# ECE276A: Sensing and Estimation in Robotics
# Final Exam Practice Problems Solutions

## Problem 1

Consider a camera with position $\mathbf{p} = [1,\ 1,\ 0]^\top$, roll 0°, pitch 0°, yaw 45°, focal length $f = 0.2$ m, image center $(c_u, c_v) = (160.5,\ 120.5)$ pixels, scaling $(s_u, s_v) = (10,\ 10)$ pixels/m, and skew-factor $s_t = 0$ pixels/m. Suppose that the camera observes a point $\mathbf{m} = [2,\ 1,\ 2]^\top$. What are the pixel coordinates of $\mathbf{m}$ assuming a noise-free perspective projection?

*Reminders*

- The rotation from the camera frame to the optical frame is given by ${}_o\mathbf{R}_r := \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

## Problem 2

Assume that we have obtained $T$ measurement pairs $(x_t, y_t)$ from the linear model:

$$y_t = \theta_1 x_t + \theta_2, \quad t = 1, \ldots, T \tag{1}$$

Derive estimates of the parameters $\theta_1$ and $\theta_2$ such that the following error is minimized (least squares estimate):

$$E(\theta_1, \theta_2) = \sum_{t=1}^{\top} (y_t - \theta_1 x_t - \theta_2)^2 \tag{2}$$

(a) Define $\mathbf{y} := [y_1, \ldots, y_T]^\top$ and $\boldsymbol{\theta} := [\theta_1,\ \theta_2]^\top$. Show that the set of equations (1) can written in matrix form as:

$$\mathbf{y} = A\boldsymbol{\theta}$$

for a suitably defined matrix $A$

(b) Write the error function in matrix form in terms of $\mathbf{y}$, $A$, and $\boldsymbol{\theta}$

(c) Compute the gradient of the matrix form error function and solve the least squares estimate of the parameters $\boldsymbol{\theta}$ by finding the point where the gradient is zero

## Problem 3

Inspired by the recent success of deep learning, you use a neural network with one layer to approximate the motion model of your robot:

$$x_{t+1} = \sigma(ax_t + bu_t) + \eta$$

where $a, b \in \mathbb{R}$ are the (known) parameters that your neural network learned, $\eta \sim \mathcal{N}(0, 1)$ is a Gaussian motion noise, and $\sigma(x) := (1 + e^{-x})^{-1}$ is the logistic sigmoid function. You guess that your robot is located at position $\mu_0 = 1$ and place a Gaussian distribution with covariance 2 on your guess. You apply control input $u_0 = 2$ to your robot and use the extended and unscented

Kalman filters to predict the robot motion. **Compute the mean $\mu_{EKF}$ and covariance $\Sigma_{EKF}$ of the EKF after the single prediction step and compare those to the mean $\mu_{UKF}$ and covariance $\Sigma_{UKF}$ of the UKF**. Your answer should only involve numbers, the function $\sigma$, and the constants $a$, $b$.

*Reminders*

- To approximate the distribution of a random vector $\mathbf{s} = g(\mathbf{y})$ for known function $g$ and $d$-dimensional random variable $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, the unscented Kalman filter chooses mean and covariance weights:

$$
\begin{aligned}
v^{(0)} &< 1, \qquad & v^{(i)} &= \frac{1 - v^{(0)}}{2d}, \; i = 1, \ldots, 2d \\
w^{(0)} &\geq v^{(0)}, \qquad & w^{(i)} &= \frac{1 - v^{(0)}}{2d}, \; i = 1, \ldots, 2d
\end{aligned}
$$

  and uses the following sigma points:

$$
\mathbf{y}^{(0)} = \boldsymbol{\mu}, \quad \mathbf{y}^{(i)} = \boldsymbol{\mu} \pm \sqrt{\frac{d}{1 - v^{(0)}}} \left[\sqrt{\Sigma}\right]_i, \quad i = 1, \ldots, d.
$$

  A common choice of weights is $v^{(0)} = 0$ and $w^{(0)} = 2$.

