

ECE171A: Linear Control System Theory

Lecture 3: System Modeling

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Outline

System Modeling

Solving First-Order LTI ODEs

State-Space Models

Examples

Solving ODEs in MATLAB

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System Modeling

- ▶ A **model** is a mathematical representation of a dynamical system
- ▶ Models allow us to make predictions about how a system will behave
- ▶ There may be multiple models for a single dynamical system
- ▶ All models are approximations of the real system behavior
- ▶ Whether we choose a simple coarse model or a complex precise model depends on the questions we wish to answer

System Modeling

- ▶ Dynamic behavior can be described in several ways:
 - ▶ **ordinary differential equations** (ODEs) in continuous time
 - ▶ **partial differential equations** (PDEs) when the system behavior is determined by other variables in addition to time
 - ▶ **difference equations** (DEs) in discrete time
- ▶ The relationships among the variables and their derivatives in these equations may be **linear** or **nonlinear**
- ▶ The coefficients of these equations may be **invariant** or **varying**
- ▶ We will focus on linear time-invariant (LTI) ordinary differential equations (ODEs)

Why LTI ODEs?

- ▶ Many practically relevant systems can be modeled as LTI ODEs:
 - ▶ Electric circuits (e.g., RLC circuits), mechanical systems (e.g., spring-mass systems, rigid-body robots), thermal systems, fluid systems, etc.
- ▶ Many techniques have been developed for LTI ODE analysis and design:
 - ▶ Classical control analysis tools: step, impulse, and frequency response, Bode/Nyquist/Nichols plots, gain/phase margins
 - ▶ Classical control design tools: loop shaping, PID control, pole placement
 - ▶ Optimal estimation and control: Kalman filter and linear quadratic regulator (LQR)
 - ▶ Robust control design: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design and μ analysis for structural uncertainty
- ▶ LTI ODE techniques provide a foundation for nonlinear system analysis and control (e.g., via linearization)

Differential Equations

- ▶ A **differential equation** is any equation involving a function and its derivatives

- ▶ Example: $\frac{d}{dt}y(t) = -y(t)$

- ▶ A **solution to a differential equation** is any function that satisfies the equation and its initial conditions

- ▶ Example: a solution to the differential equation above is:

$$y(t) = e^{-t}$$

- ▶ Another solution is

$$y(t) = 2e^{-t}$$

- ▶ A general solution is

$$y(t) = e^{-t}y(0).$$

where $y(0) \in \mathbb{R}$ is the initial value of $y(t)$ at $t = 0$.

- ▶ When the variable is time t , we will use short-hand derivative notation:

$$\frac{d}{dt}y(t) \equiv \dot{y}(t) \qquad \frac{d^2}{dt^2}y(t) \equiv \ddot{y}(t) \qquad \cdots \qquad \frac{d^n}{dt^n}y(t) \equiv y^{(n)}(t)$$

Ordinary Differential Equations

- ▶ **n th-order linear time-invariant ordinary differential equation:**

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \dots + a_1\frac{d}{dt}y(t) + a_0y(t) = u(t)$$

- ▶ If $u(t) \equiv 0$, then the n th-order LTI ODE is called **homogeneous**
- ▶ A **particular solution** is a solution $y(t)$ that contains no arbitrary constants
- ▶ A **general solution** is a solution $y(t)$ that contains n arbitrary constants
- ▶ An **initial value problem** is an LTI ODE with initial value constraints:

$$y(t_0) = y_0, \quad \dot{y}(t_0) = y_1, \quad \dots, \quad y^{(n-1)}(t_0) = y_{n-1}.$$

Theorem: Existence and Uniqueness of Solutions

Let $u(t)$ be continuous on an interval $\mathcal{I} = [t_1, t_2]$. Then, for any $t_0 \in \mathcal{I}$, a solution $y(t)$ of the initial value problem exists on $\mathcal{I}$ and is unique.

