ECE171A: Linear Control System Theory
Lecture 7: Stability

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Outline

Equilibria

Stability

Linearization

Outline

Equilibria

Stability

Linearization

Nonlinear Systems

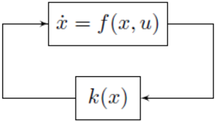
- **Nonlinear state-space model**:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases} \qquad \text{v.s.} \qquad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

- **Control problem**: design a function $\mathbf{u} = \mathbf{k}(\mathbf{x})$, called **feedback control law**, such that:
  - **Regulation problem**: the state converges to zero: $\mathbf{x}(t) \to \mathbf{0}$
  - **Servo problem**: the state tracks a reference signal: $\mathbf{x}(t) \to \mathbf{r}(t)$

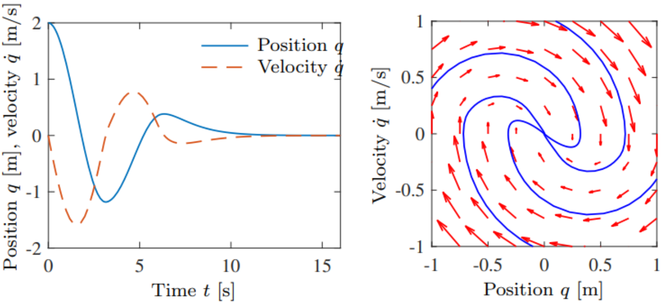
- **Closed-loop system**:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x})) = \mathbf{F}(\mathbf{x})$$



Phase Portrait

- The state trajectory $\mathbf{x}(t)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ may be visualized as a *time plot* or a *phase portrait*
- **Time plot**: plots state components $x_i(t)$ as a function of time t
- **Vector field**: plots the vector $\mathbf{F}(\mathbf{x})$ as an arrow at different states $\mathbf{x}$ in $\mathbb{R}^n$
- **Phase portrait**: plots state components relative to each other, e.g., x_2 vs x_1, by following the vector field associated with different initial conditions



(a) Time plot (b) Phase portrait

Equilibrium Points

An **equilibrium point** $\mathbf{x}_\mathrm{e} \in \mathbb{R}^n$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ satisfies:

$$\mathbf{F}(\mathbf{x}_\mathrm{e}) = 0.$$

- An equilibrium point is a stationary operating condition for the system
- If started at an equilibrium point, a system remains there for all time:

$$\mathbf{x}(t_0) = \mathbf{x}_\mathrm{e} \qquad \Rightarrow \qquad \mathbf{x}(t) = \mathbf{x}_\mathrm{e}, \quad \text{for all } t \geq t_0$$

- Nonlinear dynamical systems $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ can have zero, one, or more equilibria
- Linear dynamical systems $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ can have one ($\mathbf{x}_\mathrm{e} = \mathbf{0}$ when $\mathbf{A}$ is nonsingular) or infinitely many (null space of $\mathbf{A}$ when $\mathbf{A}$ is singular) equilibria

Example: Pendulum

- Consider a pendulum with mass m, length l, and angle θ under the influence of gravity acceleration g:
$$ml^2\ddot{\theta} = mgl\sin\theta$$

- State-space model with $x_1 = \theta$, $x_2 = \dot{\theta}$:
$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(x_1) \end{bmatrix}$$

- An ODE system with two state variables $\mathbf{x} \in \mathbb{R}^2$ is called **planar dynamical system**

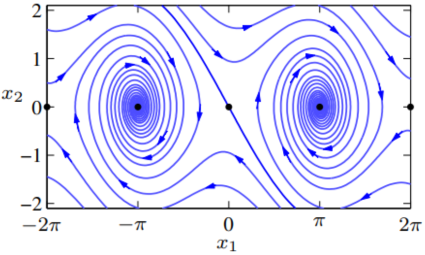
- Pendulum equilibria:
$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}_{\mathrm{e}} = \begin{bmatrix} \pm k\pi \\ 0 \end{bmatrix}, k = 0, 1, 2 \ldots$$

Example: Pendulum

- Equilibria: $\mathbf{x}_{\mathrm{e}} = \begin{bmatrix} \pm k\pi \\ 0 \end{bmatrix}$, $k = 0, 1, 2 \ldots$

