

# ECE171A: Linear Control System Theory

## Lecture 5: Transfer Function

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# LTI ODE Solution

- ▶ Consider the LTI ODE system:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, & \mathbf{x}(t_0) &= \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

- ▶ The system output satisfies the **convolution equation**:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

- ▶ **Observations:**

- ▶ Using the convolution equation directly for control design can be challenging
- ▶ A simpler relationship between  $\mathbf{u}(t)$  and  $\mathbf{y}(t)$  can be obtained by transforming the LTI ODE from the time domain to the complex domain using a **Laplace transform**

# Laplace Transform

The **Laplace transform**  $\mathcal{L}$  maps a real function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  to a complex function  $F : \mathbb{C} \mapsto \mathbb{C}$ :

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt$$

- ▶ The Laplace transform  $\mathcal{L}$  converts an LTI ODE in the time domain into a linear algebraic equation in the complex domain
- ▶ Example:

$$\begin{array}{ccc} \ddot{y}(t) + y(t) = 0 & \xrightarrow{\mathcal{L}} & s^2 Y(s) - sy(0) - \dot{y}(0) + Y(s) = 0 \\ & & \downarrow \\ y(t) = y(0) \cos(t) + \dot{y}(0) \sin(t) & \xleftarrow{\mathcal{L}^{-1}} & Y(s) = \frac{sy(0) + \dot{y}(0)}{s^2 + 1} \end{array}$$

# Outline

Complex Numbers and Rational Functions

Polynomial and Rational Functions in MATLAB

Laplace Transform

Transfer Function

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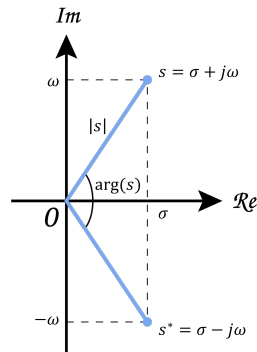
## Complex Numbers $\mathbb{C}$

- ▶ The **space of real numbers** is denoted by  $\mathbb{R}$
- ▶ The **space of complex numbers** is denoted by  $\mathbb{C}$
- ▶ A **complex number** has the form:

$$s = \sigma + j\omega,$$

where  $\sigma, \omega \in \mathbb{R}$  and  $j = \sqrt{-1}$

- ▶ Cartesian coordinates:  $s = \sigma + j\omega$ 
  - ▶ The **real part** of  $s$  is  $\text{Re}(s) = \sigma$
  - ▶ The **imaginary part** of  $s$  is  $\text{Im}(s) = \omega$
- ▶ Polar coordinates:  $s = re^{j\theta} = r(\cos(\theta) + j \sin(\theta))$ 
  - ▶ The **magnitude** of  $s$  is  $|s| = r = \sqrt{\sigma^2 + \omega^2}$
  - ▶ The **phase** of  $s$  is  $\arg(s) = \angle s = \theta = \text{atan2}(\omega, \sigma)$
- ▶ The **complex conjugate** of  $s = \sigma + j\omega$  is  $s^* = \sigma - j\omega$



## Complex Polynomial

- ▶ A **complex polynomial** of order  $n$  is a function  $a : \mathbb{C} \mapsto \mathbb{C}$ :

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_2 s^2 + a_1 s + a_0$$

where  $a_0, a_1, \dots, a_n \in \mathbb{C}$  are constants.

- ▶ A **root** of a complex polynomial  $a(s)$  is a number  $\lambda \in \mathbb{C}$  such that:

$$a(\lambda) = 0$$

- ▶ A root  $\lambda$  of **multiplicity**  $m$  of a complex polynomial  $a(s)$  satisfies:

$$\lim_{s \rightarrow \lambda} \frac{a(s)}{(s - \lambda)^m} < \infty$$

- ▶ **Fundamental theorem of algebra**: a complex polynomial  $a(s)$  of degree  $n$  has exactly  $n$  roots, counting multiplicities, and can be factorized as:

$$a(s) = a_n s^n + \dots + a_0 = a_n (s - \lambda_1) \cdots (s - \lambda_n)$$

where  $\lambda_1, \dots, \lambda_n$  are the  $n$  roots of  $a(s)$

## Complex Polynomial with Real Coefficients

- ▶ A complex polynomial of order  $n$  with real coefficients is a function:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_2 s^2 + a_1 s + a_0$$

where  $a_0, a_1, \dots, a_n \in \mathbb{R}$  are constants.

- ▶ The roots of a complex polynomial with real coefficients are either real,  $\lambda = \sigma$ , or come in complex conjugate pairs,  $\lambda = \sigma \pm j\omega$ .
- ▶ Every complex polynomial with real coefficients can be factorized into polynomials of degree one or two:

$$a(s) = a_n s^n + \dots + a_0 = a_n \prod_{i=1}^{n_1} (s - \lambda_i) \prod_{k=1}^{n_2} (s^2 + 2\zeta_k \omega_k s + \omega_k^2)$$

where  $n_1$  and  $n_2$  are the numbers of real roots and complex conjugate pairs.

- ▶ **Vieta's formulas** relate the coefficients  $a_i$  to the roots  $\lambda_i$ :

$$\sum_{i=1}^n \lambda_i = -\frac{a_{n-1}}{a_n} \quad \prod_{i=1}^n \lambda_i = (-1)^n \frac{a_0}{a_n} \quad \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \prod_{j=1}^k \lambda_{i_j} = (-1)^k \frac{a_{n-k}}{a_n}$$



## Rational Function

- ▶ A **rational function**  $F : \mathbb{C} \mapsto \mathbb{C}$  is a ratio of complex polynomials:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}$$

- ▶ Rational functions remain rational functions under addition, subtraction, multiplication, division (except by 0)
- ▶ The **characteristic equation** of a rational function  $F(s) = \frac{b(s)}{a(s)}$  is:

$$a(s) = 0$$

- ▶ A **zero**  $z \in \mathbb{C}$  of a rational function  $F(s)$  is a root of the numerator:  $b(z) = 0$
- ▶ A **pole**  $p \in \mathbb{C}$  of a rational function  $F(s)$  is a root of the characteristic equation:  $a(p) = 0$

