ECE171A: Linear Control System Theory
Lecture 2: Feedback Control Principles

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Outline

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Advantages of Feedback Control

- **Disturbance attenuation**:
  - Closed-loop control reduces the effect of disturbances and noise in the system response
- **Robustness to parameter variations**:
  - Closed-loop control reduces the sensitivity of the system response to variations in the model parameters
  - Accurate control may be achieved with imprecise components
- **Dynamic behavior shaping**:
  - Closed-loop control may widen the range in which a system behaves linearly
  - Closed-loop control allows the system output to track a desired reference signal

Disadvantages of Feedback Control

- **Increased system complexity**:
  - Sensing components are necessary for feedback control, which may be expensive and introduce noise
- **Loss of gain**:
  - The forward gain of a closed-loop system is smaller by a certain factor than the forward gain of an open-loop system
  - The gain is decreased by the same factor that reduces the sensitivity to parameter variations and disturbances
  - In practice, the advantage of increased robustness outweighs the loss of control gain
- **Potential for instability**:
  - Closed-loop control may lead to system instability, even if the open-loop system is stable

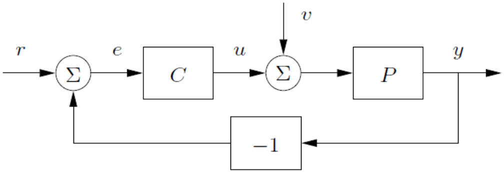
Examples of Feedback Control Use

- Feedback control was used by James Watt to make steam engines run at constant speed in spite of varying load (industrial revolution)
- Feedback control was used by electrical engineers to make water-turbine generators deliver electricity with constant frequency and voltage
- Feedback control is used to alleviate the effects of disturbances in the process industry, machine tools, engines, and vehicle cruise control
- The human body relies on feedback to keep body temperature, blood pressure, and other important variables constant
- **Servo problem**: a key application of feedback control is to make a system's output follow a desired reference signal
  - Examples: car steering, satellite tracking with an antenna, audio amplifiers, industrial robots

Outline

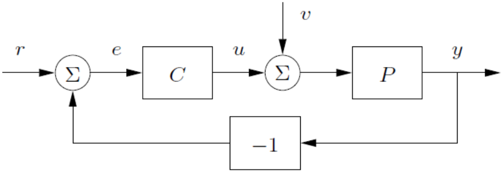
Example: Nonlinear Static System

- Feedback control has had significant impact on process control in chemical plants
- The dynamical system to be controlled is often referred to as a **plant**



- Reference signal: $r(t)$
- Controller: C
- Plant: P
- Summing point: Σ
- Input: $u(t)$
- Disturbance: $v(t)$
- Output: $y(t)$
- Error: $e(t)$

Example: Nonlinear Static System



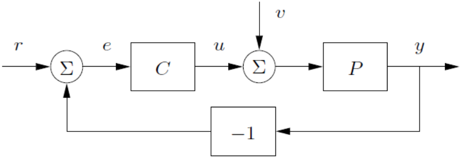
- **Plant** P: consider a static system (no ODE description):

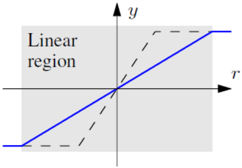
$$y = \mathrm{sat}(x) := \begin{cases} -1 & \text{if } x \le -1 \\ x & \text{if } |x| < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

- **Controller** C: consider a controller with constant gain $k > 0$:

$$u = ke$$

Dynamic Behavior Shaping





- Assume no disturbances for now: $v \equiv 0$
- **Open-loop system**: combination of C and P with no feedback:

$$y = \mathrm{sat}(kr) \quad \Rightarrow \quad \text{linear range: } |r| < 1/k$$

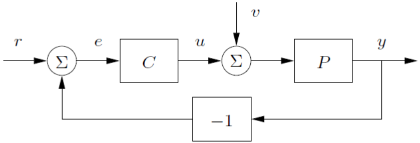
- **Closed-loop system**: combination of C and P with feedback:

$$\left.\begin{aligned} y &= \mathrm{sat}(u) \\ u &= k(r-y) \end{aligned}\right\} \quad \Rightarrow \quad y = \mathrm{sat}(k(r-y))$$

$$\Rightarrow \quad y = \mathrm{sat}\left(\frac{k}{k+1} r\right) \quad \Rightarrow \quad \text{linear range: } |r| < \frac{k+1}{k}$$