Example 1

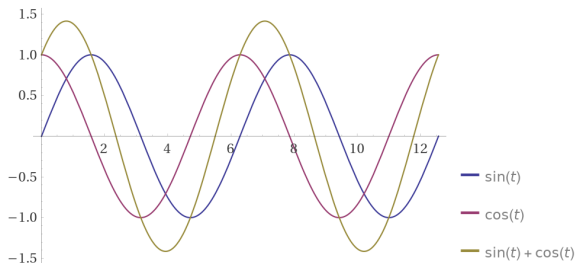
► Consider the homogeneous linear ODE: $\frac{d^2}{dt^2}y(t) + y(t) = 0$

► Two particular solutions are:

$$y_1(t) = \cos(t) \quad \Rightarrow \quad \frac{d^2}{dt^2} \cos(t) = -\cos(t)$$

$$y_2(t) = \sin(t) \quad \Rightarrow \quad \frac{d^2}{dt^2} \sin(t) = -\sin(t)$$

► In fact, any linear combination $y(t) = c_1 y_1(t) + c_2 y_2(t)$ with $c_1, c_2 \in \mathbb{R}$ is also a solution



Superposition Principle for Homogeneous Linear ODEs

Let $y_1, y_2, \dots, y_k$ be solutions to a homogeneous n th-order linear ODE on an interval $\mathcal{I}$. Then, any linear combination:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots + c_k y_k(t)$$

is also a solution, where $c_1, c_2, \dots, c_k$ are constants.

Superposition Principle for Nonhomogeneous Linear ODEs

For $i = 1, \dots, k$, let $y_{p_i}(t)$ denote particular solutions to the linear ODEs:

$$\frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \dots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = u_i(t).$$

Then, $y_p(t) = c_1 y_{p_1}(t) + c_2 y_{p_2}(t) + \dots + c_k y_{p_k}(t)$ is a particular solution of:

$$\begin{aligned} \frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \dots + a_1 \frac{d}{dt} y(t) + a_0 y(t) \\ = c_1 u_1(t) + c_2 u_2(t) + \dots + c_k u_k(t), \end{aligned}$$

where $c_1, c_2, \dots, c_k$ are constants.

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First-Order Homogeneous LTI ODE

- First-order homogeneous LTI ODE:

$$\dot{y}(t) = ay(t), \quad y(0) = b$$

- **Ansatz** (“attempt” in German): an educated guess or an additional assumption made to help solve a problem, which may later be verified
- Guess the solution to the LTI ODE above:

$$y(t) = e^{at}b$$

- Proof:

1. Initial condition: $y(0) = e^{a0}b = b$
2. Differential equation:

$$\frac{d}{dt}y(t) = \frac{d}{dt}e^{at}b = ae^{at}b = ay(t)$$

First-Order Homogeneous LTI ODE

- ▶ First-order homogeneous LTI ODE:

$$\dot{y}(t) = ay(t), \quad y(0) = b$$

- ▶ Can we obtain a solution without an Ansatz?
- ▶ Suppose $y(t) \neq 0$:

$$\frac{1}{y(t)} \frac{dy(t)}{dt} = a$$

- ▶ Recall that by the chain rule:

$$\frac{d}{dt} \log y(t) = \frac{1}{y(t)} \frac{dy(t)}{dt}$$

- ▶ Now, we can integrate both sides:

$$\frac{d}{dt} \log y(t) = a \quad \Rightarrow \quad \int_0^t \frac{d}{d\tau} \log y(\tau) d\tau = \int_0^t a d\tau$$

- ▶ By the fundamental theorem of Calculus:

$$\log y(t) - \log y(0) = at \quad \Rightarrow \quad y(t) = e^{at} b$$

First-Order Nonhomogeneous LTI ODE

- First-order nonhomogeneous LTI ODE:

$$\dot{y}(t) + ay(t) = u(t),$$

where $a \in \mathbb{R}$ is a given constant and $u(t)$ is a given function

- **Integrating factor:** multiply both sides by $\mu(t) = e^{at}$
- Since $\dot{\mu}(t) = a\mu(t)$, we have:

$$\frac{d}{dt}(\mu(t)y(t)) = \dot{\mu}(t)y(t) + \mu(t)\dot{y}(t) = \mu(t)(ay(t) + \dot{y}(t)) = \mu(t)u(t)$$