## Pole-Zero Map

- ▶ The **pole-zero form** of a rational function  $F(s)$  is:

$$F(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0} = k \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

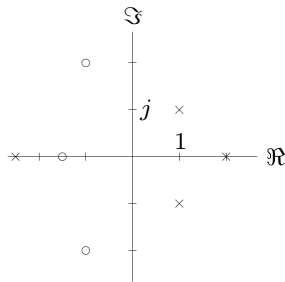
where  $k = b_m/a_n$ ,  $z_1, \dots, z_m$  are the zeros of  $F(s)$ , and  $p_1, \dots, p_n$  are the poles of  $F(s)$

- ▶ A **pole-zero map** is a plot of the poles and zeros of  $F(s)$  in the  $s$ -domain:

- ▶ Example:

$$F(s) = k \frac{(s + 1.5)(s + 1 + 2j)(s + 1 - 2j)}{(s + 2.5)(s - 2)(s - 1 - j)(s - 1 + j)}$$

- ▶  $\times$  = pole;  $\circ$  = zero;  $k$  = not available



## Example: Zeros and Poles

- ▶ Consider  $F(s) = \frac{2s+1}{3s^2+2s+1}$
- ▶  $F(s)$  has one zero:  $z = -\frac{1}{2}$
- ▶ The roots of a quadratic polynomial  $a(s) = a_2s^2 + a_1s + a_0$  are:

$$s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

- ▶  $F(s)$  has two conjugate poles:  $p_1 = -\frac{1}{3} + j\frac{\sqrt{2}}{3}$  and  $p_2 = -\frac{1}{3} - j\frac{\sqrt{2}}{3}$
- ▶ Pole-zero form of  $F(s)$ :

$$F(s) = \frac{2(s - z)}{3(s - p_1)(s - p_2)}$$

## Partial Fraction Expansion (no repeated poles)

- ▶ Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}$$

is **strictly proper** ( $m < n$ ) and has no repeated poles (all roots of  $a(s)$  have multiplicity one)

- ▶ The **residue**  $r_i$  associated with pole  $p_i$  is:

$$r_i = \lim_{s \rightarrow p_i} (s - p_i) F(s)$$

- ▶ The **partial fraction expansion** of  $F(s)$  is:

$$F(s) = \frac{r_1}{s - p_1} + \dots + \frac{r_n}{s - p_n}$$

where  $p_1, \dots, p_n$  and  $r_1, \dots, r_n$  are the poles and residues of  $F(s)$

## Example: Residues

- ▶ Consider  $F(s) = \frac{2s+1}{3s^2+2s+1}$  with zero  $z = -\frac{1}{2}$  and poles  $p_{1,2} = -\frac{1}{3} \pm j\frac{\sqrt{2}}{3}$
- ▶ The residue associated with  $p_1$  is:

$$\begin{aligned}r_1 &= \lim_{s \rightarrow p_1} (s - p_1)F(s) = \lim_{s \rightarrow p_1} \frac{2(s - z)}{3(s - p_2)} = \frac{2(p_1 + 1/2)}{3(p_1 - p_2)} \\ &= \frac{2(p_1 + 1/2)}{j2\sqrt{2}} = -j\frac{\sqrt{2}}{2} \left( \frac{1}{6} + j\frac{\sqrt{2}}{3} \right) = \frac{1}{3} - j\frac{\sqrt{2}}{12}\end{aligned}$$

- ▶ Residues associated with complex conjugate poles are also complex conjugate!
- ▶ The residue associated with  $p_2 = p_1^*$  is  $r_2 = r_1^* = \frac{1}{3} + j\frac{\sqrt{2}}{12}$
- ▶ The partial fraction expansion of  $F(s)$  is:

$$F(s) = \frac{r_1}{(s - p_1)} + \frac{r_2}{(s - p_2)}$$

## Partial Fraction Expansion (repeated poles)

- ▶ Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n (s - p_1)^{m_1} \dots (s - p_k)^{m_k}}$$

is **strictly proper** and has poles  $p_1, \dots, p_k$  with multiplicities  $m_1, \dots, m_k$

- ▶ The **residue**  $r_{i,m_i-j}$  associated with pole  $p_i$  of multiplicity  $m_i$  is:

$$r_{i,m_i-j} = \lim_{s \rightarrow p_i} \frac{1}{j!} \frac{d^j}{ds^j} [(s - p_i)^{m_i} F(s)], \quad j = 0, \dots, (m_i - 1)$$

- ▶ The **partial fraction expansion** of  $F(s)$  is:

$$\begin{aligned} F(s) &= \frac{r_{1,m_1}}{(s - p_1)^{m_1}} + \frac{r_{1,m_1-1}}{(s - p_1)^{m_1-1}} + \dots + \frac{r_{1,1}}{s - p_1} \\ &+ \frac{r_{2,m_2}}{(s - p_2)^{m_2}} + \frac{r_{2,m_2-1}}{(s - p_2)^{m_2-1}} + \dots + \frac{r_{2,1}}{s - p_2} \\ &+ \dots \\ &+ \frac{r_{k,m_k}}{(s - p_k)^{m_k}} + \frac{r_{k,m_k-1}}{(s - p_k)^{m_k-1}} + \dots + \frac{r_{k,1}}{s - p_k} \end{aligned}$$

## Partial Fraction Expansion (improper rational function)

- ▶ Assume that the rational function:

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}$$

is not strictly proper ( $m \geq n$ )

- ▶ The numerator  $b(s)$  can be divided by the denominator  $a(s)$  to obtain:

$$F(s) = \frac{b(s)}{a(s)} = c(s) + \frac{d(s)}{a(s)}$$

where  $c(s)$  is of order  $m - n$  and  $d(s)$  is of order  $k < n$

- ▶  $\frac{d(s)}{a(s)}$  is now strictly proper and has a partial fraction expansion

