ECE171A: Linear Control System Theory
Lecture 4: ODE Solutions

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Outline

Outline

Examples

Linear Properties of LTI Systems

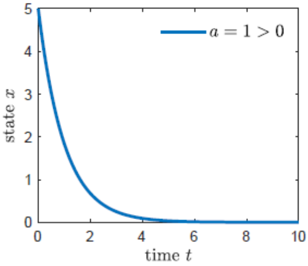
LTI ODE Solution

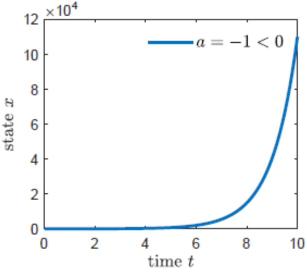
Example 1: Scalar System

- Consider the scalar system:

$$\dot{x} = -ax, \qquad x(0) = x_0$$

- Its unique solution is $x(t) = e^{-at}x_0$





Example 2: Decoupled Two-Dimensional System

- Consider the system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Its unique solution is:

$$x_1(t) = e^{-at} x_1(0), \qquad x_2(t) = e^{-bt} x_2(0)$$

- Note the vector form of the solution:

$$\mathbf{x}(t) = \begin{bmatrix} e^{-at} & 0 \\ 0 & e^{-bt} \end{bmatrix} \mathbf{x}(0)$$

Example 3: Double Integrator

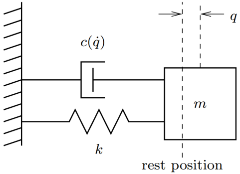
- Consider the system with constant $a \in \mathbb{R}$:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ a \end{bmatrix}$$

- Interpret the system state as position $x_1(t)$ and velocity $x_2(t)$
- Determine the velocity solution first
- The unique solution is:

$$\begin{aligned} x_1(t) &= x_1(0) + x_2(0)t + \frac{1}{2}at^2 \\ x_2(t) &= x_2(0) + at \end{aligned}$$

Example 4: Damped Oscillator (Spring-Mass System)



$m =$ mass

$F =$ external force

$c =$ friction (damper)

$k =$ spring stiffness

$q =$ position

- **System model**: from Newton's second law:

$$m\ddot{q} + c\dot{q} + kq = F$$

- **Free response**: let $F = 0$:

$$m\ddot{q} + c\dot{q} + kq = 0 \quad \Rightarrow \quad \ddot{q} + \frac{c}{m}q + \frac{k}{m}q = 0$$

- Introduce damping ratio ζ and natural frequency ω_0 parameters:

$$2\zeta\omega_0 = \frac{c}{m}, \quad \omega_0^2 = \frac{k}{m} \quad \Rightarrow \quad \ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = 0$$

Example 4: Damped Oscillator (Spring-Mass System)

- State variables:

$$x_1 = q, \qquad x_2 = \frac{\dot{q}}{\omega_0}$$

- State-space model:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega_0 x_2 \\ -\omega_0 x_1 - 2\zeta\omega_0 x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Assume $\zeta < 1$ (underdamped oscillator) and define the damped frequency:

$$\omega_{\mathrm{d}} = \omega_0\sqrt{1 - \zeta^2}$$

- The unique solution is:

$$x_1(t) = e^{-\zeta\omega_0 t}\left(x_1(0)\cos(\omega_{\mathrm{d}} t) + a_1 \sin(\omega_{\mathrm{d}} t)\right)$$
$$x_2(t) = e^{-\zeta\omega_0 t}\left(x_2(0)\cos(\omega_{\mathrm{d}} t) + a_2 \sin(\omega_{\mathrm{d}} t)\right)$$

where a_1, a_2 are constants depending on the initial conditions $x_1(0)$, $x_2(0)$:

$$a_1 = \frac{1}{\omega_{\mathrm{d}}}(\omega_0\zeta x_1(0) + x_2(0)), \qquad a_2 = -\frac{1}{\omega_{\mathrm{d}}}(\omega_0^2 x_1(0) + \omega_0\zeta x_2(0))$$

Example 4: Damped Oscillator (Spring-mass System)

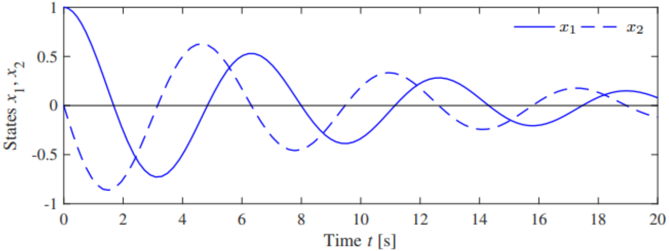


Figure 5.1: Response of the damped oscillator to the initial condition $x_0 = (1, 0)$. The solution is unique for the given initial conditions and consists of an oscillatory solution for each state, with an exponentially decaying magnitude.