Equilibrium 1 (unstable) **Equilibrium 2** (stable)

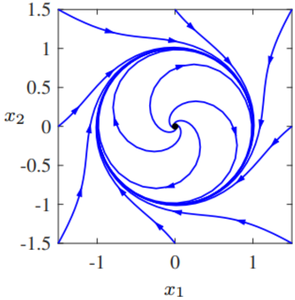
- Phase portrait:



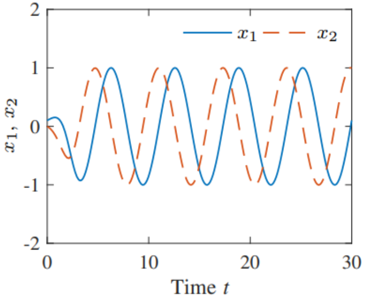
Limit Cycles

- Besides an equilibrium point, nonlinear systems may exhibit a stationary periodic solution called **limit cycle**
- A limit cycle corresponds to an oscillatory periodic trajectory in the time domain and a circular trajectory in the phase domain
- Example:

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$



(a)



(b)

Outline

Equilibria

Stability

Linearization

Stability

- **Aleksandr Lyapunov** made many important contributions to the theory of dynamical system stability
- An equilibrium point is **stable** if, when the system is started near the equilibrium point, its state remains near the equilibrium point over time
- An equilibrium point is **asymptotically stable** if, when the system is started near the equilibrium point, its state converges to the equilibrium point

A. Lyapunov

Stable Equilbrium

An equilibrium $\mathbf{x}_e$ of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is **stable** if, for all t_0 and all $\epsilon > 0$, there exists δ such that:

$$\|\mathbf{x}(t_0) - \mathbf{x}_e\| < \delta \qquad \Rightarrow \qquad \|\mathbf{x}(t) - \mathbf{x}_e\| < \epsilon, \quad \forall t \geq t_0$$

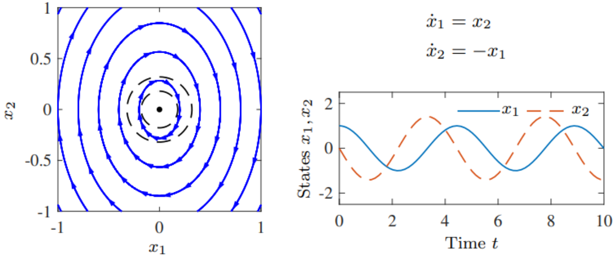


Figure: The equilibrium point $\mathbf{x}_e = \mathbf{0}$ is stable since all trajectories that start near $\mathbf{x}_e$ remain near $\mathbf{x}_e$

Asymptotically Stable Equilibrium

An equilibrium $\mathbf{x}_\mathrm{e}$ of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is **asymptotically stable** if

- $\mathbf{x}_\mathrm{e}$ is a stable equilibrium,
- for all t_0 there exists δ such that:

$$\|\mathbf{x}(t_0) - \mathbf{x}_\mathrm{e}\| < \delta \qquad \Rightarrow \qquad \lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}_\mathrm{e}\| = 0$$

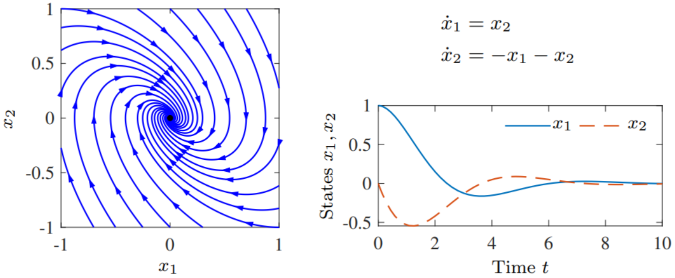


Figure: The equilibrium point $\mathbf{x}_\mathrm{e} = \mathbf{0}$ is asymptotically stable since all trajectories that start near $\mathbf{x}_\mathrm{e}$ converge to $\mathbf{x}_\mathrm{e}$ as $t \to \infty$

