ECE171A: Linear Control System Theory
Lecture 8: System Response

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Outline

System Response to Test Input Signals

Impulse Response

Step Response

Exponential Response

Frequency Response

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System Response to Test Input Signals

Impulse Response

Step Response

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System Response

- Consider an LTI ODE system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

- The system output satisfies the **convolution equation**:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

- The response $\mathbf{y}(t)$ is evaluated by separating out the short-term response from the long-term response

- **Transient response**: the response after an input is applied and before the output settles at its final value

- **Steady-state response**: the portion of the output response that reflects the long-term behavior of the system under the given input
  - For constant inputs, the steady-state response will often be constant (e.g., **step response**)
  - For periodic inputs, the steady-state response will often be periodic (e.g., **frequency response**)

System Response

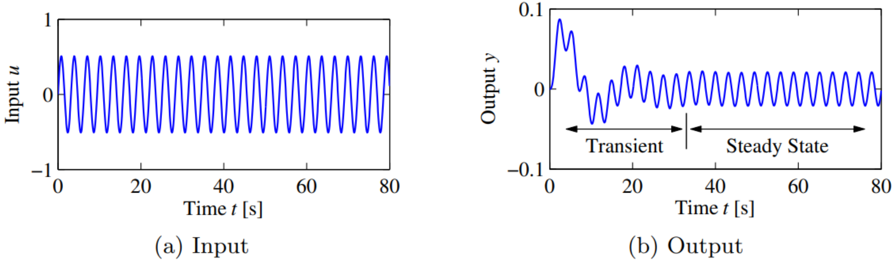


Figure 6.8: Transient versus steady-state response. The input to a linear system is shown in (a), and the corresponding output with $x(0) = 0$ is shown in (b). The output signal initially undergoes a transient before settling into its steady-state behavior.

Test Input Signals

- The transient and steady-state response of a system are often studied for specific test input signals

Test Signal	$u(t)$	$U(s)$
Impulse	$u(t)=\delta(t)=\begin{cases}\infty, & t=0,\\ 0, & t\neq 0\end{cases}$	$U(s)=1$
Step	$u(t)=H(t)=\int_{-\infty}^{t}\delta(\tau)d\tau=\begin{cases}1, & t\geq 0,\\ 0, & t<0\end{cases}$	$U(s)=\frac{1}{s}$
Ramp	$u(t)=tH(t)=\begin{cases}t, & t\geq 0,\\ 0, & t<0\end{cases}$	$U(s)=\frac{1}{s^2}$
Parabola	$u(t)=\frac{t^2}{2}H(t)=\begin{cases}\frac{t^2}{2}, & t\geq 0,\\ 0, & t<0\end{cases}$	$U(s)=\frac{1}{s^3}$
Sine	$u(t)=\begin{cases}\sin(\omega t), & t\geq 0,\\ 0, & t<0\end{cases}$	$U(s)=\frac{\omega}{s^2+\omega^2}$
Cosine	$u(t)=\begin{cases}\cos(\omega t), & t\geq 0,\\ 0, & t<0\end{cases}$	$U(s)=\frac{s}{s^2+\omega^2}$
Exponential	$u(t)=\begin{cases}e^{s_0 t}, & t\geq 0,\\ 0, & t<0\end{cases}$	$U(s)=\frac{1}{s-s_0}$

MATLAB Test Input Functions

- SYS = zpk(Z,P,K) creates a continuous-time zero-pole-gain (zpk) model SYS with zeros Z, poles P, and gains K:

```
fbksys = zpk([-4],[-8.8426, -2.0787 + 1.7078i, -2.0787 -1.7078i],8);
```

- Y = step(SYS,T): computes the step response Y of SYS at times T

```
t = 0:0.01:5;
step(fbksys,t);
```

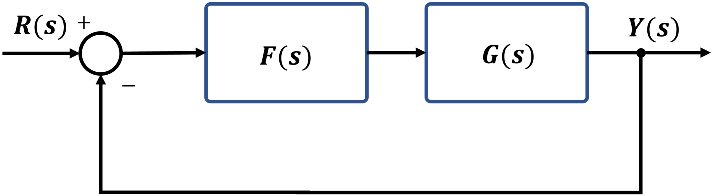
- Y = impulse(SYS, T): computes the impulse response Y of SYS at times T

```
t = 0:0.01:5;
impulse(fbksys,t);
```

- Y = lsim(SYS,U,T): computes the output response Y of SYS with input U at times T

```
[u,t] = gensig('square',4,10,0.1);
lsim(fbksys,u,t);
```

Steady-State Error



- Consider a feedback system with controller $F(s)$ and plant $G(s)$
- The forward-path gain $F(s)G(s)$ is a rational function of the form:

$$F(s)G(s) = k\frac{(s-z_1)\cdots(s-z_m)}{s^q(s-p_{q+1})\cdots(s-p_n)}$$

where $0 \leq q \leq n$ explicitly denotes the number of poles equal to zero:

$$p_1 = p_2 = \cdots = p_q = 0$$

Steady-State Error

- We will examine the steady-state error for test signals of the form $r(t) = \frac{t^d}{d!}$ for $t \geq 0$, such as step ($d = 0$), ramp ($d = 1$), parabola ($d = 2$), etc.

- Consider the error signal $e(t) = r(t) - y(t)$ with Laplace transform:

$$E(s) = R(s) - Y(s) = R(s) - F(s)G(s)E(s)$$

- The reference-to-error transfer function is:

$$E(s) = \frac{1}{1 + F(s)G(s)} R(s)$$

- When $r(t) = t^d/d!$ and $R(s) = 1/s^{d+1}$, the steady-state error can be obtained by the final value theorem:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{1}{(1 + F(s)G(s))s^d}$$

Steady-State Error

- When $r(t) = t^d/d!$ and $R(s) = 1/s^{d+1}$, the steady-state error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{(1+F(s)G(s))s^d}$$

- The error is determined by the **error coefficient**:

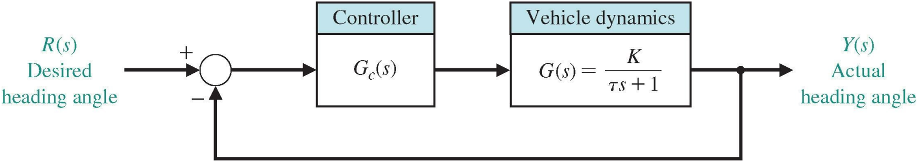
$$s^d F(s)G(s) = k\frac{s^d(s-z_1)\cdots(s-z_m)}{s^q(s-p_{q+1})\cdots(s-p_n)}$$

- Three cases are possible, assuming that the system is **stable** (all poles of $sE(s)$ are in the left-half plane):
  - If $d < q$, then $s^d F(s)G(s)$ will contain a term s^{q-d} in the denominator and $sE(s)$ will contain $q-d$ zeros at the origin. Hence, $\lim_{s\to 0} sE(s) = 0$ and **zero steady-state error** will be achieved.
  - If $d = q$, then $sE(s)$ will contain no zeros at the origin and a **constant finite steady-state error** will be achieved.
  - If $d > q$, then $sE(s)$ will have $d-q$ poles at the origin. Hence, $\lim_{s\to 0} sE(s) = \infty$ and an **infinite steady-state error** will be achieved. In other words, the system output will not track the reference input at all.