Outline

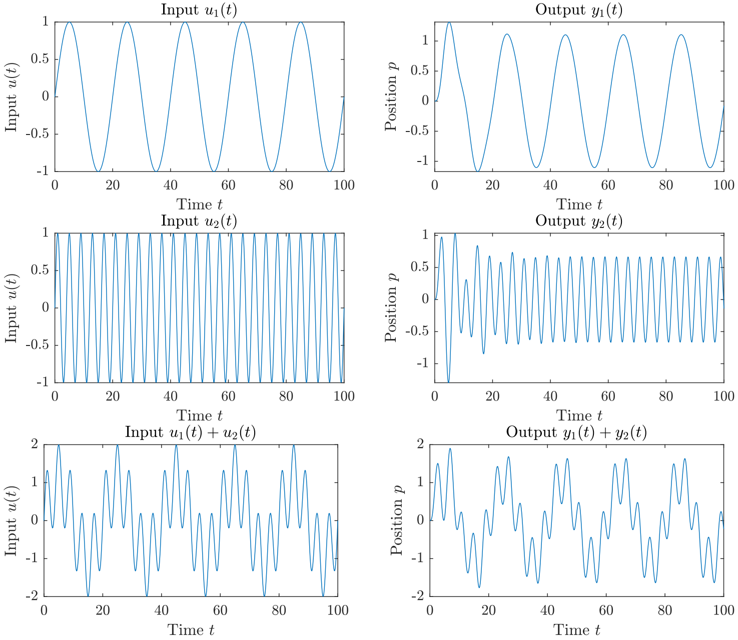
LTI ODE System

- Consider the LTI ODE system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- The output $\mathbf{y}(t)$ satisfies **linear properties**:
  - **Case 1**: Zero initial state $\mathbf{x}(0) = \mathbf{0}$: *the output* $\mathbf{y}(t)$ *is linear in input* $\mathbf{u}(t)$
  - **Case 2**: Zero input $\mathbf{u}(t) \equiv 0$: *the output* $\mathbf{y}(t)$ *is linear in initial state* $\mathbf{x}(0)$

Case 1: Zero Initial State $\mathbf{x}(0) = \mathbf{0}$



Case 1: Zero Initial State $\mathbf{x}(0) = \mathbf{0}$

Zero initial state $\mathbf{x}(0) = \mathbf{0}$: *the output* $\mathbf{y}(t)$ *is linear in input* $\mathbf{u}(t)$

$$\begin{cases} \mathbf{u}_1(t) \to \mathbf{y}_1(t) \\ \mathbf{u}_2(t) \to \mathbf{y}_2(t) \end{cases} \quad \Longrightarrow \quad \alpha\mathbf{u}_1(t) + \beta\mathbf{u}_2(t) \to \alpha\mathbf{y}_1(t) + \beta\mathbf{y}_2(t)$$

Proof:

- Denote the state trajectory for $\mathbf{u}_1(t)$ as $\mathbf{x}_1(t)$, and for $\mathbf{u}_2(t)$ as $\mathbf{x}_2(t)$:

$$\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t) + \mathbf{B}\mathbf{u}_1(t), \qquad \dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t) + \mathbf{B}\mathbf{u}_2(t)$$

- Let $\mathbf{u}(t) = \alpha\mathbf{u}_1(t) + \beta\mathbf{u}_2(t)$ and verify $\mathbf{x}(t) = \alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)$ is a solution:
  - **Initial condition**: $\mathbf{x}(0) = \alpha\mathbf{x}_1(0) + \beta\mathbf{x}_2(0) = 0$
  - **ODE**:

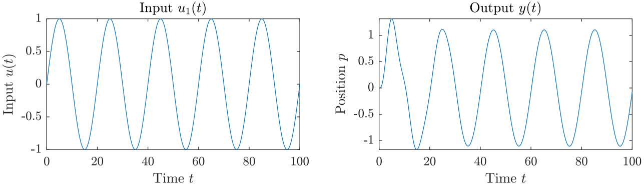
$$\begin{aligned} \dot{\mathbf{x}} = \alpha\dot{\mathbf{x}}_1 + \beta\dot{\mathbf{x}}_2 &= \alpha(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \beta(\mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2) \\ &= \mathbf{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) + \mathbf{B}(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2) \\ &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \end{aligned}$$