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## MATLAB Polynomial Functions

- ▶ Consider:

$$p(s) = (s - 11.6219)(s + 0.3110 + 2.6704j)(s + 0.3110 - 2.6704j)$$

- ▶ poly: convert roots to polynomial coefficients:

```
1 r = [11.6219, -0.3110-2.6704i, -0.3110+2.6704i]
  a = poly(r) = [1.0, -11.0, 0.0, -84.0]
```

- ▶ polyval: evaluate a polynomial, e.g.,  $p(1 - 2j)$ :

```
polyval(a, 1-2i) = -62 + 46i
```

- ▶ roots: find polynomial roots:

```
1 roots(a) = [11.6219, -0.3110-2.6704i, -0.3110+2.6704i]
```

- ▶ conv: expand the product of two polynomials, e.g.,  $(3s^2 + 2s + 1)(s + 4)$ :

```
1 conv([3, 2, 1], [1, 4]) = [3, 14, 9, 4]
```

## MATLAB Rational Functions

- ▶  $SYS = \text{zpk}(Z,P,K)$  creates a continuous-time zero-pole-gain (zpk) model  $SYS$  with zeros  $Z$ , poles  $P$ , and gains  $K$ :

```
1 dcmotor = zpk([], [-1], 200);  
  fbksys = zpk([-4], [-8.8426, -2.0787 + 1.7078i, -2.0787 -1.7078i], 8);
```

- ▶  $P = \text{pole}(SYS)$  returns the poles  $P$  of  $SYS$ :

```
sp = pole(fbksys) = [-8.8426, -2.0787 + 1.7078i, -2.0787 -1.7078i]
```

- ▶  $[Z,G] = \text{zero}(SYS)$  computes the zeros  $Z$  and gain  $G$  of  $SYS$ :

```
1 [sz,k] = zero(fbksys) = [-4, 8]
```

- ▶  $\text{pzmap}(SYS)$ : computes and plots the poles and zeros of  $SYS$

```
1 pzmap(fbksys)
```

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## Laplace Transform and Inverse Laplace Transform

- ▶ The **Laplace transform**  $F(s)$  of a function  $f(t)$  is:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt,$$

where  $s = \sigma + j\omega$  is a complex number.

- ▶ The **inverse Laplace transform**  $f(t)$  of a function  $F(s)$  is:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \lim_{\omega \rightarrow \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st} ds,$$

where  $\sigma$  is greater than the real part of all singularities of  $F(s)$ .

- ▶ **Cauchy's Residue Theorem:** If  $F(s)$  is a strictly proper rational function:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \sum_{s \text{ is a pole of } F(s)} (\text{residue of } F(s)e^{st} \text{ at } s)$$

## Laplace Transform Properties

- ▶ The Laplace transform is **linear**:

$$\begin{aligned}\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \int_0^{\infty} (\alpha f(t) + \beta g(t))e^{-st} dt \\ &= \alpha \int_0^{\infty} f(t)e^{-st} dt + \beta \int_0^{\infty} g(t)e^{-st} dt \\ &= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}\end{aligned}$$

- ▶ **Convolution**: for  $f(t)$ ,  $g(t)$  supported on  $t \in [0, \infty)$ :

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

- ▶ Convolution in time domain becomes multiplication in the complex domain:

$$\begin{aligned}\mathcal{L}\{(f * g)(t)\} &= \int_0^{\infty} \int_0^{\infty} f(\tau)g(t - \tau)e^{-st} d\tau dt \\ &= \int_0^{\infty} \int_{-\tau}^{\infty} f(\tau)g(\mu)e^{-s\tau} e^{-s\mu} d\mu d\tau \\ &\quad \underline{\underline{g(\mu)=0, \mu < 0}} \int_0^{\infty} f(\tau)e^{-s\tau} d\tau \int_0^{\infty} g(\mu)e^{-s\mu} d\mu \\ &= \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}\end{aligned}$$

## Laplace Transform Properties

► **Differentiation:**

$$\mathcal{L} \left\{ \frac{d}{dt} x(t) \right\} = s\mathcal{L} \{x(t)\} - x(0)$$

► **Proof:**

$$\int_0^{\infty} \frac{d}{dt} (x(t)e^{-st}) dt = x(t)e^{-st} \Big|_0^{\infty} = -x(0)$$

$$\begin{aligned} \int_0^{\infty} \frac{d}{dt} (x(t)e^{-st}) dt &= \int_0^{\infty} \left( \frac{d}{dt} x(t) \right) e^{-st} dt + \int_0^{\infty} x(t) \left( \frac{d}{dt} e^{-st} \right) dt \\ &= \mathcal{L} \left\{ \frac{d}{dt} x(t) \right\} - s\mathcal{L} \{x(t)\} \end{aligned}$$

► **Integration:**

$$\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} \mathcal{L} \{f(t)\}$$

► Note that  $\frac{d}{dt} \left( \int_0^t f(\tau) d\tau \right) = f(t)$

## Laplace Transform Properties

- ▶ Laplace transform of  $e^{at}$ :

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{-(s-a)t} dt = -\frac{1}{(s-a)} e^{-(s-a)t} \Big|_{t=0}^{t=\infty}$$
$$\stackrel{\text{require}}{\text{Re}(s) > a} 0 - \left( -\frac{1}{(s-a)} e^0 \right) = \frac{1}{s-a}$$

- ▶ **Delta function** (Impulse):

$$\delta_{\epsilon}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/\epsilon & \text{if } 0 \leq t < \epsilon \\ 0 & \text{if } t \geq \epsilon \end{cases} \quad \delta(t) = \lim_{\epsilon \rightarrow 0} \delta_{\epsilon}(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

- ▶ **Sifting property**: for any  $f(t)$  continuous at  $\tau \in (a, b)$ :

$$\int_a^b f(t) \delta(t - \tau) dt = f(\tau)$$

- ▶ Laplace transform of  $\delta(t)$ :

$$\mathcal{L}\{\delta(t)\} = \int_0^{\infty} \delta(t) e^{-st} dt = e^{-st} \Big|_{t=0}^{\infty} = 1$$