## Problem 4

You are using a robot equipped with a camera to localize a chair in your room. The robot is located at position $\mathbf{p}_0 = [-1,\ 1,\ 0]^\top$ with orientation $R_0 = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In other words, $\mathbf{p}_0$ and $R_0$ specify the position and orientation of the robot frame of reference at time $t = 0$ with respect to the world frame. Assume that the frames of reference of the robot and the camera coincide. Your camera is calibrated and has an intrinsic calibration matrix $K = \begin{bmatrix} 1 & 0 & 100 \\ 0 & 1 & 100 \end{bmatrix}$. You are guessing that your chair is located at $\boldsymbol{\mu}_0 = [1,\ 1,\ 0]^\top$ and place a Gaussian distribution with identity covariance on your guess. You rotate your robot 30° counter-clockwise while translating it by $\mathbf{p}_\Delta = \frac{1}{2}[\sqrt{3} - 1,\ \sqrt{3} + 1,\ 0]^\top$. In other words, $\mathbf{p}_\Delta$ and $R_\Delta := R_z(30°)$ specify the position and orientation of the robot frame of reference at time $t = 1$ with respect to the robot frame of reference and time $t = 0$. You run your chair-detection algorithm on the image received at the new robot pose and detect the chair at pixel location $\mathbf{z} = [100,\ 100]^\top$. You know that your algorithm reports detections perturbed by Gaussian noise with zero mean and identity covariance. Use this measurement and the extended Kalman filter to update your prior guess about the chair's position. **Compute the updated mean $\boldsymbol{\mu}$ and covariance $\Sigma$ of the chair position**.

*Reminders*:

- The pixel coordinates of a point $\mathbf{m} \in \mathbb{R}^3$ observed by a camera with position $\mathbf{p} \in \mathbb{R}^3$, orientation $R \in SO(3)$, and intrinsic parameters $K \in \mathbb{R}^{2 \times 3}$ are:

$$
\mathbf{z} = K\pi({}_oR_r R^\top(\mathbf{m} - \mathbf{p})) \; \in \mathbb{R}^2
$$

where ${}_oR_r = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ and $\pi(x) := \frac{1}{x_3}\mathbf{x} \in \mathbb{R}^3$

- A rotation of $\theta$ radians around the $z$-axis can be represented by a rotation matrix:

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The a posteriori covariance of the update step of the Kalman filter is

$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - \Sigma_{t+1|t}H^\top(H\Sigma_{t+1|t}H^\top + V)^{-1}H\Sigma_{t+1|t}$$

## Problem 5

Suppose that the pose of a moving robot with respect to the world frame is given by the following function of time $t$:

$$T(t) = \begin{bmatrix} \cos\frac{t\pi}{3} & 0 & -\sin\frac{t\pi}{3} & t \\ 0 & 1 & 0 & 0 \\ \sin\frac{t\pi}{3} & 0 & \cos\frac{t\pi}{3} & 2t \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3)$$

1. Find the axis-angle representations of the robot orientation at time $t = 1$.
2. Find the quaternion representations of the robot orientation at time $t = 1$ and of the inverse of this orientation.
3. Compute the linear and the angular velocity of the robot with respect to the robot frame and with respect to the world frame at time $t = 1$.
4. Let $\mathbf{p}_W = (9, 0, 0)$ be a point with coordinates specified in the world frame. Compute the coordinates $\mathbf{p}_R$ of the point $\mathbf{p}_W$ in the robot frame at time $t = 1$.

## Problem 6

Let $T \in SE(3)$ be the pose of a camera in the world frame. In other words, $T$ specifies a transformation from the camera optical frame to the world frame. Suppose that the camera is calibrated so that $K = I \in \mathbb{R}^{3\times 3}$. Let $\underline{\mathbf{m}} \in \mathbb{R}^4$ be the homogeneous world frame coordinates of a landmark observed by the camera. Consider a camera observation model based on a *spherical perspective projection* function so that the homogeneous coordinates of the projection of $\mathbf{m}$ in the optical frame are:

$$\mathbf{z} = \pi_s\left(T^{-1}\underline{\mathbf{m}}\right) \in \mathbb{R}^3 \qquad\qquad \pi_s(\mathbf{q}) := \frac{1}{\|\mathbf{q}\|_2}\mathbf{q} \in \mathbb{R}^4.$$

Determine the Jacobian $J \in \mathbb{R}^{3\times 6}$ of the pixel observation $\mathbf{z} \in \mathbb{R}^3$ with respect to the six degrees of freedom of the camera pose $T \in SE(3)$.