- Hence, the output corresponding to $\mathbf{u} = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2$ is:

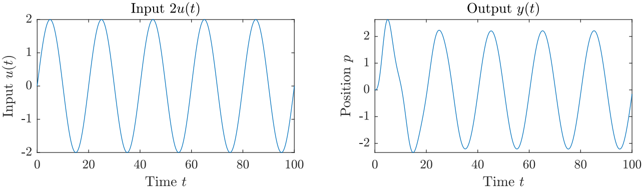
$$\begin{aligned} \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} &= \mathbf{C}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) + \mathbf{D}(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2) \\ &= \alpha(\mathbf{C}\mathbf{x}_1 + \mathbf{D}\mathbf{u}_1) + \beta(\mathbf{C}\mathbf{x}_2 + \mathbf{D}\mathbf{u}_2) = \alpha\mathbf{y}_1 + \beta\mathbf{y}_2 \end{aligned}$$

Case 1: Zero Initial State $\mathbf{x}(0) = \mathbf{0}$

- Consider an LTI ODE with zero initial condition
- Suppose that with input $\mathbf{u}(t)$, the output is $\mathbf{y}(t)$



- If the input is $2\mathbf{u}(t)$, what is the output?



- If the input amplitude is doubled, then the output amplitude is also doubled

Case 2: Zero Input $\mathbf{u}(t) \equiv \mathbf{0}$

Zero input $\mathbf{u}(t) \equiv \mathbf{0}$: *the output* $\mathbf{y}(t)$ *is linear in the initial state* $\mathbf{x}(0)$

$$\begin{cases} \mathbf{x}_1(0) = \boldsymbol{\xi}_1 \to \mathbf{y}_1(t) \\ \mathbf{x}_2(0) = \boldsymbol{\xi}_2 \to \mathbf{y}_2(t) \end{cases} \implies \mathbf{x}_3(0) = \alpha\boldsymbol{\xi}_1 + \beta\boldsymbol{\xi}_2 \to \alpha\mathbf{y}_1(t) + \beta\mathbf{y}_2(t)$$

Proof:

- Denote the state trajectory for $\boldsymbol{\xi}_1$ as $\mathbf{x}_1(t)$, and for $\boldsymbol{\xi}_2$ as $\mathbf{x}_2(t)$:

$$\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t), \qquad \dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t)$$

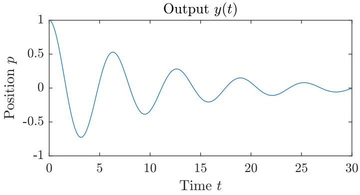
- Verify that $\mathbf{x}_3(t) = \alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)$ is a solution:
  - **Initial condition**: $\mathbf{x}_3(0) = \alpha\mathbf{x}_1(0) + \beta\mathbf{x}_2(0) = \alpha\boldsymbol{\xi}_1 + \beta\boldsymbol{\xi}_2$
  - **ODE**:

$$\dot{\mathbf{x}}_3 = \alpha\dot{\mathbf{x}}_1 + \beta\dot{\mathbf{x}}_2 = \alpha\mathbf{A}\mathbf{x}_1 + \beta\mathbf{A}\mathbf{x}_2 = \mathbf{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \mathbf{A}\mathbf{x}_3$$

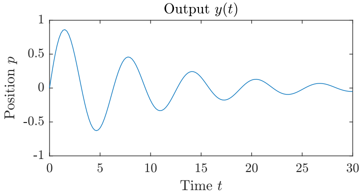
- Hence, the output corresponding to $\mathbf{x}_3(t) = \alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)$ is:

$$\mathbf{y}_3(t) = \mathbf{C}\mathbf{x}_3(t) = \mathbf{C}(\alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)) = \alpha\mathbf{y}_1(t) + \beta\mathbf{y}_2(t)$$

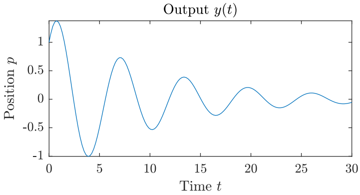
Case 2: Zero Input $\mathbf{u}(t) \equiv \mathbf{0}$



(a) Initial condition $\mathbf{x}(0) = \boldsymbol{\xi}_1$



(b) Initial condition $\mathbf{x}(0) = \boldsymbol{\xi}_2$



(c) Initial condition $\mathbf{x}(0) = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2$

Outline

Homogeneous LTI ODE Solution

- Consider the homogeneous scalar LTI ODE:

$$\dot{x} = ax, \qquad x(0) = x_0$$

- Its solution is:

$$x(t) = e^{at} x_0$$

- Consider the homogeneous vector LTI ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

- What is the solution?