## Laplace Transform Properties

- ▶ Heaviside step function:

$$H(t) = \int_{-\infty}^t \delta(\tau) d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow \mathcal{L}\{H(t)\} = \frac{1}{s}$$

- ▶ Ramp function:

$$tH(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow \mathcal{L}\{H(t)\} = \frac{1}{s^2}$$

- ▶ Parabola function:

$$\frac{t^2}{2}H(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow \mathcal{L}\{H(t)\} = \frac{1}{s^3}$$



## Laplace Transform Properties

	$t$ domain	$s$ domain
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
convolution	$(f * g)(t)$	$F(s)G(s)$
multiplication	$f(t)g(t)$	$\frac{1}{2\pi j} \int_{\text{Re}(\sigma)-j\infty}^{\text{Re}(\sigma)+j\infty} F(\sigma)G(s-\sigma)d\sigma$
scaling, $a > 0$	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
$s$ -domain derivative	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
time-domain derivative	$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0)$
$s$ -domain integration	$\frac{1}{t} f(t)$	$\int_s^\infty F(\sigma) d\sigma$
time-domain integration	$\int_0^t f(\tau) d\tau = (H * f)(t)$	$\frac{1}{s} F(s)$
$s$ -domain shift	$e^{at} f(t)$	$F(s-a)$
time-domain shift, $a > 0$	$f(t-a)H(t-a)$	$e^{-as} F(s)$

▶ Heaviside step function  $H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$

▶ Convolution:  $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

# Laplace Transform Properties

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. $e^{at}$	$\frac{1}{s-a}$
3. $t^n, n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}}, n=1,2,3,\dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t \sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t \cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at) - at \cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at) + at \cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at) - at \sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at) + at \sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s \sin(b) + a \cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s \cos(b) - a \sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$
19. $e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20. $e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21. $e^{at} \sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22. $e^{at} \cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23. $t^n e^{at}, n=1,2,3,\dots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$
25. $u_c(t) = u(t-c)$ <u>Heaviside Function</u>	$\frac{e^{-cs}}{s}$	26. $\delta(t-c)$ <u>Dirac Delta Function</u>	$e^{-cs}$
27. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	28. $u_c(t)g(t)$	$e^{-cs}\mathcal{L}\{g(t+c)\}$
29. $e^{ct}f(t)$	$F(s-c)$	30. $t^n f(t), n=1,2,3,\dots$	$(-1)^n F^{(n)}(s)$
31. $\int_t^\infty f(u)du$	$\int_s^\infty F(u)du$	32. $\int_0^t f(v)dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	34. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} f(t) dt}{1-e^{-sT}}$
35. $f'(t)$	$sF(s) - f(0)$	36. $f''(t)$	$s^2 F(s) - sf(0) - f'(0)$

# Laplace Transform Properties

$f(t)$	$F(s)$
$\int_{-\infty}^t f(t) dt$	$\frac{F(s)}{s} + \frac{1}{s} \int_{-\infty}^0 f(t) dt$
Impulse function $\delta(t)$	1
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
$\frac{1}{\omega} [(\alpha - a)^2 + \omega^2]^{1/2} e^{-at} \sin(\omega t + \phi)$	$\frac{s+\alpha}{(s+a)^2 + \omega^2}$
$\phi = \tan^{-1} \frac{\omega}{\alpha - a}$	
$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2} t, \zeta < 1$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$\frac{1}{a^2 + \omega^2} + \frac{1}{\omega\sqrt{a^2 + \omega^2}} e^{-at} \sin(\omega t - \phi)$	$\frac{1}{s[(s+a)^2 + \omega^2]}$
$\phi = \tan^{-1} \frac{\omega}{-a}$	
$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t + \phi)$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
$\phi = \cos^{-1} \zeta, \zeta < 1$	
$\frac{\alpha}{a^2 + \omega^2} + \frac{1}{\omega} \left[ \frac{(\alpha - a)^2 + \omega^2}{a^2 + \omega^2} \right]^{1/2} e^{-at} \sin(\omega t + \phi)$	$\frac{s+\alpha}{s[(s+a)^2 + \omega^2]}$
$\phi = \tan^{-1} \frac{\omega}{\alpha - a} - \tan^{-1} \frac{\omega}{-a}$	

# Initial and Final Value Theorems

## Initial Value Theorem

Suppose that  $f(t)$  has a Laplace transform  $F(s)$ . Then:

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

## Final Value Theorem

Suppose that  $f(t)$  has a Laplace transform  $F(s)$ . Suppose that every pole of  $F(s)$  is either in the open left-half plane or at the origin of  $\mathbb{C}$ . Then:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

## Example: Spring-Mass-Damper

- ▶ Consider a spring-mass-damper system:

$$M \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = 0$$

- ▶ This is an example of a second-order system with **natural frequency**  $\omega_n = \sqrt{k/M}$  and **damping ratio**  $\zeta = b/(2\sqrt{kM})$ :

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = 0$$

- ▶ Laplace transform:

$$(s^2 Y(s) - sy(0) - \dot{y}(0)) + 2\zeta\omega_n(sY(s) - y(0)) + \omega_n^2 Y(s) = 0$$

- ▶ Natural response:

$$Y(s) = \frac{(s + 2\zeta\omega_n)y(0) + \dot{y}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

## Example: Spring-Mass-Damper

- ▶ Consider the natural response with  $\omega_n^2 = k/M = 2$  and  $2\zeta\omega_n = b/M = 3$ :

$$\begin{aligned} Y(s) &= \frac{(s+3)y(0) + \dot{y}(0)}{s^2 + 3s + 2} = \frac{(s+3)y(0) + \dot{y}(0)}{(s+1)(s+2)} \\ &= \frac{2y(0) + \dot{y}(0)}{s+1} - \frac{y(0) + \dot{y}(0)}{s+2} \end{aligned}$$