Unstable Equilibrium

- An equilibrium point is **unstable** if it is not stable

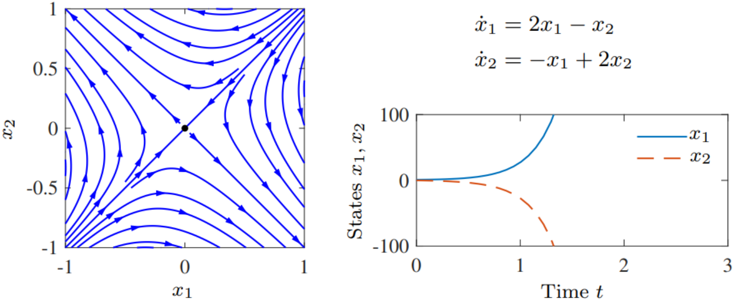
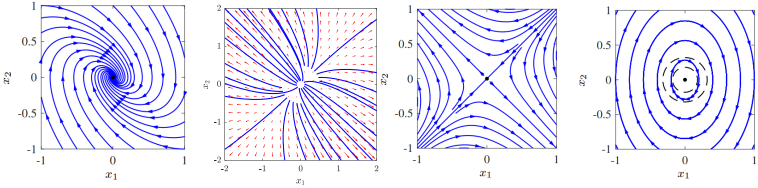


Figure: The equilibrium point $\mathbf{x}_{\mathrm{e}} = \mathbf{0}$ is unstable since not all trajectories that start near $\mathbf{x}_{\mathrm{e}}$ remain near $\mathbf{x}_{\mathrm{e}}$

Sink, Source, Saddle

- Equilibrium points have names based on their stability type
- **Sink**: an asymptotically stable equilibrium point
- **Source**: an unstable equilibrium point with all trajectories leading away
- **Saddle**: an unstable equilibrium point with some trajectories leading away
- **Center**: a stable but not asymptotically stable equilibrium point



(a) Sink (b) Source (c) Saddle (d) Center

LTI ODE Stability

- Consider an LTI ODE system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

- An **eigenvalue** of $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a complex number $\lambda \in \mathbb{C}$ such that:

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0$$

- The stability of $\mathbf{x}_{\mathrm{e}} = \mathbf{0}$ is determined by the eigenvalues of $\mathbf{A}$

Example

- System:

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{x}$$

- Solution:

$$x_i(t) = e^{\lambda_i t} x_i(0), \qquad i = 1, 2$$

- $\mathbf{x}_{\mathrm{e}} = \mathbf{0}$ is stable if $\lambda_i \leq 0$, and asymptotically stable if $\lambda_i < 0$

LTI ODE Stability

Lyapunov Stability of LTI ODE Systems

The following statements about an LTI ODE system, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, are equivalent:

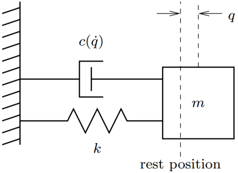
- $\mathbf{x}_\mathrm{e} = \mathbf{0}$ is a **unique** equilibrium and is **asymptotically stable**
- all eigenvalues λ_i of $\mathbf{A}$ have strictly negative real parts: $\mathrm{Re}(\lambda_i) < 0$

- If any eigenvalue λ_i of $\mathbf{A}$ has $\mathrm{Re}(\lambda_i) > 0$, then $\mathbf{x}_\mathrm{e} = \mathbf{0}$ is an **unstable** equilibrium
- If $\mathrm{Re}(\lambda_i) \leq 0$ for all eigenvalues but some $\mathrm{Re}(\lambda_i) = 0$, then $\mathbf{x}_\mathrm{e} = \mathbf{0}$ may or may not be stable

Example: Second-Order System

- Second-order system:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$$



- State-space model with $x_1 = y$ and $x_2 = \dot{y}/\omega_n$:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \omega_n \\ -\omega_n & -2\zeta\omega_n \end{bmatrix} \mathbf{x}$$

- Eigenvalues of $\mathbf{A}$:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\left(\begin{bmatrix} \lambda & -\omega_n \\ \omega_n & \lambda + 2\zeta\omega_n \end{bmatrix}\right) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- If $\zeta > 0$, the eigenvalues have negative real parts and the origin is asymptotically stable