Observation 1: Feedback control **widens** the linear range of the system by a factor of $k+1$ compared to the open-loop system

Robustness to Parameter Variations

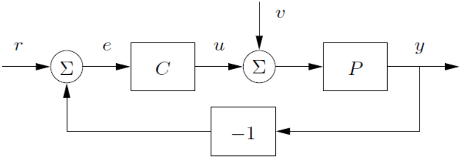


Parameter sensitivity: quantifies change in the system behavior due to change in the system parameters

- **Open-loop system**:
  - In the linear range: $y = kr$
  - It follows that $\dfrac{dy}{dk} = r = \dfrac{y}{k} \quad \Rightarrow \quad \dfrac{\Delta y}{y} = \dfrac{\Delta k}{k}$
  - **Sensitivity**: 10% change in k leads to 10% change in output
- **Closed-loop system**:
  - In the linear range: $y = \dfrac{k}{k+1} r$
  - It follows that $\dfrac{dy}{dk} = \dfrac{1}{(k+1)^2} r = \dfrac{1}{(k+1)} \dfrac{y}{k} \quad \Rightarrow \quad \dfrac{\Delta y}{y} = \dfrac{1}{(k+1)} \dfrac{\Delta k}{k}$
  - **Sensitivity**: for $k = 100$, 10% change in k leads to $\approx 0.1\%$ change in output

Observation 2: Feedback control **reduces the sensitivity** to gain variations by a factor of $k + 1$

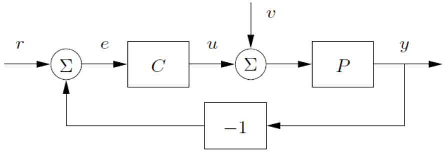
Disturbance Attenuation



- Suppose now that the system is subject to a disturbance signal v
- Assume $r \equiv 0$ for simplicity
- **Open-loop system**:
  - With $r \equiv 0$, $y = \mathrm{sat}(v)$
  - In the linear range, disturbances are passed through with no attenuation
- **Closed-loop system**:
  - With $r \equiv 0$, $y = \mathrm{sat}(v - ky) \quad \Rightarrow \quad y = \mathrm{sat}\left(\frac{v}{k+1}\right)$
  - In the linear range, disturbances are attenuated by a factor of $k + 1$

Observation 3: Feedback control **reduces the effect of disturbances** in the linear range by a factor of $k + 1$

Summary



- Static plant P:

$$y = \mathrm{sat}(x) := \begin{cases} -1 & \text{if } x \leq -1 \\ x & \text{if } |x| < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

- Constant-gain controller C:

$$u = ke, \qquad k > 0$$

Feedback control

- 1) *increases* the range of linearity of the system
- 2) *decreases* the sensitivity of the system response to parameter variations
- 3) *attenuates* the effect of disturbances

The **trade-off** is that

- 1) output sensing is required
- 2) the closed-loop gain is decreased by a factor of $k + 1$:

open-loop:

$$y = \mathrm{sat}(kr)$$

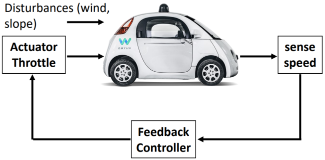
closed-loop:

$$y = \mathrm{sat}\left(\frac{k}{k+1} r\right)$$

Outline

Example: Cruise Control System

- A cruise controller aims to maintain constant velocity in the presence of disturbances caused by the road slope, friction, air drag, etc.