## Problem 7

Consider a system with an unknown state $x \in \mathbb{R}$. The system is equipped with a sensor, whose measurements $z_t \in \mathbb{R}$ will be used to estimate $x$. Suppose that the observation model describing the sensor is:

$$z_t = v_t \sqrt{x}$$

where $v_t$ is a multiplicative measurement noise, which follows a Gaussian distribution $\mathcal{N}(0,1)$ with zero mean and variance 1. Assume also that the prior distribution of $x$ is *Inverse Gamma* with shape $\alpha > 0$ and scale $\beta > 0$.

1. Derive a closed-form expression for the update step of the Bayes filter applied to this system
2. What is the distribution of $x$ given measurements $z_0, \ldots, z_t$?

*Reminders*:

- The normal distribution with mean $\mu$ and variance $\sigma^2$ has pdf $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ for $x \in (-\infty, \infty)$.
- The inverse gamma distribution with shape $\alpha > 0$ and scale $\beta > 0$ has pdf $p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right)$ for $x \in (0, \infty)$.
- The gamma function is defined as $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$ and if $z$ is a positive integer, then $\Gamma(z) = (z-1)!$.

## Problem 8

Complete each of the following statements with one sentence, possibly containing mathematical expressions.

1. Gaussian Naïve Bayes models the joint distribution $p(y, \mathbf{x})$ of an example $\mathbf{x} \in \mathbb{R}^d$ and its label $y \in \{1, \ldots, K\}$ as:
2. The space of $3 \times 3$ skew-symmetric matrices is defined as:
3. Let $(x, y, z) \in \mathbb{R}^3$ be a point in the optical frame of a monocular camera. The 3D-to-2D perspective projection operation transforms $(x, y, z)$ to:
4. The prediction step of the Bayes filter is:
5. Consider a joint Gaussian distribution of the form:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{z} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\eta} \end{pmatrix}, \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Sigma}\mathbf{H}^\top \\ \mathbf{H}\boldsymbol{\Sigma} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}^\top + \mathbf{V} \end{bmatrix}\right).$$

   The distribution of $\mathbf{x}$ conditioned on $\mathbf{z}$ is:

## Problem 9

Consider a rigid body with position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$ in the world frame. Let $\mathbf{m} \in \mathbb{R}^3$ be the body-frame coordinates of a point attached to the rigid body. Suppose that the body is undergoing pure rotation (no translation) with constant angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ (body-frame coordinates).

1. What are the coordinates of the point $\mathbf{m}$ in the world frame at time $t$?
2. Suppose that a range sensor with position $\mathbf{a} \in \mathbb{R}^3$ and quaternion orientation $\mathbf{q}$ in the world frame is measuring the squared distance $z(t)$ to the point $\mathbf{m}$ at time $t$ without any noise. What is the observation model for this range sensor? Simplify the relationship between $z(t)$ and $\mathbf{m}$ as much as possible before moving on to the next part.
3. Determine the derivative of the range measurement $z(t)$ with respect to time $t$.
4. Determine the derivative of the range measurement $z(t)$ at time $t$ with respect to the three degrees of freedom $\boldsymbol{\theta} \in \mathbb{R}^3$ of the initial body orientation $R = \exp(\hat{\boldsymbol{\theta}})$.

## Problem 10

Consider a Kalman filter applied to a discrete-time system with motion model:

$$x_{t+1} = \frac{1}{2}x_t + w_t, \qquad w_t \sim \mathcal{N}(0, a^2), \tag{3}$$

and observation model:

$$z_t = x_t + v_t, \qquad v_t \sim \mathcal{N}(0, b^2). \tag{4}$$

Suppose that the noise terms $w_t$ and $v_t$ are independent of each other, independent across time, and independent of the system state $x_t$.

