ECE171A: Linear Control System Theory
Lecture 14: Lead-Lag Compensation

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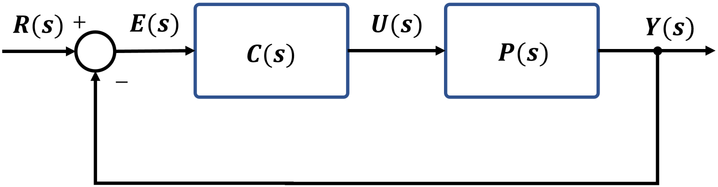
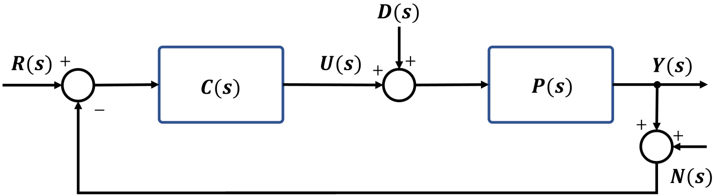
Loop Shaping



- **Loop shaping**: a trial and error procedure to choose a controller $C(s)$ that gives a loop transfer function $L(s) = C(s)P(s)$ with a desired shape
- **Backward method**:
  - Determine a desired loop transfer function $L(s)$
  - Compute the controller as $C(s) = L(s)/P(s)$
- **Forward method**:
  - Adjust proportional gain $C(s) = k_{\mathrm{p}}$ to obtain desired closed-loop bandwidth
  - Add stable poles and zeros to $C(s)$ until a desired shape of $L(s)$ is obtained

Design Considerations



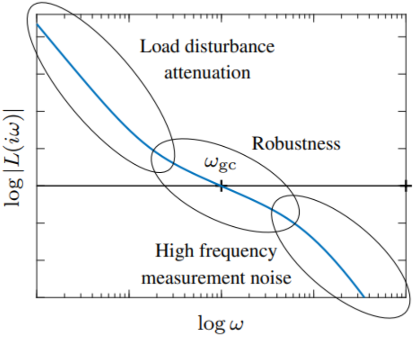
- Tracking error with input disturbance and measurement noise:

$$E(s) = \underbrace{\frac{1}{1+L(s)}}_{\text{Sensitivity } S(s)} R(s) - \frac{P(s)}{1+L(s)} D(s) + \underbrace{\frac{L(s)}{1+L(s)}}_{\text{Complementary Sensitivity } T(s)} N(s)$$

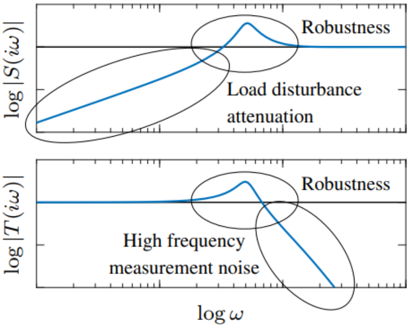
- We need a loop transfer function $L(s) = C(s)P(s)$ that leads to good **closed-loop performance** and good **stability margins**
  - $|L(s)|$ should be large at low frequencies $s = j\omega$ to ensure good reference tracking and low sensitivity to input disturbances (associated with low ω)
  - $|L(s)|$ should be small at high frequencies $s = j\omega$ to ensure low sensitivity to measurement noise (associated with high ω)

Design Considerations

- An ideal loop transfer function $L(j\omega)$ should have the shape below:
  - Unit gain at gain crossover: $|L(j\omega_g)| = 1$
  - Large gain at $\omega < \omega_g$
  - Small gain at $\omega > \omega_g$



(a) Gain plot of loop transfer function



(b) Gain plot of sensitivity functions

- The **phase margin is inversely proportional to the slope of** $L(j\omega)$ **around gain crossover frequency** ω_g (transition from high gain at low ω to low gain at high ω cannot be too fast)

Loop Shaping via Lead and Lag Compensation

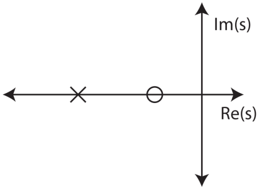
- Loop shaping is a trial-and-error procedure
- Start with a Bode plot of the plant transfer function $P(s)$
- Adjust the **proportional gain** to choose the gain crossover frequency ω_g (compromise between disturbance attenuation and measurement noise)
- Add left-half-plane poles and zeros to $C(s)$ to shape $L(s)$
- The behavior around ω_g can be changed by **lead compensation**
- The loop gain at low frequencies can be increased by **lag compensation**