Homogeneous LTI ODE Solution

Theorem

The homogeneous vector linear time-invariant ordinary differential equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0$$

Definition

The exponential function of a matrix $\mathbf{X} \in \mathbb{R}^{n \times n}$ is defined as:

$$e^{\mathbf{X}} = \mathbf{I} + \mathbf{X} + \frac{1}{2}\mathbf{X}^2 + \frac{1}{3!}\mathbf{X}^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{X}^k,$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

- **Note**: the solution $\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0$ is linear in the initial condition $\mathbf{x}_0$

Proof

- **Initial condition**:

$$\mathbf{x}(t_0) = e^{\mathbf{A}(t_0 - t_0)}\mathbf{x}_0 = e^{\mathbf{0}}\mathbf{x}_0 = \mathbf{x}_0$$

- **ODE**:

$$\begin{aligned}
\frac{d}{dt}\mathbf{x}(t) &= \frac{d}{dt}\left(e^{\mathbf{A}(t-t_0)}\mathbf{x}_0\right) \\
&= \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}(t-t_0) + \frac{1}{2}\mathbf{A}^2(t-t_0)^2 + \frac{1}{3!}\mathbf{A}^3(t-t_0)^3 \cdots\right)\mathbf{x}_0 \\
&= \left(\mathbf{0} + \mathbf{A} + \mathbf{A}^2(t-t_0) + \frac{1}{2!}\mathbf{A}^3(t-t_0)^2 + \cdots\right)\mathbf{x}_0 \\
&= \mathbf{A}\left(\mathbf{I} + \mathbf{A}(t-t_0) + \frac{1}{2!}\mathbf{A}^2(t-t_0)^2 + \frac{1}{3!}\mathbf{A}^3(t-t_0)^3 \cdots\right)\mathbf{x}_0 \\
&= \mathbf{A}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 \\
&= \mathbf{A}\mathbf{x}(t)
\end{aligned}$$

Example: Double Integrator

- Consider a second-order scalar LTI ODE:

$$\ddot{q} = u, \qquad q(0) = q_0, \;\; \dot{q}(0) = v_0$$

- It is called **double integrator** because $u(t)$ is integrated twice before it affects q

- **State-space model**: let $\mathbf{x} = (q, \dot{q})$:

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u, \qquad \mathbf{x}(0) = \mathbf{x}_0 := \begin{bmatrix} q_0 \\ v_0 \end{bmatrix}$$

- Matrix exponential of $\mathbf{A}$:

$$\mathbf{A}^2 = \mathbf{0} \qquad \Rightarrow \qquad e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

- When $u(t) \equiv 0$, the solution of the double integrator system is:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} q_0 + t v_0 \\ v_0 \end{bmatrix}$$

Example: Undamped Oscillator

- Consider a spring-mass system with zero damping:

$$\ddot{q} + \omega_0^2 q = u, \qquad q(0) = q_0, \;\; \dot{q}(0) = v_0$$

- **State-space model**: let $\mathbf{x} = (q, \dot{q}/\omega_0)$:

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u, \qquad \mathbf{x}(0) = \mathbf{x}_0 := \begin{bmatrix} q_0 \\ v_0 \end{bmatrix}$$

- Matrix exponential of $\mathbf{A}t$: $e^{\mathbf{A}t} = \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}$

- This can be verified by differentiation:

$$\frac{d}{dt} e^{\mathbf{A}t} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix} = \mathbf{A} e^{\mathbf{A}t}$$

- When $u(t) \equiv 0$, the solution of the undamped oscillator is:

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}_0 = \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix} \begin{bmatrix} q_0 \\ v_0 \end{bmatrix}$$

Where Does the Homogeneous LTI ODE Solution Come From?