Control System Type

- The results on the previous slide indicate that the number q of poles at the origin in $F(s)G(s)$ determines the type of reference inputs that the closed-loop system is able to track
- The number q of poles at the origin in $F(s)G(s)$ is called **system type**
- A system of type q can track polynomial reference signals of degree q or less to within a constant finite steady-state error
- During control design, the controller gain $F(s)$ can be chosen to achieve a certain number of poles at the origin if the process $G(s)$ does not have the required number of poles to track a desired reference signal
- It appears that having more integrators $(1/s)$ in $F(s)G(s)$ is better since it allow tracking higher-order reference signals. However, the larger q is, the harder it is to stabilize the system since integrators slow the response down

Example: Mobile Robot Heading Angle Control



- Consider a heading-angle steering control system for a mobile robot:

$$\text{Heading dynamics: } G(s) = \frac{K}{\tau s + 1} \qquad \text{Control gain: } G_c(s) = K_1 + \frac{K_2}{s}$$

- What is the steady-state error of the closed-loop system for a step input and a ramp input?

Example: Mobile Robot Heading Angle Control

- If $K_2 = 0$:
  - the forward path gain is: $G_c(s)G(s) = \frac{KK_1}{\tau(s+1/\tau)}$
  - the system is type 0 with error coefficient:

$$K_p = \lim_{s \to 0} G_c(s)G(s) = KK_1$$

  - the steady-state error for a step input is:

$$\lim_{t \to \infty} e(t) = \frac{1}{1 + K_p} = \frac{1}{1 + KK_1}$$

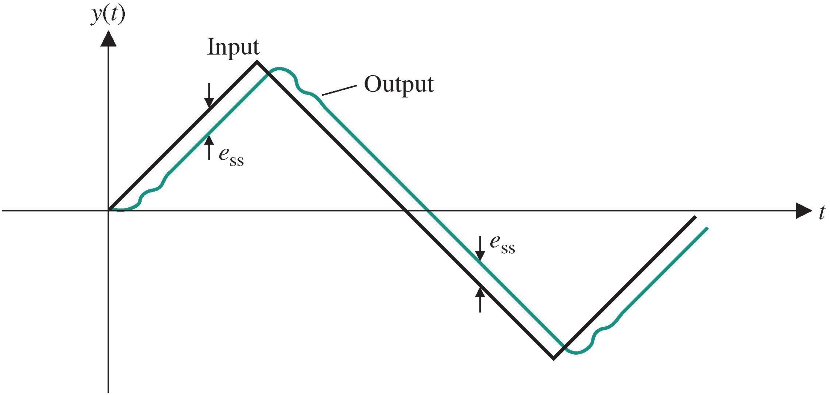
- If $K_2 > 0$:
  - the forward path gain is: $G_c(s)G(s) = \frac{KK_1(s+K_2/K_1)}{\tau s(s+1/\tau)}$
  - the system is type 1 with error coefficient:

$$K_v = \lim_{s \to 0} sG_c(s)G(s) = KK_2$$

  - the steady-state error for a ramp input is:

$$\lim_{t \to \infty} e(t) = \frac{1}{K_v} = \frac{1}{KK_2}$$

Example: Mobile Robot Heading Angle Control



- Transient response of the heading-angle steering control system to a triangular wave reference input
- The response shows the effect of the non-zero steady-state error $e_{ss} = 1/(KK_2)$

Outline

System Response to Test Input Signals

Impulse Response

Step Response

Exponential Response

Frequency Response

Impulse Response

- LTI ODE System:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

- **Impulse response**: response to an impulse input $u(t) = \delta(t)$:

$$\begin{aligned} y(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)d\tau + \mathbf{D}\delta(t) \\ &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t) \end{aligned}$$

- The impulse response with zero initial conditions reveals the transfer function:

$$Y(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + G(s)U(s)$$

$$\mathbf{x}(0) = \mathbf{0}, \quad U(s) = 1 \qquad \Rightarrow \qquad Y(s) = G(s)$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = g(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t)$$

Example: Impulse Response

- LTI ODE System:

$$\dot{y} + 10y = 9u$$

- Transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{9}{s+10}$$

- The impulse response with zero initial conditions is obtained with $U(s) = 1$:

$$Y(s) = G(s) \qquad \Rightarrow \qquad y(t) = \mathcal{L}^{-1}\{G(s)\} = 9e^{-10t}$$

- Steady-state impulse response:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s) = 0$$

Impulse Response

- Let the impulse response with zero initial conditions of an LTI ODE be:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t)$$

- Any input $u(t)$ can be decomposed into an infinite set of shifted impulses:

$$u(t) = \int_0^t \delta(t-\tau)u(\tau)d\tau$$

- By the principle of superposition, the forced response to any input $u(t)$ is the convolution of the input with the impulse response:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)}_{\text{natural response}} + \underbrace{\int_0^t g(t-\tau)u(\tau)d\tau}_{\text{forced response}}$$

Example: Impulse Response

- System:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0$$

- Input:

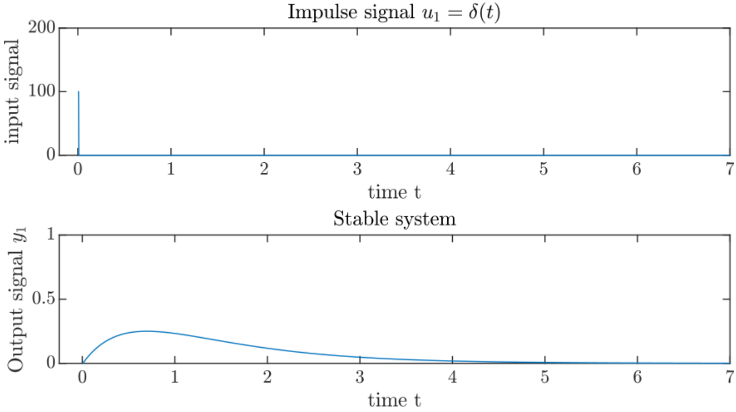
$$u(t) = \begin{cases} 1/\epsilon & \text{if } 0 \leq t < \epsilon \\ 0 & \text{else} \end{cases}$$