Stability Analysis in the Complex Domain

- LTI ODE Transfer Function:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \qquad \Longleftrightarrow \qquad G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- The eigenvalues s of $\mathbf{A}$ satisfy $\det(s\mathbf{I} - \mathbf{A}) = 0$ and hence are related to the poles $G(s)$
- If $G(s)$ contains a pole p in the right half-plane of $\mathbb{C}$, then the natural respose $y(t)$ contains a term re^{pt}, which will go to infinity
- If all poles of $G(s)$ are in the left half-plane of $\mathbb{C}$, then all terms re^{pt} in the natural response $y(t)$ will settle to a steady-state value
- The poles of $G(s)$ are the same as the eigenvalues of $\mathbf{A}$ if both are in **minimal form**:
  - $G(s)$ does not have pole-zero cancelations,
  - the state-space model is controllable and observable.

BIBO Stability

- A signal $y(t)$ is **bounded** if $|y(t)| \leq M$ for some constant M and all t
- An LTI ODE system is **bounded-input bounded-output (BIBO) stable** if every bounded input $u(t)$ leads to a bounded output $y(t)$
- A system is **BIBO unstable** if there exists at least one bounded input that produces an unbounded output

BIBO Stability of LTI ODE Systems

An LTI ODE system with transfer function $G(s)$ is:

- **BIBO stable**, if all poles of $G(s)$ are in the open left half-plane (OLHP) of $\mathbb{C}$,
- **marginally BIBO stable**, if all poles of $G(s)$ are in the closed left half-plane of $\mathbb{C}$ and all poles with zero real part are simple (multiplicity 1),
- **BIBO unstable**, otherwise.

$$\begin{aligned} \text{Lyapunov stability} \quad &\Rightarrow \quad \text{BIBO stability} \\ \text{BIBO stability, controllability, observability} \quad &\Rightarrow \quad \text{Lyapunov stability} \end{aligned}$$

No Pole-Zero Cancellation!

- **Important**: common poles and zeros in $G(s)$ should not be canceled before checking BIBO stability!
- A canceled pole will not show up in the forced response but will still appear in the natural response when the initial conditions are non-zero

Example

- Consider the system: $\ddot{y} + 2\dot{y} - 3y = \dot{u} - u$
- Transfer function: $G(s) = \frac{Y(s)}{U(s)} = \frac{s-1}{s^2+2s-3} = \frac{s-1}{(s+3)(s-1)}$
- Total response:

$$Y(s) = \frac{s+2}{s^2+2s-3}y(0) + \frac{1}{s^2+2s-3}\dot{y}(0) + \underbrace{\frac{s-1}{s^2+2s-3}}_{G(s)}U(s)$$

- With bounded $u(t) \equiv 0$ but non-zero initial conditions $y(t)$ is unbounded:

$$y(t) = \frac{y(0)}{4}(3e^t + e^{-3t}) + \frac{\dot{y}(0)}{4}(e^t - e^{-3t})$$

BIBO Stability Without Computing Poles

- A system with transfer function $G(s) = \frac{b(s)}{a(s)}$ is BIBO stable if all poles are in the OLHP of $\mathbb{C}$
- Computing the poles $p_1, \ldots, p_n$ might not always be easy, e.g., for high-order or symbolic characteristic polynomial:

$$a(s) = a_n s^n + \ldots + a_1 s + a_0 = a_n(s - p_1)\cdots(s - p_n)$$

- Whether the poles are in the OLHP can be verified from the coefficients of $a(s)$ rather than from the actual pole values
- **Vieta's formulas** relate the coefficients of a polynomial to its roots

$$\sum_{i=1}^{n} p_i = -\frac{a_{n-1}}{a_n} \qquad \prod_{i=1}^{n} p_i = (-1)^n \frac{a_0}{a_n} \qquad \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} p_{i_j} = (-1)^k \frac{a_{n-k}}{a_n}$$

Necessary Condition for BIBO Stability of LTI ODE Systems

If all poles of a transfer function $G(s) = b(s)/a(s)$ are in the open left half-plane of $\mathbb{C}$, then all coefficients of the characteristic polynomial $a(s)$ will be **non-zero** and **have the same sign**.