- ▶ Poles:  $p_1 = -1$  and  $p_2 = -2$

- ▶ Zeros:  $z_1 = -\frac{\dot{y}(0)}{y(0)} - 3$

- ▶ Residues:

$$\begin{aligned} r_1 &= \left. \frac{(s+3)y(0) + \dot{y}(0)}{(s+2)} \right|_{s=-1} \\ &= 2y(0) + \dot{y}(0) \end{aligned}$$

$$\begin{aligned} r_2 &= \left. \frac{(s+3)y(0) + \dot{y}(0)}{(s+1)} \right|_{s=-2} \\ &= -y(0) - \dot{y}(0) \end{aligned}$$

## Example: Spring-Mass-Damper

### ► Spring-Mass-Damper Pole-Zero Map

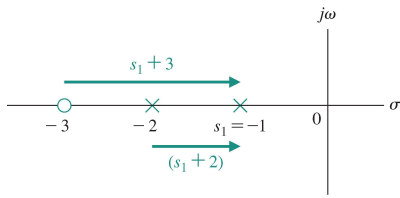
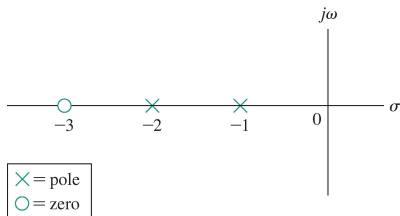
► Let the initial conditions be  $y(0) = 1$  and  $\dot{y}(0) = 0$

► The poles and zeros are:

$$p_1 = -1, \quad p_2 = -2, \quad z_1 = -3$$

► The residues are:

$$r_1 = \left. \frac{(s+3)}{(s+2)} \right|_{s=-1} = 2$$
$$r_2 = \left. \frac{(s+3)}{(s+1)} \right|_{s=-2} = -1$$



## Example: Spring-Mass-Damper

- ▶ The time-domain **natural response** of the spring-mass-damper system can be obtained using an inverse Laplace transform:

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2y(0) + \dot{y}(0)}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{y(0) + \dot{y}(0)}{s+2}\right\} \\ &= (2y(0) + \dot{y}(0))e^{-t} - (y(0) + \dot{y}(0))e^{-2t}\end{aligned}$$

- ▶ The **steady-state** response can be obtained via the Final Value Theorem:

$$\begin{aligned}\lim_{t \rightarrow \infty} y(t) &= \lim_{s \rightarrow 0} sY(s) \\ &= \lim_{s \rightarrow 0} \frac{(s^2 + 3s)y(0) + s\dot{y}(0)}{s^2 + 3s + 2} = 0\end{aligned}$$



## Example: Spring-Mass-Damper

- ▶ The poles of the system are the roots of the characteristic equation:

$$a(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

- ▶ The natural response is determined by the poles:
  - ▶ **Overdamped** ( $\zeta > 1$ ): the poles are real:

$$p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \quad p_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

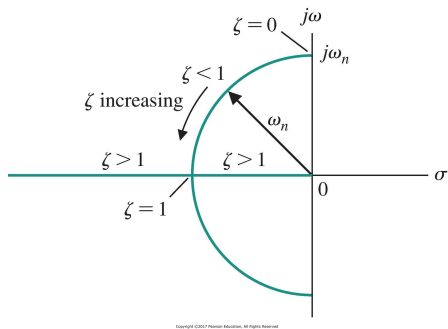
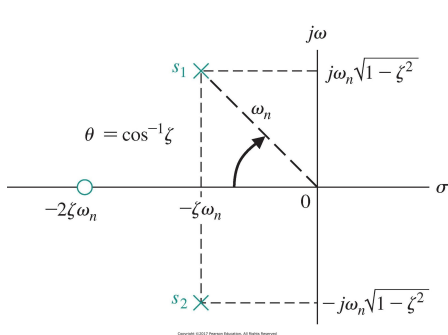
- ▶ **Critically damped** ( $\zeta = 1$ ): the poles are repeated and real:

$$p_1 = p_2 = -\omega_n$$

- ▶ **Underdamped** ( $\zeta < 1$ ): the poles are complex:

$$p_1 = -\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2} \quad p_2 = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$$

## Example: Spring-Mass-Damper Locus of Roots



- ▶ s-domain plot of the poles ( $\times$ ) and zeros ( $\circ$ ) of  $Y(s)$  with  $\dot{y}(0) = 0$
- ▶ For constant  $\omega_n$ , as  $\zeta$  varies, the complex conjugate roots follow a circular locus
- ▶ The poles and zeros can be expressed either in Cartesian coordinates or Polar coordinates (e.g., magnitude  $\omega_n$  and angle  $\theta = \cos^{-1}(\zeta)$ )

## Example: Spring-Mass-Damper Response

- ▶ The time-domain natural response can be obtained by determining the residues and applying an inverse Laplace transform:

- ▶ **Overdamped** ( $\zeta > 1$ ):

$$y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t}$$

where  $p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$ ,  $p_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$ ,  $r_1 = \frac{p_2 y(0) + \dot{y}(0)}{p_2 - p_1}$ ,  
and  $r_2 = -\frac{p_1 y(0) + \dot{y}(0)}{p_2 - p_1}$

- ▶ **Critically damped** ( $\zeta = 1$ ):