- Simulate with $\epsilon = 0.01$:

```
sys = ss(A, B, C, D); % create an LTI system
y = lsim(sys,u,t,x0); % simulate response to input u
```

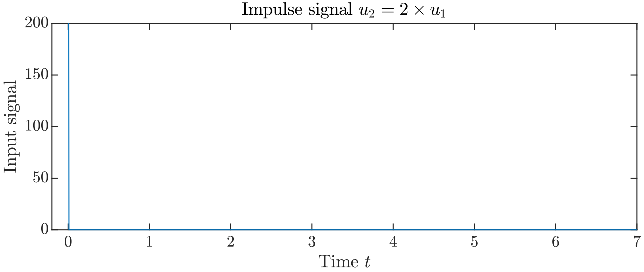
Example: Impulse Response

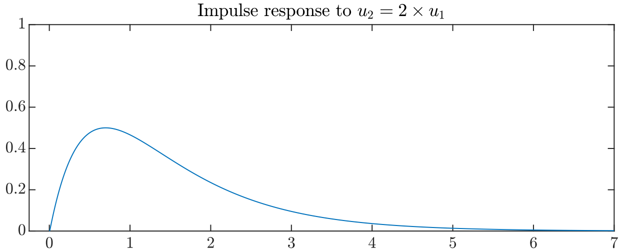
- **Case 1**: $u_1(t) = \delta(t)$



Example: Impulse Response

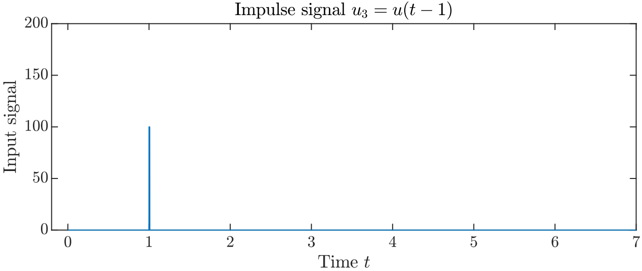
- **Case 2**: scale the input: $u_2(t) = 2u_1(t) = 2\delta(t)$

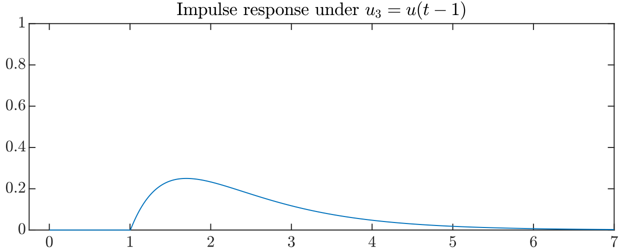




Example: Impulse Response

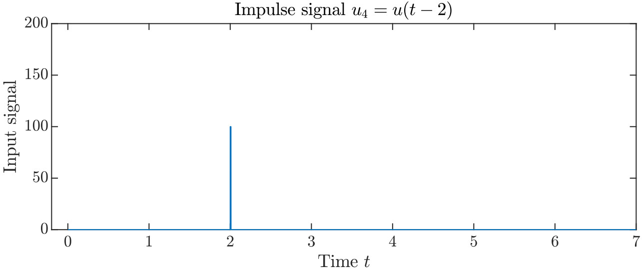
- **Case 3**: shift the input: $u_3(t) = u_1(t-1) = \delta(t-1)$

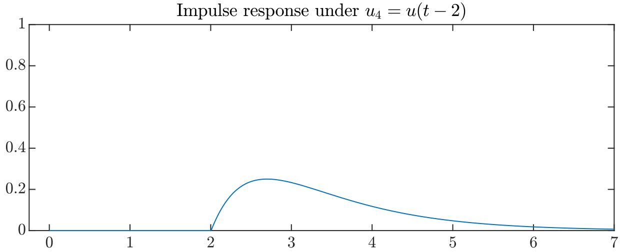




Example: Impulse Response

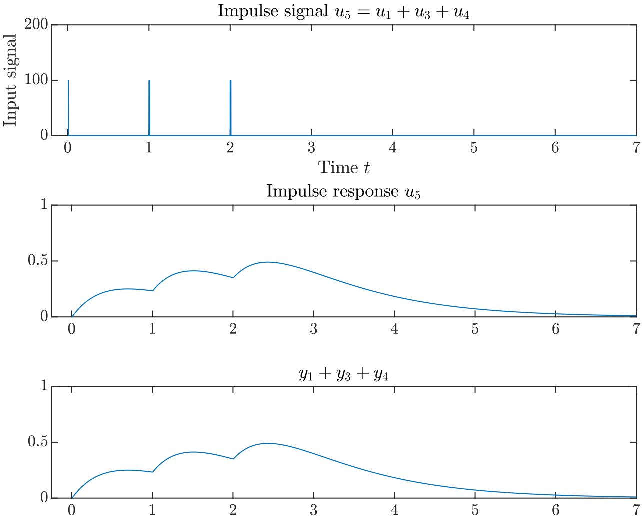
- **Case 4**: shift the input: $u_4(t) = u_1(t-2) = \delta(t-2)$





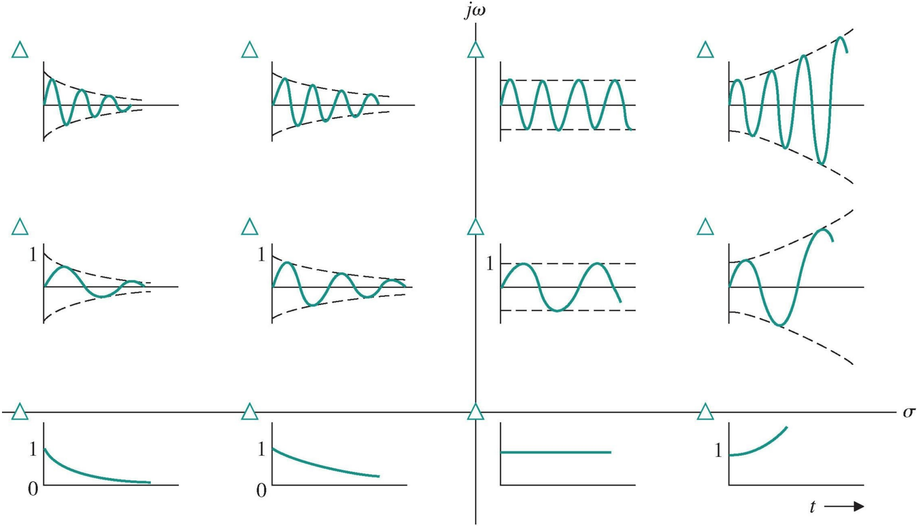
Example: Impulse Response

- **Case 5**: sum three inputs: $u_5(t) = u_1(t) + u_3(t) + u_4(t)$



Impulse Response vs s-Plane Pole Locations

- Impulse response of an abstract control system for various transfer function pole locations in the s-plane