1. What is the predicted state variance $\sigma^2_{t+1|t}$ at time $t+1$ as a function of the predicted state variance $\sigma^2_{t|t-1}$ at time $t$?
2. Denote the function above by $\sigma^2_{t+1|t} = f(\sigma^2_{t|t-1})$. Does the function $f$ have a fixed point? In other words, what is the solution $\sigma^2_\infty$ to the equation $\sigma^2_\infty = f(\sigma^2_\infty)$?
3. What is the Kalman gain corresponding to $\sigma^2_\infty$ when $a = b$? What is the Kalman gain corresponding to $\sigma^2_\infty$ when $a = 2b$? Based on these computations, describe intuitively the behavior of the Kalman filter when the motion noise increases relative to the measurement noise.

# Solutions

## Problem 1

To compute the pixel coordinates of $\mathbf{m}$, we need the camera projection function, the camera intrinsic parameters, and the camera extrinsic parameters. The camera intrinsic parameters are given by the matrix:

$$K = \begin{bmatrix} s_u f & s_t f & c_u \\ 0 & s_v f & c_v \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

The camera extrinsic parameters are its position $\mathbf{p} = [1,\ 1,\ 0]^\top$ and orientation:

$$R_z(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\theta = 45°$ is the yaw angle in radians. The projection function $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as $\pi(\mathbf{x}) := \frac{1}{x_3}\mathbf{x}$. Given all this information, the pixel coordinates $\mathbf{z}$ can be obtained by transforming $\mathbf{m}$ from the world frame to the camera frame, then to the optical frame via ${}_oR_r$, then projecting it to the image plane via $\pi$, and finally to the image array via $K$. In detail:

$$\begin{aligned}
\mathbf{z} &= K\pi\left({}_oR_r R_z(\theta)^\top (\mathbf{m} - \mathbf{p})\right) \\
&= \begin{bmatrix} 2 & 0 & 160.5 \\ 0 & 2 & 120.5 \end{bmatrix} \pi\left(\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right) \\
&= \begin{bmatrix} 2 & 0 & 160.5 \\ 0 & 2 & 120.5 \end{bmatrix} \pi\left(\begin{pmatrix} \sqrt{2}/2 \\ -2 \\ \sqrt{2}/2 \end{pmatrix}\right) = \begin{pmatrix} 162.5 \\ 120.5 - 4\sqrt{2} \end{pmatrix}
\end{aligned}$$

## Problem 2

(a) Let $\mathbf{x} \in \mathbb{R}^\top$ be a vector with elements $x_t$ and $\mathbf{1} \in \mathbb{R}^\top$ be a vector with elements equal to 1. The equations in (1) can be written in matrix form as follows:

$$\mathbf{y} = A\boldsymbol{\theta} = \begin{bmatrix} \mathbf{x} & \mathbf{1} \end{bmatrix} \boldsymbol{\theta}$$

so that $A := \begin{bmatrix} \mathbf{x} & \mathbf{1} \end{bmatrix} \in \mathbb{R}^{T\times 2}$

(b) The error function can be written as follows:

$$E(\boldsymbol{\theta}) = \left\| \begin{pmatrix} y_1 - \theta_1 x_1 - \theta_2 \\ \vdots \\ y_T - \theta_1 x_T - \theta_2 \end{pmatrix} \right\|_2^2 = \|\mathbf{y} - A\boldsymbol{\theta}\|_2^2 = (\mathbf{y} - A\boldsymbol{\theta})^\top (\mathbf{y} - A\boldsymbol{\theta})$$

(c) Taking the gradient of $E(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ and setting it equal to zero leads to:

$$0 = \nabla_{\boldsymbol{\theta}} E(\boldsymbol{\theta}) = -A^\top(\mathbf{y} - A\boldsymbol{\theta}) \quad \Rightarrow \quad A^\top A\boldsymbol{\theta} = A^\top \mathbf{y}.$$

The above equation can be solved in closed-form as follows:

$$\boldsymbol{\theta} = \left(A^\top A\right)^{-1} A^\top \mathbf{y} = \left(\begin{bmatrix} \mathbf{x}^\top \\ \mathbf{1}^\top \end{bmatrix} \begin{bmatrix} \mathbf{x} & \mathbf{1} \end{bmatrix}\right)^{-1} \begin{bmatrix} \mathbf{x}^\top \\ \mathbf{1}^\top \end{bmatrix} \mathbf{y} = \frac{1}{T\mathbf{x}^\top\mathbf{x} - 2\mathbf{1}^\top\mathbf{x}} \begin{bmatrix} T & -\mathbf{1}^\top\mathbf{x} \\ -\mathbf{1}^\top\mathbf{x} & \mathbf{x}^\top\mathbf{x} \end{bmatrix} \begin{bmatrix} \mathbf{x}^\top\mathbf{y} \\ \mathbf{1}^\top\mathbf{y} \end{bmatrix}$$