Lead and Lag Compensation

- Consider a controller with transfer function:

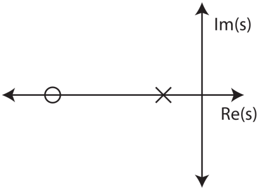
$$C(s) = k\frac{s+z}{s+p} \qquad z > 0,\ p > 0$$

- **Lead compensator**: $z < p$
  - Adds **phase lead** in the frequency range $\omega \in [z, p]$
  - Provides **additional phase margin** at ω_g
  - Equivalent to PD control with filtering
  - Root locus branches move left



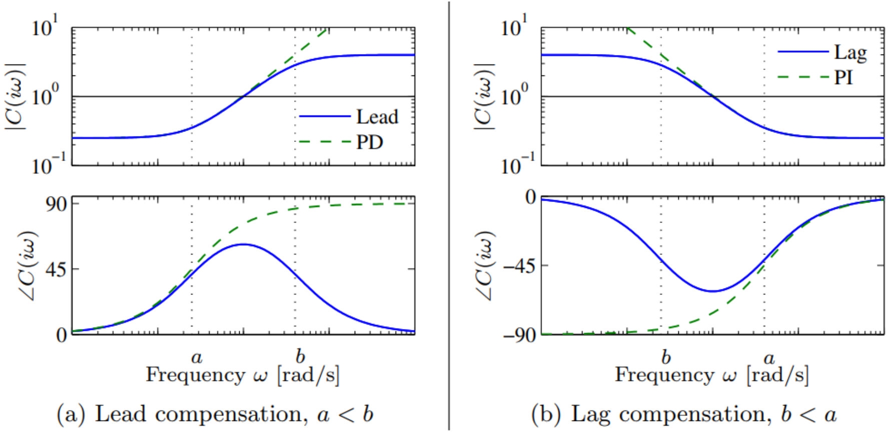
(when we want a zero)

- **Lag compensator**: $z > p$
  - **Increases the gain at low frequencies** leading to improved tracking and disturbance attenuation
  - PI control is a special case with $p = 0$
  - Root locus branches move right



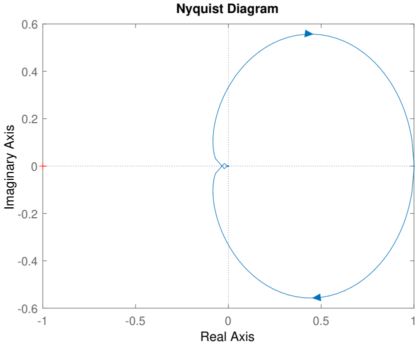
(when we want a pole)