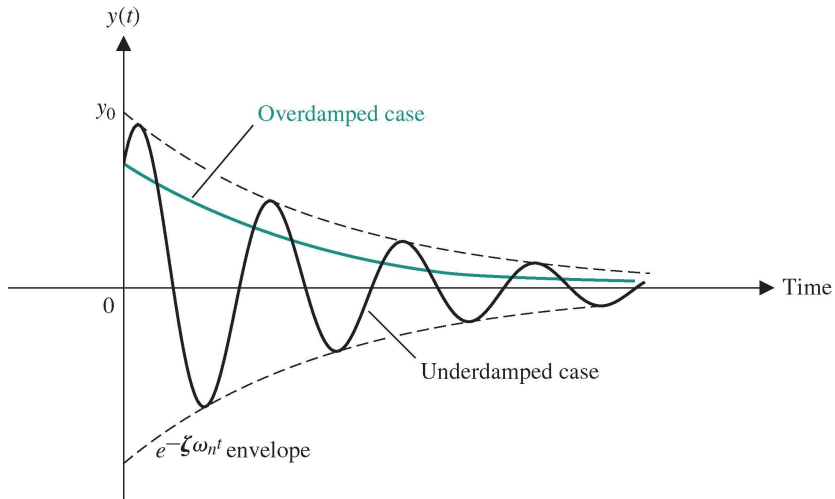
$$y(t) = y(0)e^{-\omega_n t} + (\dot{y}(0) + \omega_n y(0))te^{-\omega_n t}$$

- ▶ **Underdamped** ( $\zeta < 1$ ):

$$y(t) = e^{-\zeta\omega_n t} \left( c_1 \cos(\omega_n \sqrt{1 - \zeta^2} t) + c_2 \sin(\omega_n \sqrt{1 - \zeta^2} t) \right)$$

where  $c_1 = y(0)$  and  $c_2 = \frac{\dot{y}(0) + \zeta\omega_n y(0)}{\omega_n \sqrt{1 - \zeta^2}}$

## Example: Spring-Mass-Damper Natural Response with $\dot{y}(0) = 0$



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# Outline

Complex Numbers and Rational Functions

Polynomial and Rational Functions in MATLAB

Laplace Transform

Transfer Function

## Laplace Transform of LTI ODE

- ▶ Consider an LTI ODE with zero initial conditions:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \dots + b_0 u$$

- ▶ Let  $Y(s) = \mathcal{L}\{y(t)\}$  and  $U(s) = \mathcal{L}\{u(t)\}$

- ▶ Recall that  $\mathcal{L}\left\{\frac{d^n}{dt^n} y(t)\right\} = s^n Y(s) - \sum_{k=1}^n s^{n-k} \frac{d^{k-1}}{dt^{k-1}} y(t) \Big|_{t=0}$

- ▶ Laplace transform of the LTI ODE:

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0) Y(s) = (b_m s^m + b_{m-1} s^{m-1} + \dots + b_0) U(s)$$

- ▶ **Transfer function:** ratio of Laplace transform of output to Laplace transform of input with zero initial conditions:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

## Transfer Function

### Transfer Function

The transfer function  $G(s)$  of a single-input single-output LTI ODE is the ratio of the Laplace transform  $Y(s)$  of the output  $y(t)$  to the Laplace transform  $U(s)$  of the input  $u(t)$  with zero initial conditions:

$$G(s) = \frac{Y(s)}{U(s)}$$

### Relative Degree

The relative degree of a single-input single-output LTI ODE with transfer function  $G(s)$  is the difference  $r = n - m$  between the number of poles  $n$  and number of zeros  $m$  of  $G(s)$ .

- ▶ If  $r > 0$ , the transfer function is called *strictly proper*.
- ▶ If  $r \geq 0$ , the transfer function is called *proper*.
- ▶ If  $r < 0$ , the transfer function is called *improper* (there is no state space realization).

## Example

- ▶ A vehicle with position  $p(t)$  and acceleration input  $u(t)$  satisfies:

$$m\ddot{p}(t) = u(t)$$

- ▶ The transfer function of this system is:

$$G(s) = \frac{P(s)}{U(s)} = \frac{1}{ms^2}$$

- ▶ The transfer function is strictly proper with relative degree  $r = 2$



## Example: Second-order LTI ODE

- ▶ Consider a second-order system with natural frequency  $\omega_n$ , damping ratio  $\zeta$ , and input  $u(t)$ :

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2y(t) = u(t)$$

- ▶ Laplace transform:

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + 2\zeta\omega_n(sY(s) - y(0)) + \omega_n^2Y(s) = U(s)$$

- ▶ Transfer function (set  $y(0) = \dot{y}(0) = 0$ ):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- ▶ Total response:

$$Y(s) = \underbrace{\frac{(s + 2\zeta\omega_n)y(0) + \dot{y}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{\text{natural response}} + \underbrace{G(s)U(s)}_{\text{forced response}}$$

## Transfer Function of State-Space Model

- ▶ Consider an LTI ODE system in state-space:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- ▶ Laplace transform:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

- ▶ The response  $\mathbf{Y}(s)$  of LTI ODE system consists of **natural response** due to the initial conditions  $\mathbf{x}(0)$  and **forced response** due to the input  $\mathbf{U}(s)$ :

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + \underbrace{\left(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)}_{\mathbf{G}(s)}\mathbf{U}(s)$$

The transfer function of an LTI ODE system in state-space form is:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

## Response to Periodic Signals

- ▶ The idea of a transfer function comes from looking at the response of an LTI ODE system to periodic input signals with fundamental frequency  $\omega_f$ :

$$u(t) = \sum_{k=0}^{\infty} (a_k \sin(k\omega_f t) + b_k \cos(k\omega_f t))$$

- ▶ **Euler's formula:**  $e^{j\omega} = \cos \omega + j \sin \omega$

- ▶ The exponential function  $e^{st}$  with  $s = j\omega$  can represent periodic signals:

$$\sin(\omega t) = \text{Im}(e^{j\omega t}) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$

$$\cos(\omega t) = \text{Re}(e^{j\omega t}) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$

- ▶ Thanks to linearity (**superposition**), it suffices to compute the response to  $u(t) = e^{st}$  and then reconstruct the response to a cosine or sine by combining the responses corresponding to  $s = j\omega$  and  $s = -j\omega$