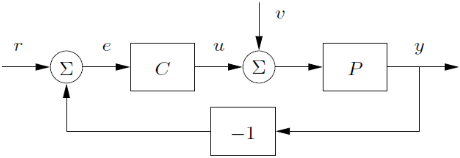


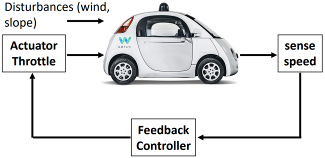
- **Variables**:
  - Desired speed (reference): $r(t)$
  - Actual speed (output): $y(t)$
  - Engine force (input): $u(t) = F_{\text{engine}}(t)$
  - Mass (parameter): m
  - Disturbances:
    - Road slope: $F_{\text{slope}}(t) = -mg\sin(\theta(t))$
    - Air drag: $F_{\text{drag}}(t) = -\delta y(t)$

- **System model**:

$$m\frac{d}{dt}y(t) = u(t) - \delta y(t) - mg\sin(\theta)$$

Example: Cruise Control System





- **Plant** P:

$$m\frac{d}{dt}y(t) = u(t) - \delta y(t) - mg\sin(\theta(t))$$

- **Controller** C: design $u(t)$ using reference $r(t)$ and output $y(t)$
- **Performance criteria**:
  - **Stable response**
    - Steady-state velocity approaches desired velocity
    - Smooth response with no overshoot or oscillations
  - **Disturbance rejection**
    - Effect of disturbances (road slope and air drag) approaches zero over time
  - **Robustness**
    - The system response is invariant to variations in the parameters (e.g., mass m)

Closed-Loop Control

- **System model**:

$$m\frac{d}{dt}y(t) = u(t) - \delta y(t) - mg\sin(\theta(t))$$

- **Closed-loop control**:
  - $u(t)$ designed using the error signal $e(t) = r(t) - y(t)$
  - **P (Proportional)** control:

$$u(t) = k_{\mathrm{p}}e(t)$$

  - **I (Integral)** control:

$$u(t) = k_{\mathrm{i}}\int_0^t e(t)dt$$

  - **D (Derivative)** control:

$$u(t) = k_{\mathrm{d}}\frac{d}{dt}e(t)$$

  - **PID control**:

$$u(t) = k_{\mathrm{p}}e(t) + k_{\mathrm{i}}\int_0^t e(t)dt + k_{\mathrm{d}}\frac{d}{dt}e(t)$$

Open-Loop Control

- **System model**:

$$m\frac{d}{dt}y(t) = u(t) - \delta y(t) - mg\sin(\theta(t))$$

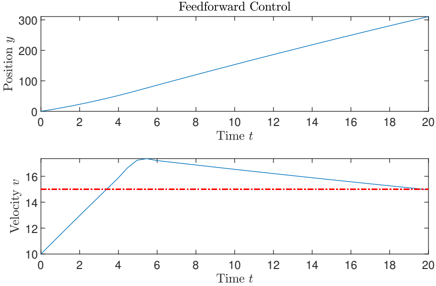
- **Open-loop control**:
  - $u(t)$ is designed using reference $r(t)$ and initial condition $y(0) = y_0$ but no measurements of the output $y(t)$
  - Approximate the error using y_0 and some function $a(t)$ to be designed:

$$e(t) \approx r(t) - a(t)y_0$$

  - Use PID control with the approximate error

Open-Loop P Control Simulation

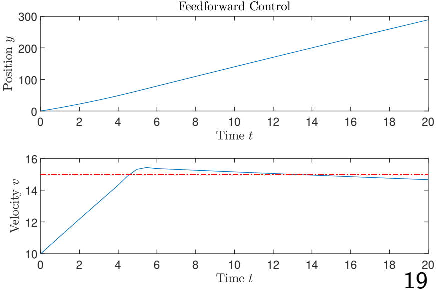
- Parameters: $r(t) \equiv 15$ m/s, $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$, $\theta = 0°$
- Matlab ODE45 function: `[t,y] = ode45(odefun,tspan,y0)`