Example

- Consider an LTI ODE system with transfer function $G(s) = b(s)/a(s)$ and characteristic polynomial $a(s)$ shown below. Is this system BIBO stable?
  - $a(s) = s^3 - 2s^2 + s + 1$
  - $a(s) = s^4 + s^2 + s + 1$
  - $a(s) = s^3 + 2s^2 + 2s + 1$
  - $a(s) = s^3 + 2s^2 + s + 12$

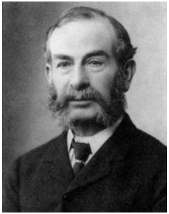
Necessary and Sufficient Condition for BIBO Stability

- In the 1870s-1890s, **Edward Routh** and **Adolf Hurwitz** independently developed a method for determining the locations in $\mathbb{C}$ but not the actual values of the roots of a complex polynomial with constant real coefficients

- Characteristic polynomial:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$$

- **Routh-Hurwitz method**
  - construct a table with $n+1$ rows from the coefficients a_i of $a(s)$
  - relate the **number of sign changes in the first column** of the table to the **number of roots in the closed right half-plane**



E. Routh

A. Hurwitz

Routh Table

- $a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	a_0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	$b_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$	$b_{n-3} = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$	b_{n-5}	$\cdots$	0
s^{n-3}	$c_{n-1} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$	$c_{n-3} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}}$	c_{n-5}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
s^0	a_0	0	0	$\cdots$	0

- Any row can be multiplied by a positive constant without changing the result

Routh-Hurwitz BIBO Stability Criterion

Theorem: Routh-Hurwitz

Consider a Routh table constructed from a polynomial $a(s)$. The number of sign changes in the first column of the Routh table is equal to the number of roots of $a(s)$ in the closed right half-plane of $\mathbb{C}$.

Corollary

An LTI ODE system with transfer function $G(s) = b(s)/a(s)$ is **BIBO stable** if and only if there are no sign changes in the first column of the Routh table of $a(s)$.

- There are two special cases related to the Routh table:
  1. The first element of a row is 0 but some of the other elements are not
     - **Solution**: replace the 0 with an arbitrary small ϵ
  2. All elements of a row are 0
     - **Solution**: replace the zero row with the coefficients of $\frac{dA(s)}{ds}$, where $A(s)$ is an **auxiliary polynomial** with coefficients from the row just above the zero row

Example: Second-order System

- Consider the characteristic polynomial of a second-order system:

$$a(s) = as^2 + bs + c$$

- The Routh table is:

s^2	a	c
s^1	b	0
s^0	$-\frac{1}{b}(0 - bc) = c$	0

- A necessary and sufficient condition for BIBO stability of a second-order system is that all coefficients of the characteristic polynomial are non-zero and have the same sign.

Example: Third-order System

- Consider the characteristic polynomial of a third-order system:

$$a(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

- The Routh table is:

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$-\frac{1}{a_2}(a_3 a_0 - a_1 a_2)$	0
s^0	a_0	0

- A necessary and sufficient condition for BIBO stability of a third-order system is that all coefficients of the characteristic polynomial are non-zero, have the same sign, and $a_1 a_2 > a_0 a_3$.

- If $a_1 a_2 = a_0 a_3$, one pair of roots lies on the imaginary axis in the s plane and the system is marginally stable. This results in an all zero row in the Routh table.

Example: Higher-order System

- Consider the characteristic polynomial of a fifth-order system:

$$a(s) = s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240$$

- The Routh table is:

s^5	1	10	152
s^4	1	72	240
s^3	-62	-88	0
s^2	70.6	240	0
s^1	122.6	0	0
s^0	240	0	0

- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**

- The roots of $a(s)$ are:

$$a(s) = (s+3)(s+1 \pm j\sqrt{3})(s-2 \pm j4)$$

Example: Special Case 1

- Consider the polynomial:

$$a(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

- The Routh table is:

s^5	1	2	11
s^4	2	4	10
s^3	$\cancel{0}\ \epsilon$	6	0
s^2	$c_4 = \frac{1}{\epsilon}(4\epsilon - 12)$	10	0
s^1	$d_4 = \frac{1}{c_4}(6c_4 - 10\epsilon)$	0	0
s^0	10	0	0