Outline

System Response to Test Input Signals

Impulse Response

Step Response

Exponential Response

Frequency Response

Step Response

- LTI ODE system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad\qquad G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- **Exponential response**: SISO LTI system response to $u(t) = e^{s_0 t}$ for $t \geq 0$ such that $s_0 \in \mathbb{C}$ is not an eigenvalue of $\mathbf{A}$:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}(0) - (s_0\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right)}_{\text{transient response}} + \underbrace{\left(\mathbf{C}(s_0\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)e^{s_0 t}}_{\text{steady-state response}}$$

- **Step response**: the response to a step input $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t > 0 \end{cases}$ is a special case of $u(t) = e^{s_0 t}$ with $s_0 = 0$:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}(\mathbf{x}(0) + \mathbf{A}^{-1}\mathbf{B})}_{\text{transient response}} \quad + \quad \underbrace{G(0)}_{\text{steady-state response}}$$

Example: Step Response

- LTI ODE System:

$$\dot{y} + 10y = 9u$$

- Transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{9}{s+10}$$

- The step response with zero initial conditions is obtained with $U(s) = \frac{1}{s}$:

$$Y(s) = \frac{G(s)}{s} = \frac{9}{s(s+10)} = \frac{9}{10s} - \frac{9}{10(s+10)}$$

- Time-domain step response:

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \underbrace{0.9}_{\text{steady-state}} - \underbrace{0.9e^{-10t}}_{\text{transient}}$$

Example: Stable System Step Response

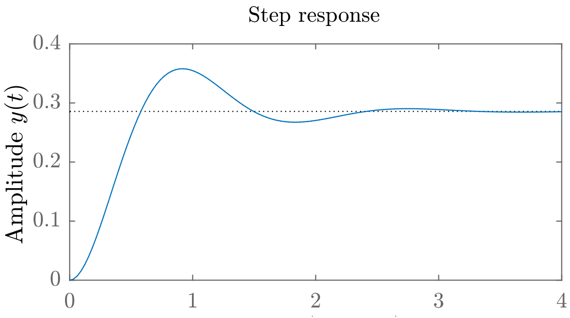
- LTI system:

$$\mathbf{A} = \begin{bmatrix} -1 & 4 \\ -3 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0.$$

- Transfer function:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{4}{s^2 + 3s + 14}$$

- Step response:



Example: Unstable System Step Response

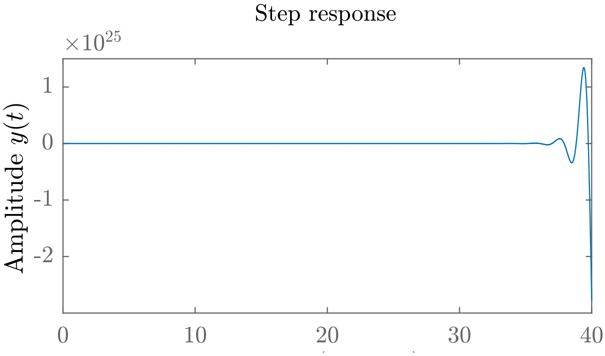
- LTI system:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0.$$

- Transfer function:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{4}{s^2 - 3s + 14}$$

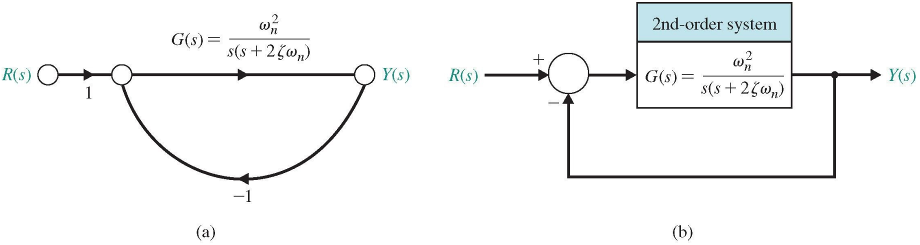
- Step response:



Step Response Performance Measures

- The step response of a feedback control system is evaluated using several performance criteria:
  - **Rise time**
  - **Percent overshoot**
  - **Settling time**
  - **Steady-state error**

Second-Order Feedback Control System



- Consider a second-order feedback control system
- Closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with **natural frequency** ω_n and **damping ratio** ζ

Second-Order System Poles

- Transfer function: $T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
- Transfer function poles:

$$p = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Response	Damping ratio	Poles
Underdamped	$\zeta < 1$	$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
Critically damped	$\zeta = 1$	$-\omega_n, -\omega_n$
Overdamped	$\zeta > 1$	$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

- The natural frequency ω_n and damping ratio ζ of a pole p can be obtained as:

$$\omega_n = |p| \qquad\qquad \zeta = -\cos(\angle p)$$

Underdamped Second-Order System Impulse Response

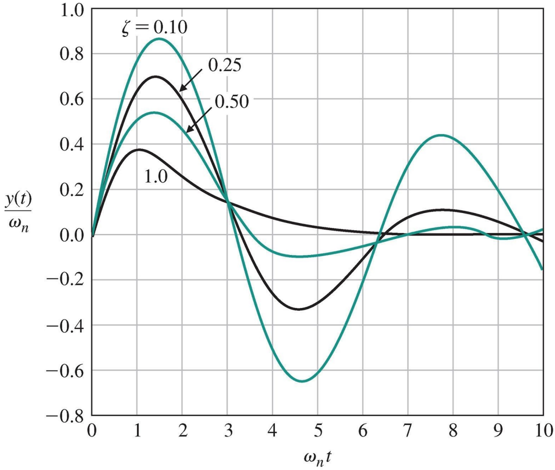
- Consider the underdamped and critically damped cases ($0 \leq \zeta \leq 1$)
- Impulse input: $r(t) = \delta(t)$, $R(s) = 1$
- Impulse response (s domain): reveals the transfer function:

$$Y(s) = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\alpha)^2 + \omega_d^2}$$

  where we introduced the terms:
  - **damping constant**: $\alpha = \zeta\omega_n$
  - **damped frequency**: $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- Impulse response (t domain):