Lead and Lag Compensation

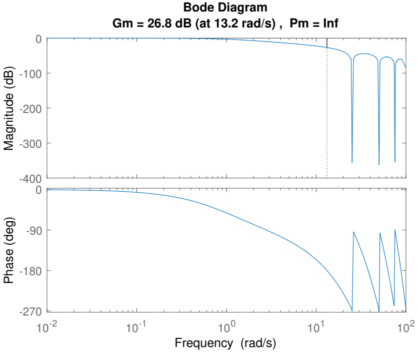


Example 1

- Plant: $P(s) = \dfrac{4(1 - e^{-s/4})}{s(s+1)}$



(a) Nyquist plot



(b) Bode plot

Example 1: Tracking Performance

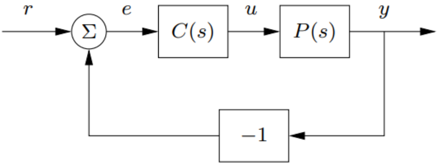
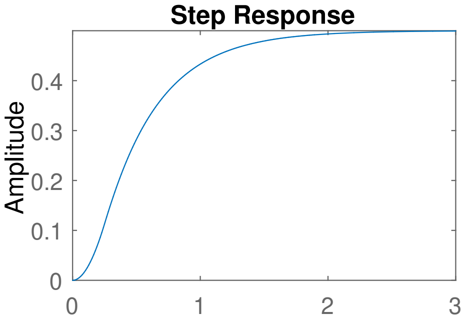
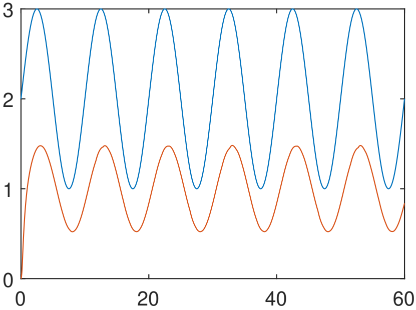


Figure: Proportional control: $C(s) = 1$



(a) Step response



(b) Frequency response at $\omega = \pi/5$

Example 1: Lag Compensation

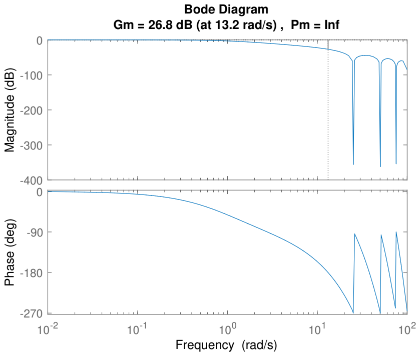


Figure: $P(s)$

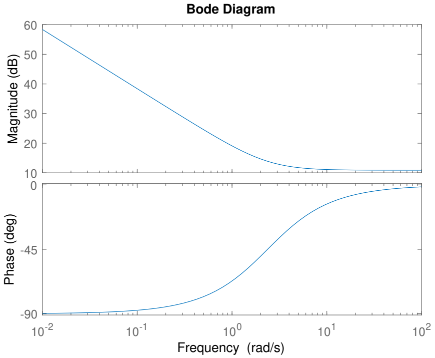


Figure: Lag compensator $C(s) = 3.5 + \dfrac{8.3}{s}$ (PI)

Example 1: Lag Compensation

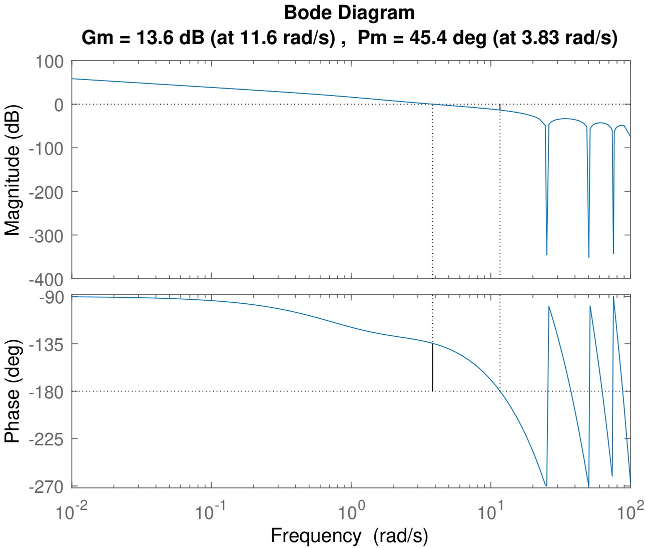


Figure: Margins for $L(s) = C(s)P(s)$

Example 1: Lag Compensation

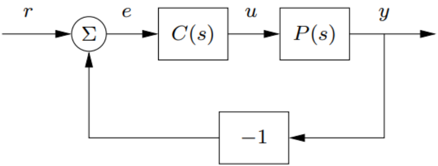
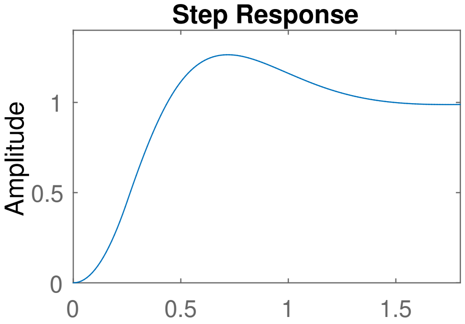
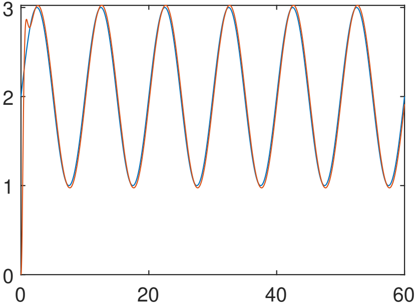