- For $0 < \epsilon \ll 1$, we see that $c_4 < 0$ and $d_4 > 0$
- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**

Example: Special Case 1

- Consider the polynomial:

$$a(s) = s^4 + s^3 + 2s^2 + 2s + 3$$

- The Routh table is:

s^4	1	2	3
s^3	1	2	0
s^2	$\cancel{0} \to \epsilon$	3	0
s^1	$2 - \frac{3}{\epsilon}$	0	0
s^0	3	0	0

- For $0 < \epsilon \ll 1$, we see that $2 - \frac{3}{\epsilon} < 0$

- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**

Example: Special Case 2

- Consider the polynomial:

$$a(s) = s^3 + 2s^2 + 4s + 8$$

- The Routh table is:

s^3	1	4
s^2	2	8
s^1	0	0
s^0	8	0

- There is an all-zero row at s^1
- The auxiliary polynomial is: $A(s) = 2s^2 + 8 = 2(s + j2)(s - j2)$
- There are two roots on the $j\omega$-axis and the system is **marginally stable**

Example: Special Case 2

- Consider the polynomial:

$$a(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$$

- The Routh table is:

s^5	1	2	1
s^4	1	2	1
s^3	0	0	0
s^2	1	1	0
s^1	0	0	0
s^0	1	0	0

- There is an all-zero row at s^3 and s^1
- The auxiliary polynomial at the s^3 row is:

$$A(s) = s^4 + 2s^2 + 1 = (s^2 + 1)^2 = (s + j)(s - j)(s + j)(s - j)$$

- There are repeated roots on the $j\omega$-axis and the system is **unstable**

Example: Special Case 2

- Consider the polynomial:

$$a(s) = s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4$$

- The Routh table is:

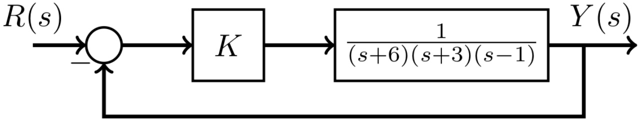
s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	0
s^1	$\cancel{0}$ 8	0	0
s^0	4	0	0

- There is an all-zero row at s^1 with auxiliary polynomial

$$A(s) = 4s^2 + 4 = 4(s^2 + 1) = 4(s + j)(s - j)$$

- There are two roots on the $j\omega$-axis and the system is **marginally stable**

Example: Parametric System



- The Routh-Hurwitz stability criterion can be used to determine the range of system parameters for which the system is stable
- Transfer function: $T(s) = \frac{K}{s^3+8s^2+9s+(K-18)}$
- Characteristic polynomial: $a(s) = s^3 + 8s^2 + 9s + (K - 18)$

Example: Parametric System

- Characteristic polynomial: $a(s) = s^3 + 8s^2 + 9s + (K - 18)$
- The Routh table is:

s^3	1	9
s^2	8	$(K-18)$
s^1	$\frac{90-K}{8}$	0
s^0	$(K-18)$	0

- There will be no sign changes in the first column of the Routh table if $(90 - K) > 0$ and $(K - 18) > 0$
- The system is BIBO stable if and only if $18 < K < 90$

Outline

Equilibria

Stability

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Nonlinear Systems

- Most practical systems are nonlinear:
  - No control input:
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$
  - With control input:
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u})$$

- Common approach for nonlinear system analysis and control design:
  - Approximate the system by a linear one around an equilibrium point $\mathbf{x}_{\mathrm{e}}$
  - Study the behavior of the approximate linear model:
    - analyze the closed-loop system stability if a control law is given,
    - design a control law using the linear open-loop model.
  - Verify the results in the original closed-loop nonlinear system

Linearization

- **Linearization**: linear approximation of a function $f(x)$ in the neighborhood of a point x_e usually based on a Taylor series expansion