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}\,t) \\ &= \left(\frac{\alpha^2}{\omega_d} + \omega_d\right) e^{-\alpha t} \sin(\omega_d t) \end{aligned}$$

Underdamped Second-Order System Impulse Response



- As the damping ζ decreases, the poles approach the imaginary axis and the response becomes increasingly oscillatory

Underdamped Second-order System Step Response

- Step response (s domain): obtained with $R(s) = \frac{1}{s}$:

$$Y(s) = \frac{G(s)}{s(1+G(s))} = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{1}{s} - \frac{(s+\zeta\omega_n)+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

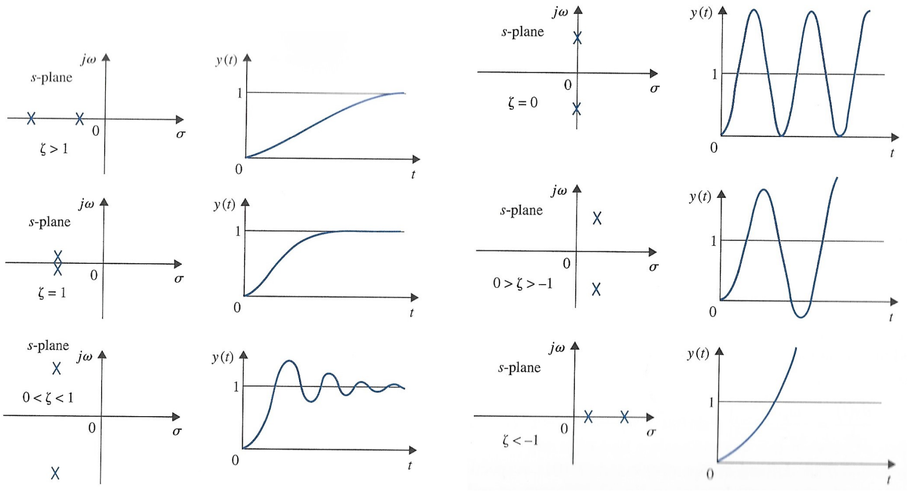
- Step response (t domain):

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}t + \cos^{-1}(\zeta)) \\ &= 1 - e^{-\alpha t}\left(\cos(\omega_d t) + \frac{\alpha}{\omega_d}\sin(\omega_d t)\right) \end{aligned}$$

- The derivative of the step response is equal to the impulse response:

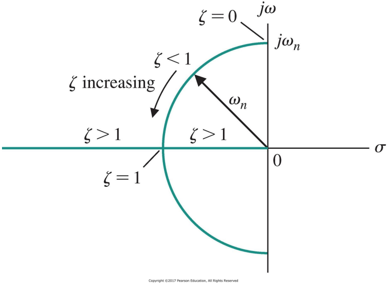
$$\frac{d}{dt}y(t) = \left(\frac{\alpha^2}{\omega_d} + \omega_d\right) e^{-\alpha t}\sin(\omega_d t)$$

Second-Order System Step Response



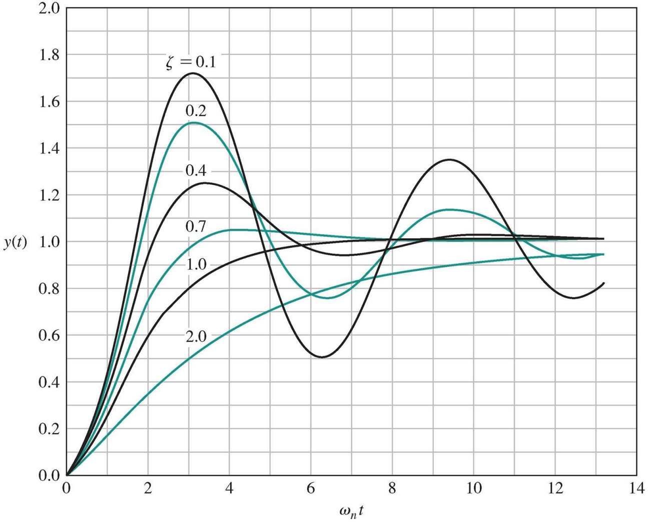
Second-Order System Step Response

- If the poles are complex, the step response has oscillations and overshoot
- As the poles move toward the real axis, maintaining a fixed distance from the origin (ζ increasing for fixed ω_n), the oscillations and overshoot decrease
- If ω_n increases, the poles move further left in the left half plane and the oscillations reduce faster
- If all poles are on the negative real axis, there are no oscillations or overshoot
- If there is a pole in the open right half plane, then the step response contains a term that goes to ∞



- For constant ω_n, as ζ varies, the complex conjugate roots follow a circular locus

Underdamped Second-Order System Step Response



- As the damping ζ decreases, the poles approach the imaginary axis and the response becomes increasingly oscillatory

Step Response Performance Measures

- **Rise time** t_r: time for the system step response $y(t)$ to go from $\delta\%$ to $1-\delta\%$ of the steady-state value

- **Peak time** t_p: time at which the system step response $y(t)$ achieves its maximum value (**defined only for underdamped systems**)

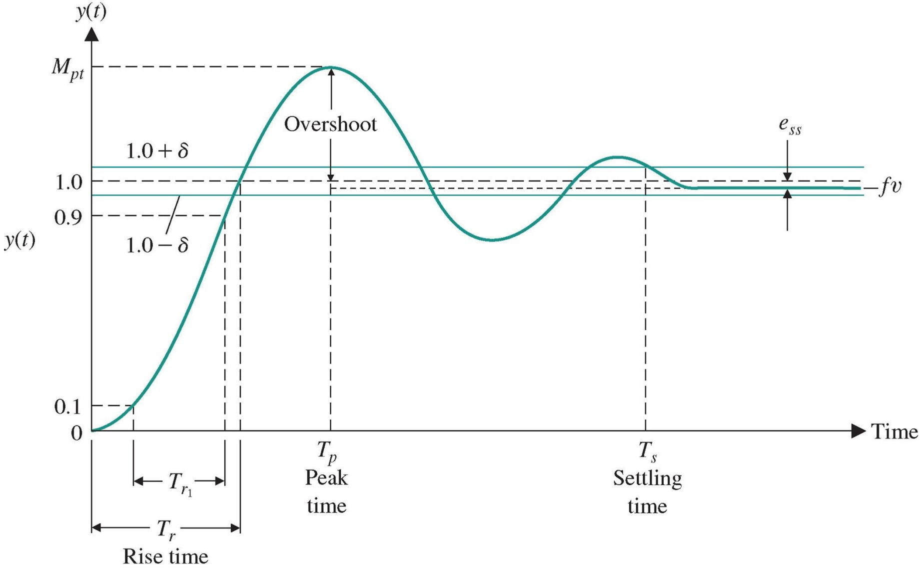
- **Percent overshoot**: the max value of the system step response, $y(t_p)$, expressed as a percentage of the steady-state value, $y(\infty) = \lim_{t\to\infty} y(t)$:

$$\text{percent overshoot} = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$