- The solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ should satisfy:

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{A}\mathbf{x}(\tau)d\tau$$

- This is an implicit equation. Replace the expression above into the integral:

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{A}\left(\mathbf{x}_0 + \int_{t_0}^{\tau} \mathbf{A}\mathbf{x}(\tau_1)d\tau_1\right)d\tau \\ &= \left(\mathbf{I} + \mathbf{A}(t-t_0)\right)\mathbf{x}_0 + \int_{t_0}^{t}\int_{t_0}^{\tau} \mathbf{A}^2\mathbf{x}(\tau_1)d\tau_1 d\tau\end{aligned}$$

- Repeat the step above:

$$\begin{aligned}\mathbf{x}(t) &= \left(\mathbf{I} + \mathbf{A}(t-t_0)\right)\mathbf{x}_0 + \int_{t_0}^{t}\int_{t_0}^{\tau} \mathbf{A}^2\left(\mathbf{x}_0 + \int_{t_0}^{\tau_1} \mathbf{A}\mathbf{x}(\tau_2)d\tau_2\right)d\tau_1 d\tau \\ &= \left(\mathbf{I} + \mathbf{A}(t-t_0) + \frac{1}{2}\mathbf{A}^2(t-t_0)^2\right)\mathbf{x}_0 + \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{\tau_1} \mathbf{A}^3\mathbf{x}(\tau_2)d\tau_2 d\tau_1 d\tau \\ &= \ldots = \left(\mathbf{I} + \mathbf{A}(t-t_0) + \frac{1}{2!}\mathbf{A}^2(t-t_0)^2 + \frac{1}{3!}\mathbf{A}^3(t-t_0)^3 + \cdots\right)\mathbf{x}_0\end{aligned}$$

Nonhomogeneous LTI ODE Solution

Theorem

The linear time-invariant ordinary differential equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Proof

- **Initial condition**:

$$\mathbf{x}(t_0) = e^{\mathbf{A}(t_0 - t_0)}\mathbf{x}_0 + \int_{t_0}^{t_0} e^{\mathbf{A}(t_0 - \tau)}\mathbf{B}\mathbf{u}(\tau)d\tau = \mathbf{I}\mathbf{x}_0 + \mathbf{0} = \mathbf{x}_0$$

- **ODE**:

$$\begin{aligned}
\frac{d}{dt}\mathbf{x}(t) &= \frac{d}{dt}\left(e^{\mathbf{A}(t-t_0)}\mathbf{x}_0\right) + \frac{d}{dt}\left(\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\right) \\
&= \mathbf{A}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \frac{d}{dt}\left(e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau\right) \\
&= \mathbf{A}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \left(\mathbf{A}e^{\mathbf{A}t}\right)\left(\int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau\right) + \left(e^{\mathbf{A}t}\right)\left(e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)\right) \\
&= \mathbf{A}\left(e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\right) + \mathbf{B}\mathbf{u}(t) \\
&= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)
\end{aligned}$$

Nonhomogeneous LTI ODE Solution

- Consider the LTI ODE system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- The system output satisfies the **convolution equation**:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

- **Observations**:
  - Due to the linearity of matrix-vector multiplication and integration, the output is **jointly linear** in the initial condition $\mathbf{x}_0$ and the input $\mathbf{u}(t)$
  - The objective of control design is to choose an input signal $\mathbf{u}(t)$ to shape the output $\mathbf{y}(t)$, e.g., to achieve regulation or tracking without overshoot or oscillations and with robustness to noise
  - Using the convolution equation directly for control design can be challenging
  - We will look for a simpler relationship between $\mathbf{u}(t)$ and $\mathbf{y}(t)$ by transforming the LTI ODE from the time domain to the frequency domain using a **Laplace transform**

LTI Difference Equation Solution

Theorem

The linear time-invariant difference equation:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k$$

has a unique solution:

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{A}^{k-j-1}\mathbf{B}\mathbf{u}_j$$

Proof:

- **Base case** (time $k = 1$): $\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 + \mathbf{B}\mathbf{u}_0$
- **Induction hypothesis** (time k): $\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{A}^{k-j-1}\mathbf{B}\mathbf{u}_j$
- **Induction step** (time $k+1$):

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k = \mathbf{A}\left(\mathbf{A}^k\mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{A}^{k-j-1}\mathbf{B}\mathbf{u}_j\right) + \mathbf{B}\mathbf{u}_k$$

$$= \mathbf{A}^{k+1}\mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{A}^{k-j}\mathbf{B}\mathbf{u}_j + \mathbf{B}\mathbf{u}_k = \mathbf{A}^{k+1}\mathbf{x}_0 + \sum_{j=0}^{k} \mathbf{A}^{k-j}\mathbf{B}\mathbf{u}_j$$