- **Taylor series** of infinitely differentiable function $f(x)$ around point x_e:

$$f(x) = f(x_\mathrm{e}) + f'(x_\mathrm{e})(x - x_\mathrm{e}) + \frac{f''(x_\mathrm{e})}{2!}(x - x_\mathrm{e})^2 + \ldots + \frac{f^{(n)}(x_\mathrm{e})}{n!}(x - x_\mathrm{e})^n + \ldots$$

Examples

- Taylor series of $f(x)$ around $x_\mathrm{e} = 0$:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \end{aligned}$$

Linearization: No Control Input

- Consider a nonlinear system with equilibrium $\mathbf{x}_{\mathrm{e}}$:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \qquad\qquad \mathbf{F}(\mathbf{x}_{\mathrm{e}}) = \mathbf{0}$$

- Taylor series expansion of $\mathbf{F}(\mathbf{x})$ around $\mathbf{x}_{\mathrm{e}}$:

$$\mathbf{F}(\mathbf{x}) = \cancelto{\mathbf{0}}{\mathbf{F}(\mathbf{x}_{\mathrm{e}})} + \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{\mathrm{e}}} (\mathbf{x} - \mathbf{x}_{\mathrm{e}}) + \text{higher-order terms in } (\mathbf{x} - \mathbf{x}_{\mathrm{e}})$$

- Define a new state $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_{\mathrm{e}}$ to obtain a linear approximation around $\mathbf{x}_{\mathrm{e}}$:

$$\dot{\tilde{\mathbf{x}}} \approx \mathbf{A}\tilde{\mathbf{x}} \qquad \text{with} \qquad \mathbf{A} = \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{\mathrm{e}}}$$

Example: Inverted Pendulum

- Consider a damped inverted pendulum:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) - cx_2 \end{bmatrix}$$

- **Step 1**: find equilibrium points:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

- **Step 2**: Linearize the system around an equilibrium, e.g., $\mathbf{x}_{\mathrm{e}} = (0, 0)$

$$\begin{aligned} f_1(x_1, x_2) &= x_2 \\ f_2(x_1, x_2) &= \sin x_1 - cx_2 \approx f_2(0,0) + \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} (x_1 - 0) + \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} (x_2 - 0) \\ &= 0 + x_1 - cx_2 \end{aligned}$$

- **Step 3**: Define a new state $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_{\mathrm{e}}$ to obtain a linear model:

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix} \tilde{\mathbf{x}}$$

Example: Inverted Pendulum

- Consider a damped inverted pendulum:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) - cx_2 \end{bmatrix}$$

- **Step 1**: find equilibrium points:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

- **Step 2**: Linearize the system around an equilibrium, e.g., $\mathbf{x}_{\mathrm{e}} = (\pi, 0)$

$$\begin{aligned} f_1(x_1, x_2) &= x_2 \\ f_2(x_1, x_2) &= \sin x_1 - cx_2 \approx f_2(\pi, 0) + \left.\frac{\partial f_2}{\partial x_1}\right|_{(\pi,0)} (x_1 - \pi) + \left.\frac{\partial f_2}{\partial x_2}\right|_{(\pi,0)} (x_2 - 0) \\ &= 0 - (x_1 - \pi) - cx_2 \end{aligned}$$

- **Step 3**: Define a new state $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_{\mathrm{e}}$ to obtain a linear model:

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \tilde{\mathbf{x}}$$

Lyapunov's First Method for Stability

- Lyapunov's first method is an approach to test the stability of a nonlinear system equilibrium by considering the system's linearization

Theorem

Consider a nonlinear system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ with equilibrium $\mathbf{x}_\mathrm{e} = \mathbf{0}$.

- If all eigenvalues of $\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\Big|_{\mathbf{x}_\mathrm{e}}$ have negative real parts, then $\mathbf{x}_\mathrm{e} = 0$ is locally asymptotically stable.
- If one or more eigenvalues of $\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\Big|_{\mathbf{x}_\mathrm{e}}$ have positive real parts, then $\mathbf{x}_\mathrm{e} = 0$ is unstable.