- **Settling time** t_s: the time required for the step response to settle within $\delta\%$ of the steady-state value, i.e., for all $t \geq t_s$:

$$|y(t) - y(\infty)| \leq \frac{\delta}{100}$$

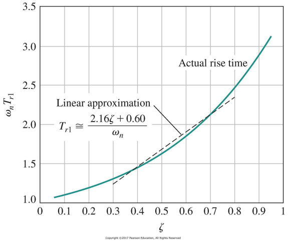
Step Response Performance Measures



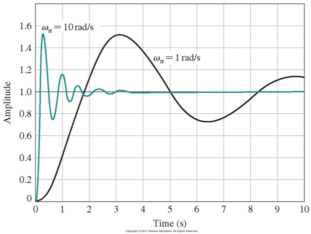
Rise Time

- **Rise time**: an exact expression for t_r is challenging to obtain
- The best linear fit to the 10%-to-90% rise time is accurate for $0.3 < \zeta < 0.8$:

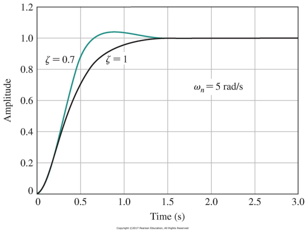
$$t_r \approx \frac{2.16\zeta + 0.6}{\omega_n}$$



(a) Rise time



(b) Effect of ω_n for $\zeta = 0.2$



(c) Effect of ζ for $\omega_n = 5$

Peak Time

- **Peak time**: obtained by setting the derivative of the step response to zero and solving for t:

$$0 = \left(\frac{\alpha^2}{\omega_d} + \omega_d\right) e^{-\alpha t} \sin(\omega_d t) \quad \Rightarrow \quad t = \frac{k\pi}{\omega_d},\ k = 0, 1, 2, \ldots$$

- The maximum overshoot occurs at the first peak:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

- The maximum value of the system step response is:

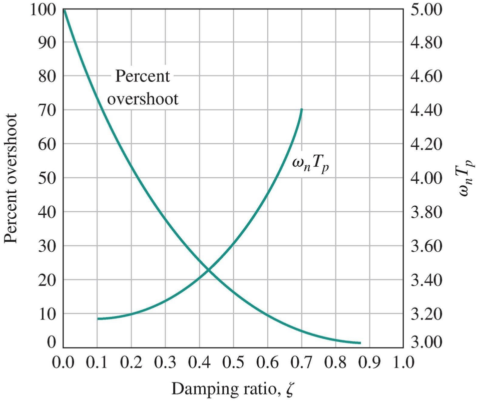
$$y(t_p) = 1 + e^{-\alpha \frac{\pi}{\omega_d}} = 1 + e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}}$$

Percent Overshoot

- **Percent overshoot**: since $y(\infty) = \lim_{t\to\infty} y(t) = 1$:

$$\text{percent overshoot} = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\% = e^{-\alpha \frac{\pi}{\omega_d}} \times 100\%$$
$$= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

- There is a trade-off between swiftness of response and percent overshoot



Settling Time

- Underdamped second-order system step response:

$$y(t) = 1 - e^{-\alpha t}\left(\cos(\omega_d t) + \frac{\alpha}{\omega_d}\sin(\omega_d t)\right)$$

- **Settling time**: since the cosine and sine terms oscillate, approximate the time required for the step response to settle within $\delta\%$ of the steady-state value by calculating the time at which the exponential term $e^{-\alpha t}$ becomes equal to $\delta/100$:

$$e^{-\alpha t_s} \approx \frac{\delta}{100} \qquad \Rightarrow \qquad t_s \approx -\frac{1}{\alpha}\ln\frac{\delta}{100}$$

- For $\delta = 2\%$, the settling time is: $t_s \approx \dfrac{4}{\alpha} = \dfrac{4}{\zeta\omega_n}$

Step Response Performance Measures

- It is desirable to achieve small t_r, small percent overshoot, and small t_s
- As ω_n increases with fixed ζ, t_r decreases, t_p decreases, the percent overshoot stays the same, and t_s decreases
- As ζ increases with fixed ω_n, t_r stays the same, t_p increases, the percent overshoot decreases, and t_s decreases
- If desired upper bounds are given:

$$t_r \leq \bar{t}_r \qquad t_p \leq \bar{t}_p \qquad \text{p.o.} \leq \bar{\text{p.o.}} \qquad t_s \leq \bar{t}_s$$

we can obtain constraints for ζ and ω_n, which determine valid regions for the transfer function poles $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ in the complex plane:

$$\frac{2.16\zeta + 0.6}{\omega_n} \leq \bar{t}_r \qquad \frac{\zeta}{\sqrt{1-\zeta^2}}\pi \geq -\ln\frac{\bar{\text{p.o.}}}{100}$$

$$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \leq \bar{t}_p \qquad \frac{4}{\zeta\omega_n} \leq \bar{t}_s$$

Effect of Additional Poles or Zeros

- So far we analyzed the step response of an underdamped second-order system with transfer function:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- What happens if the transfer function contains zeros or additional poles?