- **Case 1**:

$$k_{\mathrm{p}} = 160, \quad a(t) = \begin{cases} 1 & 0 \leq t \leq 5 \\ \frac{3}{2} & t > 5 \end{cases}$$

$$u(t) = \begin{cases} 800 & 0 \leq t \leq 5 \\ 0 & t > 5 \end{cases}$$



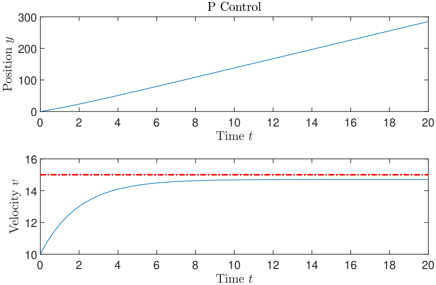
- **Case 2**:

$$k_{\mathrm{p}} = 120, \quad a(t) = \begin{cases} 1 & 0 \leq t \leq 5 \\ \frac{35}{24} & t > 5 \end{cases}$$

$$u(t) = \begin{cases} 600 & 50 \leq t \leq 5 \\ 0 & t > 5 \end{cases}$$

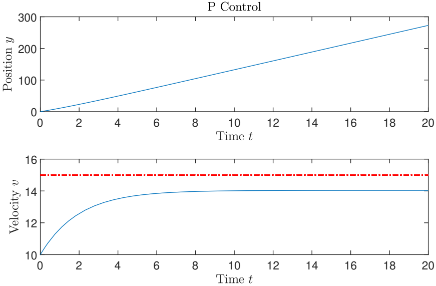
Closed-Loop P Control Simulation

- Parameters: $r(t) \equiv 15$ m/s, $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$



- **Case 1**: flat road $\theta = 0^\circ$

  $k_{\mathrm{p}} = 250 \quad u(t) = k_{\mathrm{p}} e(t)$



- **Case 2**: uphill $\theta = 2^\circ$

  $k_{\mathrm{p}} = 250 \quad u(t) = k_{\mathrm{p}} e(t)$

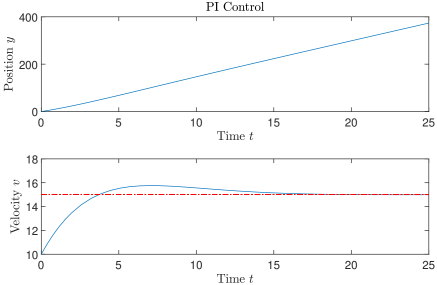
Closed-Loop PI Control Simulation

- Parameters: $r(t) \equiv 15$ m/s, $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$

- **Case 1**: flat road $\theta = 0^\circ$

  $k_\mathrm{p} = 250, \quad k_\mathrm{i} = 50$

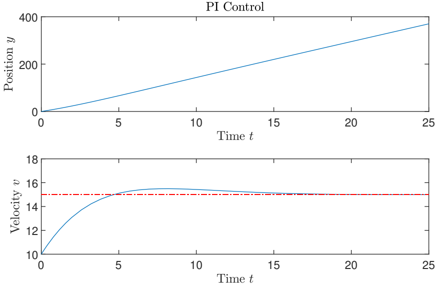
  $$u(t) = k_\mathrm{p} e(t) + k_\mathrm{i} \int_0^t e(t)dt$$



- **Case 2**: uphill $\theta = 2^\circ$

  $k_\mathrm{p} = 250, \quad k_\mathrm{i} = 50$

  $$u(t) = k_\mathrm{p} e(t) + k_\mathrm{i} \int_0^t e(t)dt$$



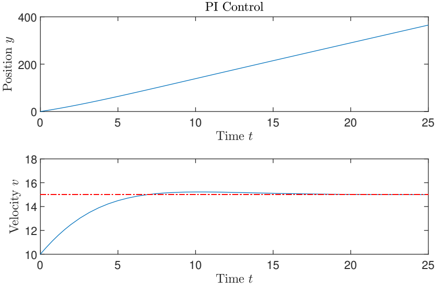
Disturbance Attenuation with PI Control

- Parameters: $r(t) \equiv 15$ m/s, $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$