Example: Inverted Pendulum Stability

- Consider a damped inverted pendulum with $c > 0$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) - cx_2 \end{bmatrix}$$

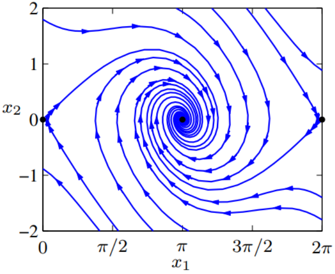
- Equilibrium $(0, 0)$: **Unstable**

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix} \tilde{\mathbf{x}} \qquad \Rightarrow \qquad \det\left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda + c \end{bmatrix} \right) = \lambda^2 + c\lambda - 1 = 0$$

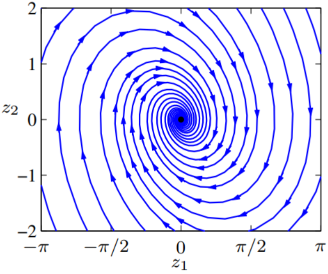
- Equilibrium $(\pi, 0)$: **Stable**

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \tilde{\mathbf{x}} \qquad \Rightarrow \qquad \det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda + c \end{bmatrix} \right) = \lambda^2 + c\lambda + 1 = 0$$

Example: Inverted Pendulum Linearization around $(\pi, 0)$



(a) Nonlinear model



(b) Linear approximation

Figure: Comparison between the phase portraits of (a) the nonlinear system and (b) its linear approximation around the origin. Notice that near the equilibrium point, the phase portraits are almost identical.

Linearization: With Control Input

- Consider a nonlinear system with equilibrium $(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})$:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}) & \qquad \mathbf{f}(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e}) &= \mathbf{0} \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}) & \qquad \mathbf{h}(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e}) &= \mathbf{y}_\mathrm{e} \end{aligned}$$

- Define new state, input, and output:

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_\mathrm{e}, \qquad \tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_\mathrm{e}, \qquad \tilde{\mathbf{y}} = \mathbf{y} - \mathbf{y}_\mathrm{e}$$

- Taylor series expansion of $\mathbf{f}(\mathbf{x}, \mathbf{u})$ and $\mathbf{h}(\mathbf{x}, \mathbf{u})$ around $(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})$:

$$\begin{aligned} \mathbf{f}(\mathbf{x}, \mathbf{u}) &\approx \cancelto{\mathbf{0}}{\mathbf{f}(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e}} \tilde{\mathbf{x}} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})} \tilde{\mathbf{u}} \\ \mathbf{h}(\mathbf{x}, \mathbf{u}) &\approx \cancelto{\mathbf{y}_\mathrm{e}}{\mathbf{h}(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})} + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e}} \tilde{\mathbf{x}} + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})} \tilde{\mathbf{u}} \end{aligned}$$

- LTI system approximation:

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\tilde{\mathbf{u}} & \qquad \mathbf{A} &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})} & \quad \mathbf{B} &= \left.\frac{\partial \mathbf{f}}{\partial u}\right|_{(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})} \\ \tilde{\mathbf{y}} &= \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}\tilde{\mathbf{u}} & \qquad \mathbf{C} &= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})} & \quad \mathbf{D} &= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_\mathrm{e}, \mathbf{u}_\mathrm{e})} \end{aligned}$$

Linearization: Summary

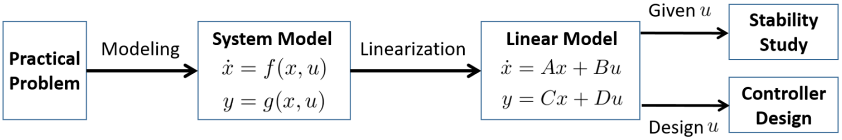


Figure: General control design approach

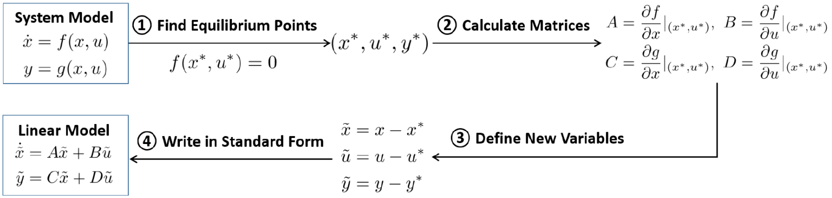


Figure: Model linearization procedure