- Let $\mu(t)y(t) = g(t)$ and $\mu(t)u(t) = h(t)$ and integrate both sides:

$$\dot{g}(t) = h(t) \quad \Rightarrow \quad \int_0^t \dot{g}(\tau) d\tau = \int_0^t h(\tau) d\tau \quad \Rightarrow \quad g(t) = \int_0^t h(\tau) d\tau + g(0)$$

- Thus, the general solution is:

$$y(t) = \frac{1}{\mu(t)} \left(\int_0^t \mu(\tau) u(\tau) d\tau + y(0) \right) = e^{-at} \left(\int_0^t e^{a\tau} u(\tau) d\tau + y(0) \right)$$

Example 2: First-Order Nonhomogeneous LTI ODE

- ▶ Consider a first-order nonhomogeneous LTI ODE with positive input:

$$\dot{y}(t) + 2y(t) = 5, \quad y(0) = 1$$

- ▶ Integrating factor: $\mu(t) = e^{2t}$

- ▶ The solution is

$$y(t) = e^{-2t} \left(\int_0^t 5e^{2\tau} d\tau + y(0) \right) = e^{-2t} \left(\frac{5}{2}e^{2t} - \frac{5}{2} + y(0) \right) = \frac{5}{2} - \frac{3}{2}e^{-2t}$$

- ▶ Initial condition: $y(0) = \frac{5}{2} - \frac{3}{2} = 1$

- ▶ Verify the LTI ODE:

$$\dot{y}(t) + 2y(t) = -\frac{3}{2}(-2)e^{-2t} + 5 - 3e^{-2t} = 5$$

Example 3: First-Order Nonhomogeneous LTI ODE

- ▶ Consider a first-order nonhomogeneous LTI ODE with negative input:

$$\dot{y}(t) + 2y(t) = -5, \quad y(0) = 1$$

- ▶ Integrating factor $\mu(t) = e^{2t}$ and solution:

$$y(t) = e^{-2t} \left(\int_0^t -5e^{2\tau} d\tau + y(0) \right) = e^{-2t} \left(-\frac{5}{2}e^{2t} + \frac{5}{2} + 1 \right)$$

- ▶ Initial condition: $y(0) = -\frac{5}{2} + \frac{7}{2} = 1$
- ▶ Check that $y(t) = \frac{7}{2}e^{-2t} - \frac{5}{2}$ is a solution:

$$\dot{y}(t) + 2y(t) = -\frac{7}{2}(-2)e^{-2t} - 7e^{-2t} - 5 = -5$$

Integration by Parts

- Indefinite integral form:

$$\int u(t)\dot{v}(t)dt = u(t)v(t) - \int \dot{u}(t)v(t)dt$$

- Definite integral form:

$$\int_a^b u(t)\dot{v}(t)dt = u(b)v(b) - u(a)v(a) - \int_a^b \dot{u}(t)v(t)dt$$

- Integration by parts is useful to find antiderivatives of terms such as te^{2t} and $e^{2t} \sin t$

Example 4: First-Order LTI ODE with Polynomial Input

- Consider a first-order nonhomogeneous LTI ODE with polynomial input:

$$\dot{y}(t) + 2y(t) = t, \quad y(0) = 1$$

- Integration by parts with $u(t) = t$ and $v(t) = \frac{1}{2}e^{2t}$:

$$\begin{aligned} y(t) &= e^{-2t} \left(\int_0^t e^{2\tau} \tau d\tau + y(0) \right) = e^{-2t} \left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4} + 1 \right) \\ &= \frac{1}{2}t - \frac{1}{4} + \frac{5}{4}e^{-2t} \end{aligned}$$

- Initial condition: $y(0) = -\frac{1}{4} + \frac{5}{4} = 1$

- Verify the LTI ODE:

$$\dot{y}(t) + 2y(t) = \frac{1}{2} + \frac{5}{4}(-2)e^{-2t} + t - \frac{1}{2} + \frac{5}{2}e^{-2t} = t$$