Effect of Poles on the Step Response

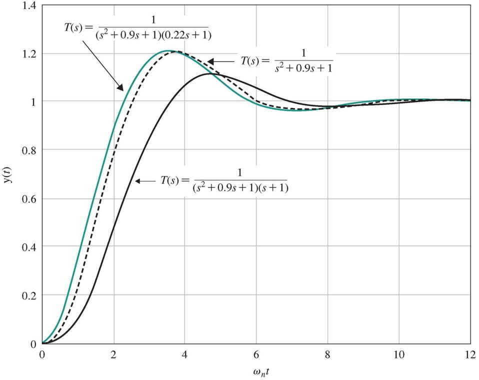
- From the partial fraction expansion of the transfer function, we know that a pole p contributes a term of the form re^{pt}
- If any pole is in the right half-plane ($\text{Re}(p) > 0$), then the step response will go to infinity (unstable system)
- If any pole is far left in the left half-plane ($\text{Re}(p) \ll 0$), then its contribution to the step response dies out quickly
- If the poles can be divided into a set that is close to the origin, and another set that is far away, then the poles that are close to the origin are called **dominant poles**. The exponential terms in the step response of the dominant poles determine the overall system response.
- Adding a left half-plane pole to the transfer function makes the response **slower** because an additional exponential term must die out before the system reaches its final value

Introducing a Pole in a Second-Order System

- Introduce a pole $s = -1/\gamma$ in the transfer function:

$$T_\gamma(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(\gamma s + 1)}$$

- If $|1/\gamma| \geq 10|\zeta\omega_n|$, then $T_\gamma(s)$ can be approximated by $T(s)$ since the contribution of the new pole to the step response is dominated by the original two poles



Introducing a Zero in a Second-Order System

- Introduce a zero $s = -a$ in the transfer function:

$$T_a(s) = \frac{(\frac{1}{a}s + 1)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- The reason for writing $(\frac{1}{a}s + 1)$ instead of $s + a$ is to maintain a steady-state value of 1

- The new transfer function can be decomposed as:

$$T_a(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{s}{a}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = T(s) + \frac{s}{a}T(s)$$

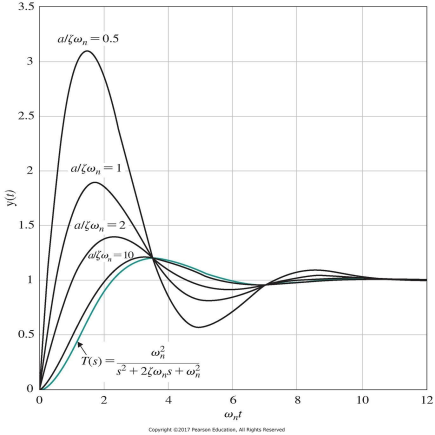
- The response of the third order system to a step $R(s) = 1/s$ is:

$$Y_a(s) = \left(T(s) + \frac{s}{a}T(s)\right)\frac{1}{s} = Y(s) + \frac{s}{a}Y(s)$$
$$y_a(t) = y(t) + \frac{1}{a}\dot{y}(t)$$

  where $Y(s)$ and $y(t)$ are the s- and t-domain step response of the original second-order system

Introducing a Zero in a Second-Order System

- Step response of a system with transfer function $T_a(s) = \frac{(\frac{1}{a}s+1)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ and $\zeta = 0.45$

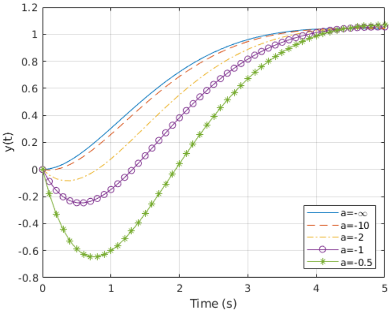


- As a increases, the zero moves farther into the left half-plane and the step response of $T_a(s)$ approaches that of the second-order system $T(s)$

Introducing a Zero in a Second-Order System

- We can see from the step-response of $T_a(s)$ that adding a zero in the left half-plane makes the step response **faster**:
  - the rise time decreases
  - the peak time decreases
  - the overshoot increases
  - the settling time does not change

- If the zero is added in the right half-plane (i.e., $a < 0$), then $\dot{y}(t)$ is subtracted from $y(t)$ to produce $y_a(t)$. The response is **slower** and can go decrease before before rising to its steady state value (**undershoot**).



Dominant Pole-Zero Approximation

- If a high-order system has a cluster of poles and zeros that are much closer (e.g., 5 times or more) to the origin than the remaining poles and zeros, then the system can be approximated by a lower order system with only those dominant poles and zeros
- **Example**: if $a \gg \zeta\omega_n > 0$ and $1/\gamma \gg \zeta\omega_n > 0$, then:

$$T_{a,\gamma}(s) = \frac{\omega_n^2(\frac{1}{a}s + 1)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(\gamma s + 1)} \approx T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Example

- Consider a control system with transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}$$

(a) Determine the steady-state error for a unit step input.

(b) Assume that the complex poles are dominant. Determine the percent overshoot and the settling time to within 2% of the steady-state value.

(c) Plot the actual system response and compare it with the estimates of part (b).

Example: Part (a)

- The error is:

$$E(s) = R(s) - Y(s) = R(s) - T(s)R(s) = (1 - T(s))R(s)$$

- The steady-state error for input $R(s) = 1/s$ is:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} (1 - T(s))$$
$$= \lim_{s \to 0} \left(1 - \frac{108(s+3)}{(s+9)(s^2+8s+36)} \right) = 1 - \frac{108(3)}{9(36)} = 0$$

Example: Part (b)

- Assuming that the complex poles are dominant:

$$T(s) = \frac{36(\frac{s}{3}+1)}{(s+9)(s^2+8s+36)} \approx \frac{36}{s^2+8s+36}$$

- The second-order system approximation has natural frequency $\omega_n = 6$ and damping ratio $\zeta = \frac{8}{2\omega_n} = \frac{2}{3}$.

- The percent overshoot is:

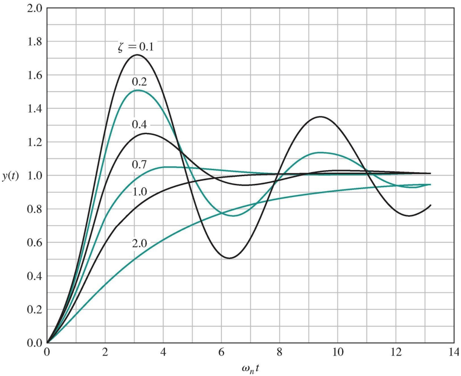
$$\text{p.o.} = 100 \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi\right) = 100 \exp\left(-\frac{2\pi}{\sqrt{5}}\right) \approx 6\%$$

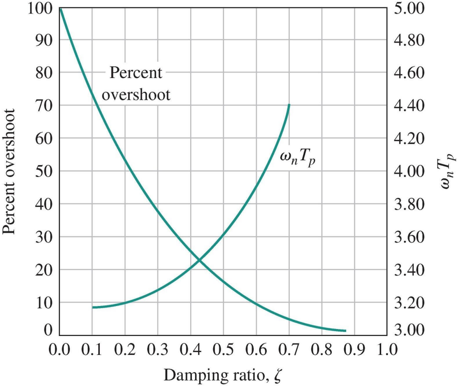
- The settling time to within 2% of the steady-state value is:

$$t_s \approx \frac{4}{\zeta\omega_n} = 1 \text{ second.}$$

Example: Part (b)