Figure: Lag compensator $C(s) = k_p + \frac{k_i}{s}$



(a) Step response



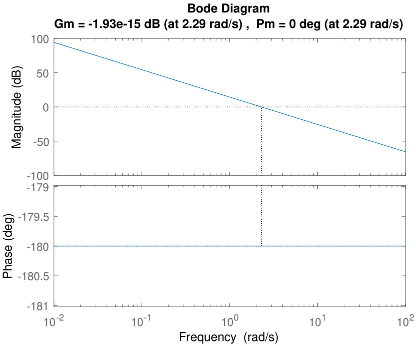
(b) Frequency response at $\omega = \pi/5$

Example 2

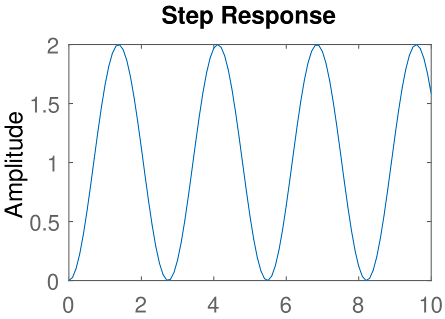
- Plant:

$$P(s) = \frac{r}{Js^2}, \qquad r = 0.25, \quad J = 0.0475$$

- Objectives:
  - Steady-state step error at most 1%
  - Tracking error with $\omega \leq 10$ rad/s at most 10%



(a) Bode plot



(b) Step response for unit negative feedback

Example 2: Lead Compensation

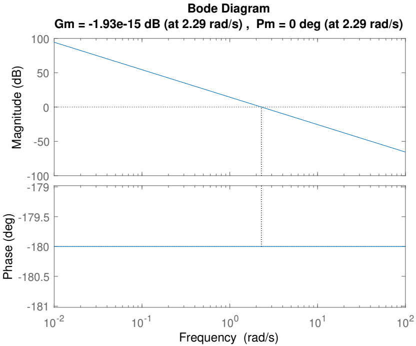


Figure: $P(s)$

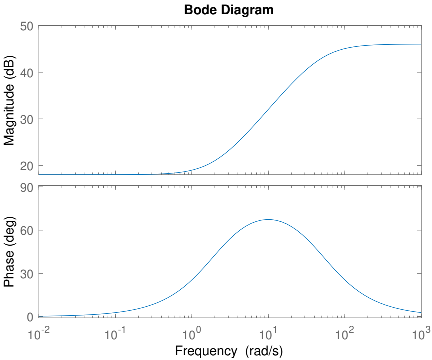


Figure: Lead compensator $C(s) = 200\dfrac{s+2}{s+50}$

Example 2: Lead Compensation

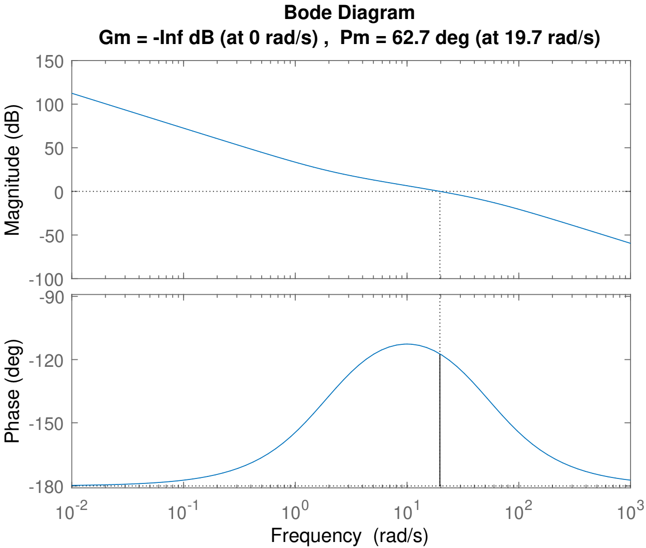
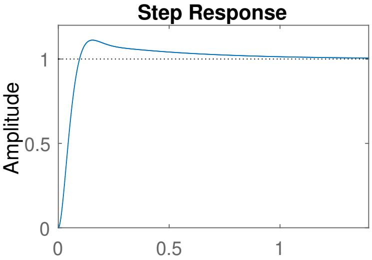
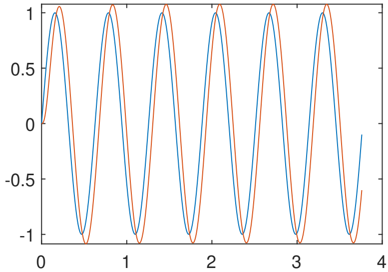