## Problem 3

The prior distribution over the robot state is $x_0 \sim \mathcal{N}(\mu_0, \sigma_0)$, where $\mu_0 = 1$ and $\Sigma_0 = 2$. The noise distribution is $\eta \sim \mathcal{N}(0, W)$ where $W = 1$. The derivative of the sigmoid function is $\sigma'(x) := \sigma(x)(1 - \sigma(x))$. The Jacobian of the motion model with respect to the state, evaluated at $\mu_0$ is:

$$A = \frac{d}{dx}\sigma(ax + bu_0)\bigg|_{x=\mu_0} = a\sigma'(a + 2b)$$

The posterior distribution approximated by the EKF is:

$$\begin{aligned}\mu_{EKF} &= \sigma(a+2b)\\ \Sigma_{EKF} &= A\Sigma_0 A^\top + W = 2a^2\sigma(a+2b)^2(1-\sigma(a+2b))^2+1\end{aligned}$$

The sigma points chosen by the UKF are $\{1, 1-\sqrt{2}, 1+\sqrt{2}\}$. The posterior distribution approximated by the UKF is:

$$\begin{aligned}\boldsymbol{\mu}_{UKF} &= \sum_{i=0}^{2d} v^{(i)}\sigma(a\mathcal{Y}_i + 2b) = \frac{1}{2}\sigma(a(1-\sqrt{2})+2b) + \frac{1}{2}\sigma(a(1+\sqrt{2})+2b)\\ \Sigma_{UKF} &= \sum_{i=0}^{2d} w^{(i)}\left(\sigma(a\mathcal{Y}_i+2b) - \mu_{UKF}\right)^2 + W\\ &= 2\left(\sigma(a+2b)-\mu_{UKF}\right)^2 + \frac{1}{2}\left(\sigma(a(1-\sqrt{2})+2b)-\mu_{UKF}\right)^2 + \frac{1}{2}\left(\sigma(a(1+\sqrt{2})+2b)-\mu_{UKF}\right)^2 + 1\end{aligned}$$

## Problem 4

The rotation matrix corresponding to the 30° counter-clockwise rotation is:

$$R_\Delta = \begin{bmatrix}\sqrt{3}/2 & -1/2 & 0\\ 1/2 & \sqrt{3}/2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

The robot pose after moving with translation $\mathbf{p}_\Delta$ and rotation $R_\Delta$ is:

$$\begin{aligned}\mathbf{p} = R_0\mathbf{p}_\Delta + \mathbf{p}_0 &= \frac{1}{2}\begin{bmatrix}\sqrt{3}/2 & 1/2 & 0\\ -1/2 & \sqrt{3}/2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}\sqrt{3}-1\\ \sqrt{3}+1\\ 0\end{pmatrix} + \begin{pmatrix}-1\\ 1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 2\\ 0\end{pmatrix}\\ R = R_0 R_\Delta &= \begin{bmatrix}\sqrt{3}/2 & 1/2 & 0\\ -1/2 & \sqrt{3}/2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{3}/2 & -1/2 & 0\\ 1/2 & \sqrt{3}/2 & 0\\ 0 & 0 & 1\end{bmatrix} = I_3\end{aligned}$$

The measurement model of the camera with position $\mathbf{p}$, orientation $R$, and intrinsic parameters $K$, observing a point $\mathbf{m} \in \mathbb{R}^3$ is:

$$\mathbf{z} = h(\mathbf{m}) + \mathbf{v} := K\pi({}_oR_r R^\top(\mathbf{m}-\mathbf{p})) + \mathbf{v},$$

where $\mathbf{v} \sim \mathcal{N}(0, I_2)$ is the measurement noise. The Jacobian of the canonical projection function $\pi(\mathbf{x})$ is:

$$\frac{d\pi}{d\mathbf{x}} = \frac{1}{x_3^2}\begin{bmatrix}x_3 & 0 & -x_1\\ 0 & x_3 & -x_2\\ 0 & 0 & 0\end{bmatrix}$$