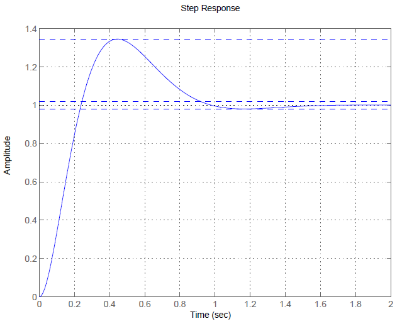
- The percent overshoot can also be determined approximately from the second-order system plots:





Example: Part (c)

- The step response of the original system is:




```
sys = tf([108 324],[1 17 108 324]);
characteristics = stepinfo(sys,
    'RiseTimeLimits',[0.05,0.95],
    'SettlingTimeThreshold', 0.02);
stepplot(sys);
hold on;
plot([0,2],[characteristics.Peak,
     characteristics.Peak],'b--');
```

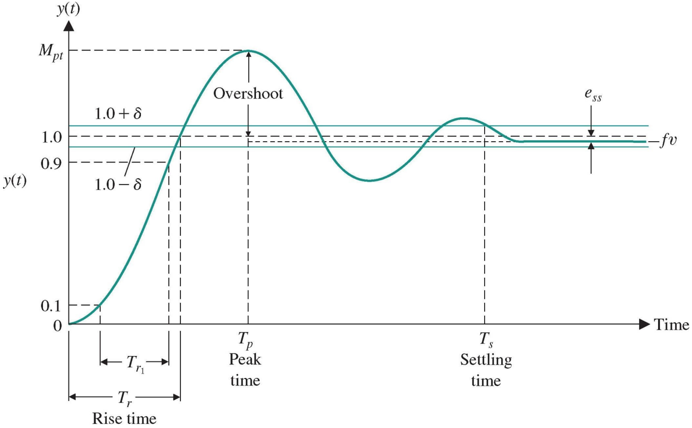
- The actual percent overshoot and settling time are:

$$\text{p.o} = 34.4\% \qquad \text{and} \qquad t_s = 1.18 \text{ second.}$$

- The difference in the actual and estimated percent overshoot is due to the term $(\frac{s}{a} + 1)$ in the numerator, which does not satisfy the requirement for an accurate dominant pole-zero approximation:

$$3 = a \not\gg \zeta\omega_n = 4$$

Step Response Performance Measures



- **Rise time**: from 10% to 90% of steady-state value: $t_r \approx \frac{2.16\zeta+0.6}{\omega_n}$
- **Peak time**: time at which the response is maximum: $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
- **Overshoot**: overshoot as percent of steady-state: $\text{p.o.} = 100\exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)\%$
- **Settling time**: response settles within 2% of steady-state: $t_s \approx \frac{4}{\zeta\omega_n}$
- **Steady-state error**: $e_{ss} = 1 - \lim_{t\to\infty} y(t) = 1 - G(0)$

Outline

System Response to Test Input Signals

Impulse Response

Step Response

Exponential Response

Frequency Response

Exponential Response

- LTI ODE System:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad\qquad G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- **Exponential response**: SISO LTI system response to $u(t) = e^{s_0 t}$ for $t \geq 0$ such that $s_0 \in \mathbb{C}$ is not an eigenvalue of $\mathbf{A}$:

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}(0) - (s_0\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right)}_{\text{transient response}} + \underbrace{\left(\mathbf{C}(s_0\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)e^{s_0 t}}_{\text{steady-state response}}$$

- The transfer function $G(s)$ is a complex number:

$$G(s) = |G(s)|e^{j\angle G(s)}$$

- Steady-state exponential response:

$$y_{ss}(t) = |G(s_0)|e^{j\angle G(s_0)}e^{s_0 t} = |G(s_0)|e^{s_0 t + j\angle G(s_0)}$$

Outline

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Frequency Response

- LTI ODE System:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- **Frequency response**: response to a sinusoidal input $u(t) = \sin(\omega t + \phi)$

Frequency Response

The steady-state response of LTI ODE system with transfer function $G(s)$ to a sinusoidal input $u(t) = \sin(\omega t + \phi)$ is a sinusoid of the **same frequency** with **amplitude scaled by** $|G(j\omega)|$ and **phase shifted by** $\angle G(j\omega)$:

$$y_{ss}(t) = |G(j\omega)| \sin(\omega t + \phi + \angle G(j\omega))$$

- The **magnitude** $|G(j\omega)|$ is determined from the ratio of the amplitudes of the output versus the input sinusoids
- The **phase** $\angle G(j\omega)$ is determined from the ratio of the time of the output versus the input zero crossings

Example: Stable System Frequency Response

- System 1:

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 4 \\ -3 & -2 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D}_1 = 0$$

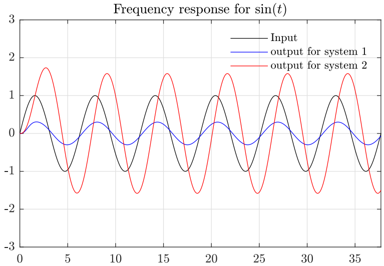
$$G_1(s) = \frac{4}{s^2 + 3s + 14} \quad |G_1(j)| = 0.3 \quad \angle G(j) = -13^\circ$$

- System 2:

$$\mathbf{A}_2 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D}_2 = 0$$

$$G_2(s) = \frac{5}{s^2 + 3s + 2} \quad |G_2(j)| = 1.58 \quad \angle G_2(j) = -71.5^\circ$$

- Response to $u(t) = \sin(t)$



Example: Stable System Frequency Response

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$$\mathbf{A}_1 = \begin{bmatrix} -1 & 4 \\ -3 & -2 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D}_1 = 0$$

$$G_1(s) = \frac{4}{s^2 + 3s + 14} \quad |G_1(0.5j)| = 0.29 \quad \angle G(0.5j) = -6.2^\circ$$

- System 2:

$$\mathbf{A}_2 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D}_2 = 0$$

$$G_2(s) = \frac{5}{s^2 + 3s + 2} \quad |G_2(0.5j)| = 2.17 \quad \angle G_2(0.5j) = -40.6^\circ$$

- Response to $u(t) = \sin(0.5t)$