Figure: Margins for $L(s) = C(s)P(s)$

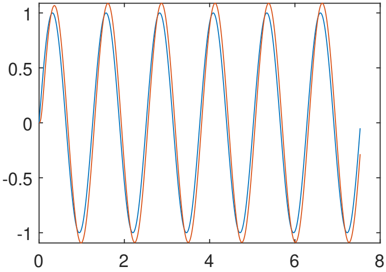
Example 2: Lead Compensation



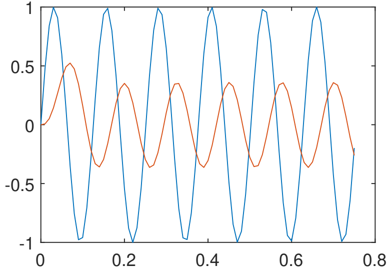
(a) Step response



(b) Frequency response $\omega = 10$



(c) Frequency response $\omega = 1$



(d) Frequency response $\omega = 50$

Example 3

- Plant:
$$P(s) = \frac{1}{s(s+1)}$$

- Objectives:
  - Percent overshoot of at most 20% $\quad\Rightarrow\quad \zeta \geq 0.5$
  - Settling time of at most 4 sec $\quad\Rightarrow\quad \zeta\omega_n \geq 1$

- Desired closed-loop poles: $s_{1,2} = -1 \pm j\sqrt{3}$

- Can we place $s_{1,2}$ on the root locus using lead-lag compensation?

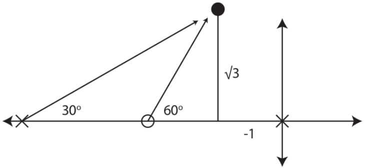
Example 3

- Is $s_1 = -1 + j\sqrt{3}$ already on the Root Locus?
- Check via the **phase condition**:

$$\angle G(s_1) = -\angle s_1 - \angle s_1 + 1 = -120^\circ - 90^\circ = -210^\circ$$

- s_1 is not on the Root Locus and lacks 30° of phase
- Need to add 30° at s_1
- Add a zero at 60° and a pole at 30°:

$$\tan 60^\circ = \frac{\sqrt{3}}{z-1} \qquad \tan 30^\circ = \frac{\sqrt{3}}{p-1}$$

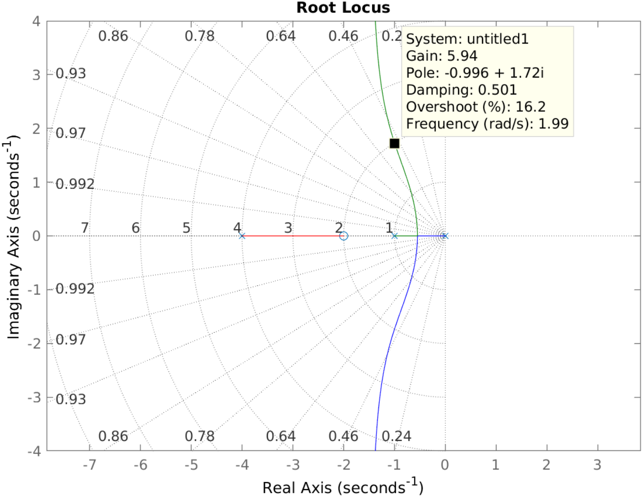


- **Lead compensator**:

$$C(s) = \frac{s+2}{s+4}$$

Example 3

- Root locus of $L(s) = C(s)P(s) = \dfrac{s+2}{s(s+1)(s+4)}$



- Final control design: $C(s) = 6\dfrac{s+2}{s+4}$