- **Case 1**: uphill $\theta = 5^\circ$

  $k_\mathrm{p} = 250, \quad k_\mathrm{i} = 50$

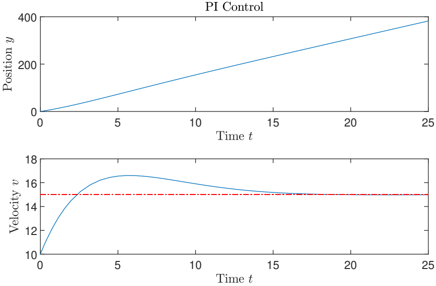
  $u(t) = k_\mathrm{p} e(t) + k_\mathrm{i} \int_0^t e(t) dt$



- **Case 2**: downhill $\theta = -5^\circ$

  $k_\mathrm{p} = 250, \quad k_\mathrm{i} = 50$

  $u(t) = k_\mathrm{p} e(t) + k_\mathrm{i} \int_0^t e(t) dt$



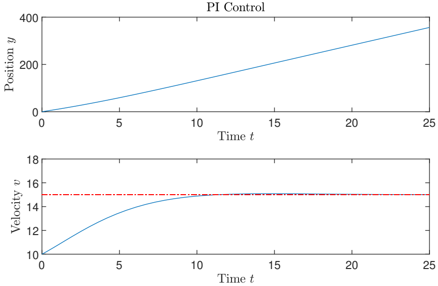
Disturbance Attenuation with PI Control

- Parameters: $r(t) \equiv 15$ m/s, $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$

- **Case 3**: uphill $\theta = 10°$

$k_\mathrm{p} = 250, \quad k_\mathrm{i} = 50$

$$u(t) = k_\mathrm{p} e(t) + k_\mathrm{i} \int_0^t e(t) dt$$

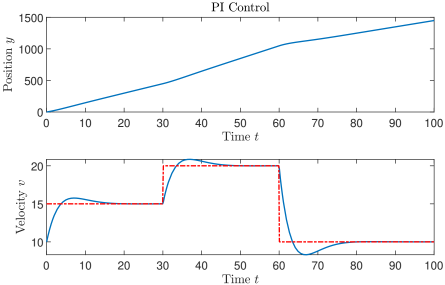


Disturbance attenuation: The same PI controller achieves *zero steady-state error*, i.e., $e(t) \to 0$, despite the presence of an *unknown* disturbance θ

Dynamic Behavior Shaping with PI Control

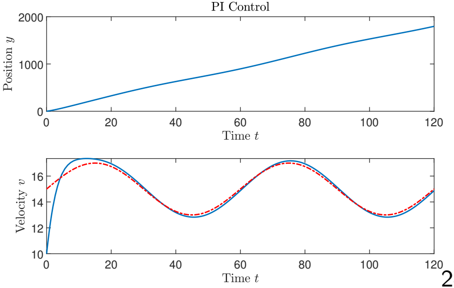
- Parameters: $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$, $\theta = 0^\circ$
- Closed-loop PI control with $k_\mathrm{p} = 250$ and $k_\mathrm{i} = 50$
- **Case 1**: piecewise-constant reference

$$r(t) = \begin{cases} 15 m/s & t \leq 30 \\ 20 m/s & 30 < t \leq 60 \\ 10 m/s & 60 < t \end{cases}$$



- **Case 2**: sinusoidal reference

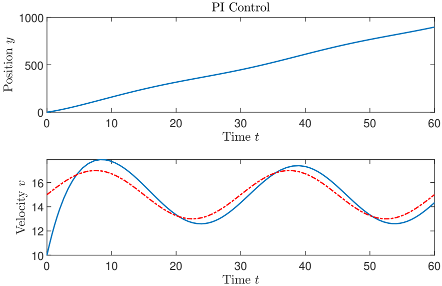
$$r(t) = 15 + 2\sin\left(\frac{2\pi}{60} t\right)$$



