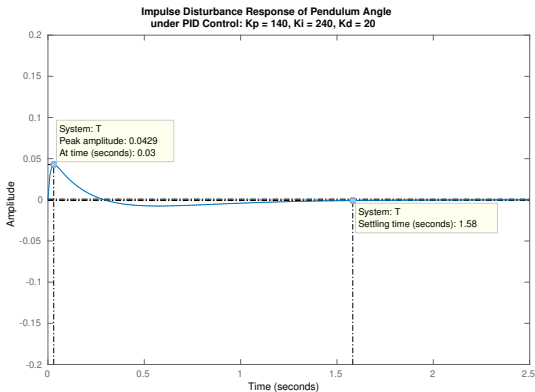
$$\frac{(s + 3)(s + 4)}{s} G(s)$$



- ▶ To achieve $t_s \leq 5$ sec, we need the real parts of the dominant closed-loop poles to be less than $-4/5 = -0.8$
- ▶ To ensure that p.o. $\leq 5\%$, we also need sufficient damping for the dominant closed-loop poles
- ▶ Placing the dominant poles near the real axis increases the damping ratio ζ
- ▶ Choose $k \approx 20$

Inverted Pendulum: PID Control

```
T = feedback(G,20*(s+3)*(s+4)/s);  
t=0:0.01:10;  
impulse(T,t);  
title({'Impulse Disturbance Response of Pendulum Angle'; 'under PID Control: Kp  
= 140, Ki = 240, Kd = 20'});
```



- ▶ **Settling time:** 1.580 sec meets the specifications
- ▶ **Peak response:** 0.043 rad meets the specifications