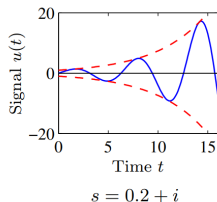
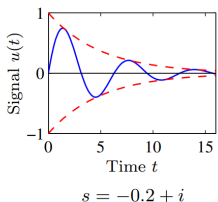
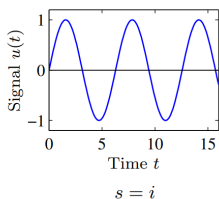
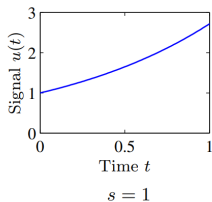
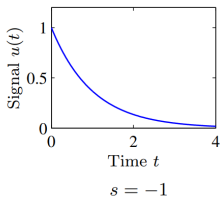
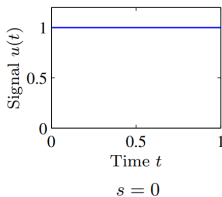
## Exponential Input $e^{st}$

- ▶ The exponential input  $e^{st}$  generalizes periodic signals to a broader class:

$$e^{st} = e^{\sigma t} e^{j\omega t} = e^{\sigma t} (\cos(\omega t) + j \sin(\omega t))$$

- ▶ Examples of exponential signals:

- ▶ Top row: exponential signals with a real exponent  $s = \sigma$
- ▶ Bottom row: exponential signals with a complex exponent  $s = j\omega$



# Frequency Domain Analysis

- ▶ Analyze LTI ODE response to sinusoidal and exponential signals

- ▶ State-space model:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, & \mathbf{x}(0) &= \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

- ▶ Convolution equation:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

- ▶ SISO system with input  $u(t) = e^{st}$  such that  $s$  is not an eigenvalue of  $\mathbf{A}$ :

$$\begin{aligned}y(t) &= \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0}_{\text{natural response}} + \underbrace{\mathbf{C}e^{\mathbf{A}t}(\mathbf{s}\mathbf{I} - \mathbf{A})^{-1} \left( e^{(\mathbf{s}\mathbf{I} - \mathbf{A})t} - \mathbf{I} \right) \mathbf{B} + \mathbf{D}e^{st}}_{\text{forced response}} \\ &= \underbrace{\mathbf{C}e^{\mathbf{A}t} (\mathbf{x}(0) - (\mathbf{s}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B})}_{\text{transient response}} + \underbrace{(\mathbf{C}(\mathbf{s}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}) e^{st}}_{\text{steady-state response}}\end{aligned}$$

## Frequency Domain Analysis

- ▶ SISO LTI ODE response to  $u(t) = e^{st}$ :

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t} (\mathbf{x}(0) - (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B})}_{\text{transient response}} + \underbrace{(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}) e^{st}}_{\text{steady-state response}}$$

The transfer function from  $u(t)$  to  $y(t)$  of a SISO LTI ODE is the coefficient of the **steady-state response to an exponential input**:

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- ▶ The transfer function represents the system dynamics in terms of the generalized frequency  $s$  instead of time  $t$
- ▶ Analyzing the system in the complex domain uncovers interesting properties

## Example

- ▶ Consider a SISO LTI ODE with state-space model:

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 1], \quad \mathbf{D} = 0$$

- ▶ Transfer function:

$$\begin{aligned} G(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = [0 \quad 1] \begin{bmatrix} s + a_1 & a_2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= [0 \quad 1] \frac{1}{s^2 + a_1s + a_2} \begin{bmatrix} s & -a_2 \\ 1 & s + a_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{s^2 + a_1s + a_2}. \end{aligned}$$

## Example

- ▶ Consider a Heaviside step input:

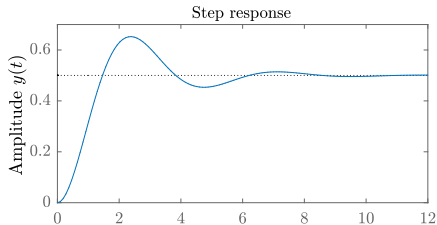
$$u(t) = H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

- ▶ Note that  $u(t) = e^{st}$  with  $s = 0$  for  $t \geq 0$ :

$$y(t) = \mathbf{C}e^{\mathbf{A}t} (\mathbf{x}(0) + \mathbf{A}^{-1}\mathbf{B}) + G(0)u(t)$$

- ▶ Suppose  $a_1 = 1$  and  $a_2 = 2$ :  $G(s) = \frac{1}{s^2+s+2}$

- ▶ The steady-state response as  $t \rightarrow \infty$  is  $G(0) = \frac{1}{2}$





## Controllable Canonical Form

- ▶ Consider a general  $n$ -th order transfer function (some of  $b_i$  may be 0):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

- ▶ To convert this transfer function to state-space form multiply by  $Z(s)/Z(s)$ :

$$G(s) = \frac{Y(s)/Z(s)}{U(s)/Z(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

- ▶ Time-domain LTI ODEs:

$$y = b_n z^{(n)} + b_{n-1} z^{(n-1)} + \dots + b_1 \dot{z} + b_0 z$$

$$u = z^{(n)} + a_{n-1} z^{(n-1)} + \dots + a_1 \dot{z} + a_0 z$$

- ▶ This suggests the following choice of state variables:

$$x_1 = z \quad x_2 = \dot{z} \quad \dots \quad x_n = z^{(n-1)}$$

## Controllable Canonical Form

- ▶ Consider a general  $n$ -th order transfer function (some of  $b_i$  may be 0):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

- ▶ The **controllable canonical form** is a state-space model with the same transfer function:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$
$$y = [(b_0 - a_0 b_n) \quad (b_1 - a_1 b_n) \quad \dots \quad (b_{n-1} - a_{n-1} b_n)] \mathbf{x} + b_n u$$

## Zero Frequency Gain

- ▶ The features of the transfer function reveal important system properties
- ▶ **Zero frequency gain:** the magnitude  $|G(0)|$  of the transfer function at  $s = 0$
- ▶ Interpretation: the ratio of the steady-state output to a step input
- ▶ LTI ODE:

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0} \quad \Rightarrow \quad G(0) = \frac{b_0}{a_0}$$

- ▶ State-space model:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \quad \Rightarrow \quad G(0) = -\mathbf{CA}^{-1}\mathbf{B} + \mathbf{D}$$