Example 5: First-Order LTI ODE with Trigonometric Input

- Consider a first-order nonhomogeneous LTI ODE with trigonometric input:

$$\dot{y}(t) + 2y(t) = \sin(t), \quad y(0) = 1$$

- Integration by parts with $u(t) = \sin(t)$ and $v(t) = \frac{1}{2}e^{2t}$:

$$\begin{aligned} I &= \int_0^t e^{2\tau} \sin(\tau) d\tau \quad \frac{u(t)=\sin(t)}{v(t)=e^{2t}/2} \quad \frac{1}{2}e^{2t} \sin(t) - \int_0^t \frac{1}{2}e^{2\tau} \cos(\tau) d\tau \\ &\quad \frac{u(t)=\cos(t)}{v(t)=e^{2t}/4} \quad \frac{1}{2}e^{2t} \sin(t) - \frac{1}{4}e^{2t} \cos(t) + \frac{1}{4} - \int_0^t \frac{1}{4}e^{2\tau} \sin(\tau) d\tau \\ &= \frac{1}{2}e^{2t} \sin(t) - \frac{1}{4}e^{2t} \cos(t) + \frac{1}{4} - \frac{1}{4}I \quad \Rightarrow \quad I = \frac{1}{5}e^{2t}(2 \sin(t) - \cos(t)) + \frac{1}{5} \end{aligned}$$

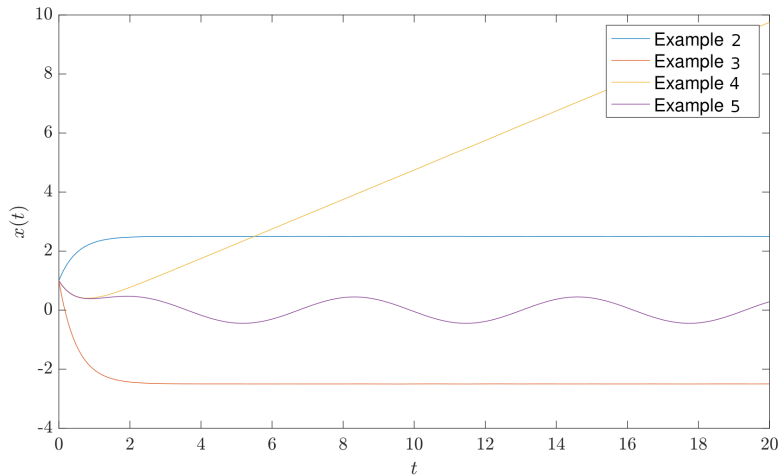
- Solution: $y(t) = e^{-2t} (I + y(0)) = \frac{1}{5}(2 \sin(t) - \cos(t)) + \frac{6}{5}e^{-2t}$

- Initial condition: $y(0) = -\frac{1}{5} + \frac{6}{5} = 1$

- Verify the LTI ODE:

$$\begin{aligned} \dot{y}(t) + 2y(t) &= \frac{1}{5}(2 \cos t + \sin t) + \frac{6}{5}(-2)e^{-2t} + \frac{1}{5}(4 \sin t - 2 \cos t) + \frac{12}{5}e^{-2t} \\ &= \sin t \end{aligned}$$

Simulation of First-Order LTI ODE Solutions



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State-Space Model

- ▶ An n th-order LTI ODE:

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \dots + a_1\frac{d}{dt}y(t) + a_0y(t) = u(t)$$

can be reformulated into a first-order vector LTI ODE of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

- ▶ Define variables:

$$x_1(t) = y(t), \quad x_2(t) = \frac{d}{dt}y(t), \quad \dots, \quad x_n(t) = \frac{d^{n-1}}{dt^{n-1}}y(t)$$

- ▶ The n th-order linear ODE specifies the following relationships:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\vdots$$

$$\dot{x}_{n-1}(t) = x_n(t)$$

$$\dot{x}_n(t) = -a_0x_1(t) - a_1x_2(t) - \dots - a_{n-1}x_n(t) + u(t)$$

State-Space Model

- ▶ Let $\mathbf{x}(t) := [x_1(t) \ x_2(t) \ \cdots \ x_n(t)]^\top$ be a vector called system **state**
- ▶ The forcing function $u(t)$ is called system **control input**
- ▶ A **state-space model** of the n th-order linear ODE is obtained by rewriting the equations in vector-matrix form:

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}}_{\mathbf{A}} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u(t)$$

- ▶ The system **output** $y(t)$ can be obtained from the state $\mathbf{x}(t)$ as:

$$y(t) = \underbrace{[1 \ 0 \ \cdots \ 0]}_{\mathbf{C}} \mathbf{x}(t)$$

State-Space Model

- ▶ An **LTI ODE state-space model** of a dynamical system is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

with:

- ▶ **state:** $\mathbf{x} \in \mathbb{R}^n$
 - ▶ **input:** $\mathbf{u} \in \mathbb{R}^m$
 - ▶ **output:** $\mathbf{y} \in \mathbb{R}^p$
 - ▶ **parameters:** $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, $\mathbf{D} \in \mathbb{R}^{p \times m}$
- ▶ **Single-input single-output (SISO) system:** $m = p = 1$
 - ▶ **Multi-input multi-output (MIMO) system:** $m, p > 1$

State-Space Model Variables

- ▶ State-space model variables:
 - ▶ **State:** consists of variables that capture information from the past motion of the system sufficient to predict the future motion
 - ▶ **Input:** consists of external effects acting on the system
 - ▶ **Output:** consists of measured variables
 - ▶ **Parameters:** describe the state evolution in the form of an update rule
- ▶ The choice of state is not unique:
 - ▶ There may be many choices of variables that are sufficient to describe the system evolution
- ▶ The choice of input and output depends on the point of view
 - ▶ Inputs in one model might be outputs of another model (e.g., the output of a cruise controller provides the input to the vehicle model)
 - ▶ Outputs are variables (often states) that can be measured and depend on what components of the system interact with external system components

Historical Perspective

- ▶ In the 1940s, when control theory emerged as a discipline, modeling was strongly influenced by *input-output models* used in electrical engineering
- ▶ An algebraic relationship, called **transfer function**, between the input and the output of an LTI ODE system can be obtained by transforming it from the time domain to the complex domain via a **Laplace transform**
- ▶ In the 1950s, a second wave of control developments, inspired by mechanics, focused on *state-space models*
- ▶ Both perspectives provide useful and often distinct information about the system behavior and offer different tools for control analysis and design
- ▶ State-space techniques generalize more directly and are easier to use for MIMO systems

Nonlinear ODEs

- ▶ In general, we may have a nonlinear ODE initial value problem:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

- ▶ A function $\mathbf{s}(t)$ is a **solution** to the initial value problem on interval $[t_0, t_f]$ if:

$$\mathbf{s}(t_0) = \mathbf{x}_0 \quad \text{and} \quad \frac{d}{dt}\mathbf{s}(t) = \mathbf{F}(\mathbf{s}(t)), \quad \forall t_0 < t < t_f$$

- ▶ If the function $\mathbf{F}(\mathbf{x})$ is well-behaved (Lipschitz continuous), then the initial value problem has a **unique solution**

- ▶ A nonlinear ODE initial value problem:

- ▶ may not have a solution (Example 5.2 in Åström & Murray: $\dot{x} = x^2$)
- ▶ may not have a unique solution (Example 5.3 in Åström & Murray: $\dot{x} = 2\sqrt{x}$)

Nonlinear Systems

► **Nonlinear state-space model:**

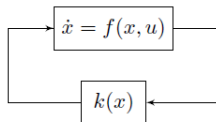
$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases} \quad \text{v.s.} \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

► **Control problem:** design a function $\mathbf{u} = \mathbf{k}(\mathbf{x})$, called **feedback control law**, such that:

- **Regulation problem:** the state converges to zero: $\mathbf{x}(t) \rightarrow \mathbf{0}$
- **Servo problem:** the state tracks a reference signal: $\mathbf{x}(t) \rightarrow \mathbf{r}(t)$

► **Closed-loop system:**

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x})) = \mathbf{F}(\mathbf{x})$$



Discrete-Time Systems

- ▶ In some situations, it is natural to describe the evolution of a system at discrete instants of time rather than continuously in time
- ▶ Time step: $k = 0, 1, 2, \dots$
- ▶ **Discrete-time nonlinear system:** modeled by nonlinear difference equation:

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) \\ \mathbf{y}_k &= \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k)\end{aligned}$$

- ▶ **Discrete-time linear system:** modeled by linear difference equation:

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k\end{aligned}$$

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System Modeling

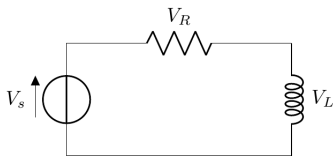
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RL Circuit



R : Resistance

L : Inductance

$V_R = Ri$: Resistor

$V_L = L \frac{di}{dt}$: Inductor

► **Kirchhoff's voltage law:**

$$V_S - V_R - V_L = 0$$

► **System model:**

$$L \frac{di}{dt} = V_S - Ri$$

► **State-space model:**

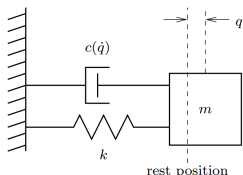
► Variables: $x = i$, $u = V_S$, $y = V_R$

► Model:

$$\dot{x} = -\frac{R}{L}x + \frac{1}{L}u$$

$$y = Rx$$

Spring-Mass System



m = mass

F = external force

c = friction (damper)

k = spring stiffness

q = deviation from rest position

- **System model:** from Newton's second law:

$$m\ddot{q} + c\dot{q} + kq = F$$

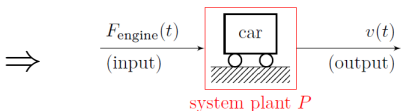
- **State-space model:**

- Variables: $x_1 = q$, $x_2 = \dot{q}$, $y = x_1 = q$, $u = F$

- Model:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}(-cx_2 - kx_1 + u) \end{bmatrix} \\ y = x_1 \end{cases} \Leftrightarrow \begin{cases} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u \end{cases}$$

Speed Control



- ▶ **Variables:** position p , velocity v , engine force F_{engine} , mass m , gravity acceleration g , road slope θ
- ▶ **System model:** from Newton's second law:

$$\begin{aligned}\dot{p} &= v \\ m\dot{v} &= F_{\text{engine}} - mg \sin \theta\end{aligned}$$

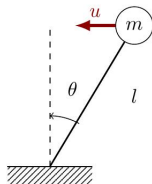
- ▶ **State-space model:**

▶ Variables: $x_1 = p$, $x_2 = \dot{p}$, $y = x_2 = v$, $u_1 = F_{\text{engine}}$, $u_2 = g \sin(\theta)$

▶ Model:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}u_1 - u_2 \end{bmatrix} \\ y = x_2 \end{cases} \Leftrightarrow \begin{cases} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m} & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{cases}$$

Inverted Pendulum



m = mass

l = length

u = external force

θ = angle

► **Torque:** $T = mgl \sin \theta - ul \cos \theta$

► **Moment of inertia:** $J = ml^2$

► **System model:** from Newton's second law:

$$ml^2\ddot{\theta} = mgl \sin \theta - ul \cos \theta$$

► **State-space model:**

► Variables: $x_1 = \theta$, $x_2 = \dot{\theta}$, $y = \theta$

► Model (nonlinear):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{mgl \sin(x_1) - ul \cos(x_1)}{ml^2} \end{bmatrix}$$

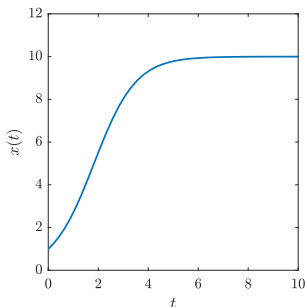
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u$$

Population Dynamics

- ▶ Population growth is a complex dynamic process that involves the interaction of one or more species with their environment and the larger ecosystem
- ▶ **Variables:** $x(t)$: species population at time t , b : birth rate, d : death rate, $r = (b - d)$: differential birth rate, k : carrying capacity of the environment
- ▶ **Logistic growth model:**

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right), \quad x \geq 0$$

- ▶ Logistic growth model simulation with $r = 1.2$ and $k = 10$:



Outline

System Modeling

Solving First-Order LTI ODEs

State-Space Models

Examples

Solving ODEs in MATLAB

Numerical ODE Solutions: Matlab ode45

- ▶ **Matlab ode45 function:**

$[t, x] = \text{ode45}(\text{odefun}, \text{tspan}, x_0)$

- ▶ **odefun:** function f defining the ode $\dot{x} = f(t, x)$

- ▶ **tspan:** time interval $[t_0, t_f]$

- ▶ **x0:** initial condition x_0

- ▶ **detailed description:**

<https://www.mathworks.com/help/matlab/ref/ode45.html>

Example 1

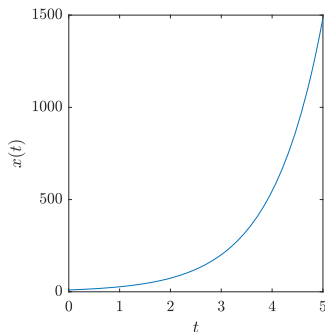
- Consider the initial value problem:

$$\dot{x} = x, \quad x(0) = 10$$

- Determine the solution for $t \in [0, 5]$

```
%----- Example 1 -----  
% \dot x = x,  
% with x(0) = 10  
%-----
```

```
f1 = @(t,x)(x); % vector field  
[ts,xs] = ode45(f1,[0,5],10);  
plot(ts,xs);
```



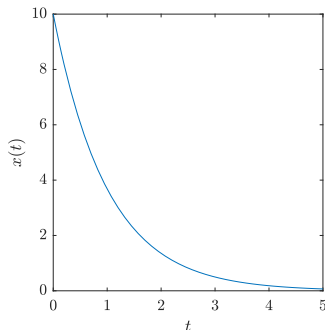
Example 2

- Consider the initial value problem:

$$\dot{x} = -x, \quad x(0) = 10$$

- Determine the solution for $t \in [0, 5]$

```
%----- Example 2 -----  
% \dot x = -x,  
% with x(0) = 10  
%-----  
f2      = @(t,x)(-x); % vector field  
[ts,xs] = ode45(f2,[0,5],10);  
plot(ts,xs);
```



Example 3

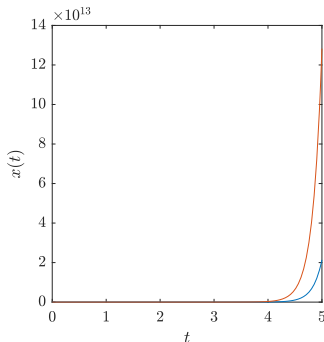
- Consider the initial value problem:

$$\ddot{z} - 3\dot{z} - 18z = 0, \quad z(0) = 3, \dot{z}(0) = 9$$

- Determine the solution for $t \in [0, 5]$
- State-space model:
 - Variables: $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$
 - Model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 18 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

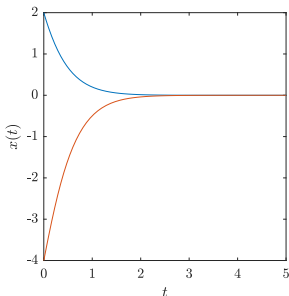
```
%----- Example 3 -----  
% \ddot{z} - 3 \dot{z} - 18 z = 0  
% with z(0) = 3, \dot{z}(0) = 9  
%-----  
[ts,xs] = ode45(@f3,[0,5],[3;9]);  
plot(ts,xs);  
  
function dx = f3(t,x)  
    dx = [0 1; 18 3]*x;  
end
```



Example 4 & 5

```
%----- Example 4 -----  
% \ddot{z} + 6 \dot{z} + 9z = 0  
% with z(0) = 2, \dot{z}(0) = -4  
%-----  
[ts,ys] = ode45(@f4,[0,5],10);
```

```
function dx = f4(t,x)  
    dotx = [0 1; -9 -6]*x;  
end
```



```
%----- Example 5 -----  
% \ddot{z} - 6 \dot{z} + 13 z = 0  
% with z(0) = 3, \dot{z}(0) = 17  
%-----  
[ts,ys] = ode45(@f5,[0,20],[3;17]);
```

```
function dx = f5(t,x)  
    dx = [0 1; -13 6]*x;  
end
```