Dynamic Behavior Shaping with PI Control

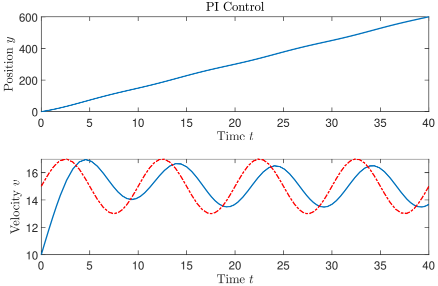
- Parameters: $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$, $\theta = 0^\circ$
- Closed-loop PI control with $k_{\mathrm{p}} = 250$ and $k_{\mathrm{i}} = 50$
- **Case 3**: sinusoidal reference

$$r(t) = 15 + 2\sin\left(\frac{2\pi}{30}t\right)$$



- **Case 4**: sinusoidal reference

$$r(t) = 15 + 2\sin\left(\frac{2\pi}{10}t\right)$$



Dynamic Behavior Shaping with PI Control

Reference tracking: The same PI controller can make the closed-loop system follow a reference signal with small tracking error

- To analyze the tracking behavior with respect to the frequency of the reference signal and to quantify the tracking error, we need to study the system behavior not only in the time domain ($t \in \mathbb{R}$) but also in the complex domain ($s = \sigma + j\omega \in \mathbb{C}$)
- The **bandwidth** of the closed-loop system provides an upper bound on the frequency of reference signals that can be tracked with small error

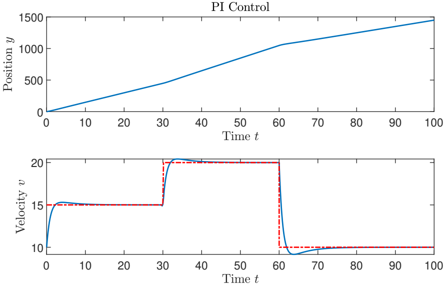
Robustness to Parameter Variations with PI Control

- Parameters: $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$, $\theta = 0^\circ$
- Closed-loop PI control with $k_\mathrm{p} = 250$ and $k_\mathrm{i} = 50$

- **Case 1**: mass change: $m = 200$ kg

  piecewise-constant reference

$$r(t) = \begin{cases} 15m/s & t \leq 30 \\ 20m/s & 30 < t \leq 60 \\ 10m/s & 60 < t \end{cases}$$



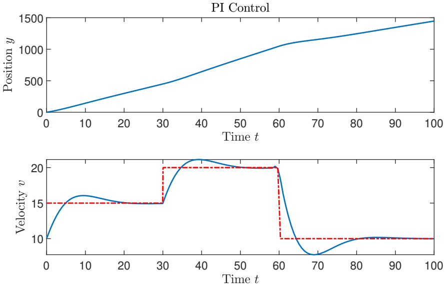
Robustness to Parameter Variations with PI Control

- Parameters: $y_0 = 10$ m/s, $m = 500$ kg, $\delta = 0.5$, $\theta = 0°$
- Closed-loop PI control with $k_\mathrm{p} = 250$ and $k_\mathrm{i} = 50$

- **Case 2**: mass change: $m = 800$ kg

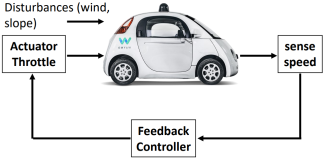
  piecewise-constant reference

$$r(t) = \begin{cases} 15m/s & t \leq 30 \\ 20m/s & 30 < t \leq 60 \\ 10m/s & 60 < t \end{cases}$$



Robustness: The same PI controller can make the closed-loop system follow a reference signal even when some system parameters are not known exactly

Summary



- Plant:

$$\frac{d}{dt}y(t) = -\frac{\delta}{m}y(t) + \frac{1}{m}u(t) - g\sin(\theta(t))$$

- Error:

$$e(t) = r(t) - y(t)$$

- Controller:

$$u(t) = k_{\mathrm{p}}e(t) + k_{\mathrm{i}}\int_0^t e(t)dt + k_{\mathrm{d}}\frac{d}{dt}e(t)$$

Feedback control

- 1) *achieves reference signal tracking*
- 2) *decreases* the sensitivity of the system response to parameter variations
- 3) *attenuates* the effect of disturbances