- ▶ Integrator:  $\dot{y} = u$

$$G(s) = \frac{1}{s} \quad \Rightarrow \quad G(0) \rightarrow \infty \quad \text{pole}$$

- ▶ Differentiator  $y = \dot{u}$

$$G(s) = s \quad \Rightarrow \quad G(0) = 0 \quad \text{zero}$$

## Transfer Function Poles

- ▶ Consider the LTI ODE:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \dots + b_0 u$$

- ▶ The response  $Y(s)$  consists of **natural response** due to the initial conditions  $x(0)$  and **forced response** due to the input  $U(s)$ :

$$Y(s) = \underbrace{\frac{c(s)}{a(s)}}_{\text{natural response}} + \underbrace{\frac{b(s)}{a(s)} U(s)}_{\text{forced response}}$$

- ▶ The transfer function  $G(s) = \frac{b(s)}{a(s)}$  and the natural response have the same denominator:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$$

- ▶ A pole  $p$  of the transfer function  $G(s)$  is a solution to the characteristic equation  $a(s) = 0$ . If  $u(t) \equiv 0$ , then  $y(t) = e^{pt}$  is a solution to the LTI ODE.

**The poles  $p$  of a transfer function  $G(s)$  correspond to the natural solutions  $y(t) = e^{pt}$  of the LTI ODE called modes.**

## Transfer Function Zeros

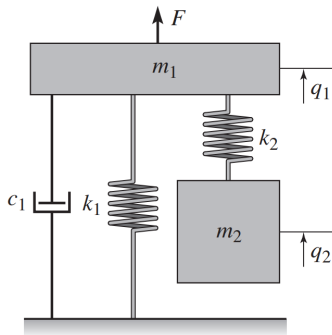
- ▶ SISO LTI ODE response to an exponential input  $u(t) = e^{st}$ :

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t} (\mathbf{x}(0) - (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B})}_{\text{transient response}} + \underbrace{(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}) e^{st}}_{\text{steady-state response}}$$

- ▶ A zero  $z$  of the transfer function  $G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$  makes  $G(z) = 0$  and hence the steady-state response to  $u(t) = e^{zt}$  is zero

**The zeros  $z$  of a transfer function  $G(s)$  block transmission of an exponential input  $u(t) = e^{zt}$ .**

## Example: Vibration Damper



**Figure:** Vibrations of the mass  $m_1$  can be damped by providing an auxiliary mass  $m_2$ , attached to  $m_1$  by a spring with stiffness  $k_2$ . The parameters  $m_2$  and  $k_2$  are chosen so that the frequency  $\sqrt{k_2/m_2}$  matches the frequency of vibration.

## Example: Vibration Damper

- ▶ Vibration damper dynamics:

$$\begin{aligned}m_1 \ddot{q}_1 + c_1 \dot{q}_1 + k_1 q_1 + k_2 (q_1 - q_2) &= f \\ m_2 \ddot{q}_2 + k_2 (q_2 - q_1) &= 0\end{aligned}$$

- ▶ The Laplace transform with zero initial conditions is:

$$\begin{aligned}(m_1 s^2 + c_1 s + k_1) Q_1(s) + k_2 (Q_1(s) - Q_2(s)) &= F(s) \\ m_2 s^2 Q_2(s) + k_2 (Q_2(s) - Q_1(s)) &= 0\end{aligned}$$

- ▶ The transfer function from  $F(s)$  to  $Q_1(s)$  is obtained by eliminating  $Q_2(s)$ :

$$G(s) = \frac{Q_1(s)}{F(s)} = \frac{m_2 s^2 + k_2}{m_1 m_2 s^4 + m_2 c_1 s^3 + (m_1 k_2 + m_2 (k_1 + k_2)) s^2 + k_2 c_1 s + k_1 k_2}$$

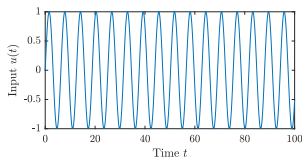
- ▶ **Blocking property:** the transfer function has **zeros** at  $s = \pm j \sqrt{k_2/m_2}$

## Example: Vibration Damper

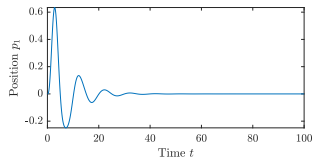
- ▶ Blocking property with parameters

$$m_1 = 1, c_1 = 1, k_1 = 1, m_2 = 1, k_2 = 1$$

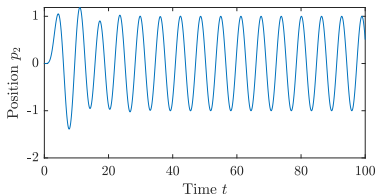
- ▶ **Case 1: external input** :  $u = \sin(\omega t)$ , with  $\omega = 1$



(a) Input  $u = \sin(t)$



(b) Position of mass 1



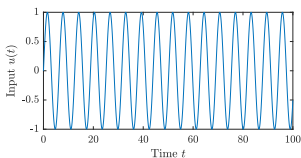
(c) Position of mass 2



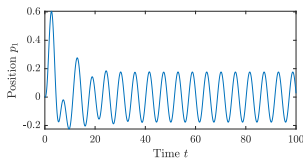
## Example: Vibration Damper

- ▶ Other frequency responses

- ▶ **Case 2: external input** :  $u = \sin(\omega t)$ , with  $\omega = 1.1$

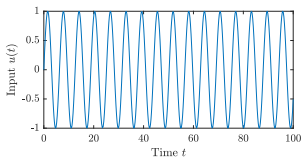


(a) Input  $u = \sin(1.1t)$

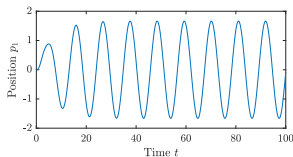


(b) Position of mass 1

- ▶ **Case 3: external input** :  $u = \sin(\omega t)$ , with  $\omega = 0.578$



(a) Input  $u = \sin(1.1t)$



(b) Position of mass 1