The Jacobian of the observation model $h(\mathbf{m}) := K\pi({}_oR_r R^\top(\mathbf{m}-\mathbf{p}))$ evaluated at $\boldsymbol{\mu}_0$ is:

$$\begin{aligned}H :&= \frac{d}{d\mathbf{m}}h(\boldsymbol{\mu}_0) = K\frac{d\pi}{d\mathbf{x}}\left({}_oR_r R^\top(\boldsymbol{\mu}_0-\mathbf{p})\right){}_oR_r R^\top\\ &= \begin{bmatrix}1 & 0 & 100\\ 0 & 1 & 100\end{bmatrix}\frac{d\pi}{dx}\left(\begin{bmatrix}0 & -1 & 0\\ 0 & 0 & -1\\ 1 & 0 & 0\end{bmatrix}\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}\right)\begin{bmatrix}0 & -1 & 0\\ 0 & 0 & -1\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}-1 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\end{aligned}$$

The Kalman gain for $\Sigma_0 = I_3$ is:

$$L = \Sigma_0 H^\top (H\Sigma_0 H^\top + I_2)^{-1} = H^\top (HH^\top + I_2)^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^\top \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1/3 & 0 \\ -1/3 & 0 \\ 0 & -1/2 \end{bmatrix}$$

The updated mean of the chair position is:

$$\begin{aligned}
\boldsymbol{\mu} = \boldsymbol{\mu}_0 + L(\mathbf{z} - h(\boldsymbol{\mu}_0)) &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} -1/3 & 0 \\ -1/3 & 0 \\ 0 & -1/2 \end{bmatrix} \left( \begin{pmatrix} 100 \\ 100 \end{pmatrix} - K\pi({}_oR_r R^\top (\boldsymbol{\mu}_0 - \mathbf{p})) \right) \\
&= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} -1/3 & 0 \\ -1/3 & 0 \\ 0 & -1/2 \end{bmatrix} \left( \begin{pmatrix} 100 \\ 100 \end{pmatrix} - \begin{bmatrix} 1 & 0 & 100 \\ 0 & 1 & 100 \end{bmatrix} \pi \left( \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} -1/3 & 0 \\ -1/3 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ 0 \end{pmatrix}
\end{aligned}$$

The updated covariance of the chair position is:

$$\Sigma = \Sigma_0 - LH\Sigma_0 = I_3 - \begin{bmatrix} -1/3 & 0 \\ -1/3 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} I_3 = \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

## Problem 5

The position $\mathbf{p}(t) \in \mathbb{R}^3$ and orientation $R(t) \in SO(3)$ of the robot with respect to the world frame are:

$$\mathbf{p}(t) = \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} \qquad R(t) = \begin{bmatrix} \cos\frac{t\pi}{3} & 0 & -\sin\frac{t\pi}{3} \\ 0 & 1 & 0 \\ \sin\frac{t\pi}{3} & 0 & \cos\frac{t\pi}{3} \end{bmatrix}$$

1. Find the axis-angle representations of the robot orientation at time $t = 1$.

   The axis-angle representation $\boldsymbol{\theta} \in \mathbb{R}^3$ of the robot orientation at time $t = 1$ can be obtained via the logarithm function:

$$\|\boldsymbol{\theta}\| = \arccos\left( \frac{\mathbf{tr}(R(1)) - 1}{2} \right) = \frac{\pi}{3} \qquad \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} = \frac{1}{2\sin\|\boldsymbol{\theta}\|} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

2. Find the quaternion representations of the robot orientation at time $t = 1$ and of the inverse of this orientation.

$$\mathbf{q} = \begin{bmatrix} \cos\left(\frac{\|\boldsymbol{\theta}\|}{2}\right) \\ \sin\left(\frac{\|\boldsymbol{\theta}\|}{2}\right) \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} \qquad \mathbf{q}^{-1} = \bar{\mathbf{q}} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

3. Compute the linear and the angular velocity of the robot with respect to the robot frame and with respect to the world frame at time $t = 1$.

The rotation kinematics for a body-frame twist $\boldsymbol{\zeta}(t) \in \mathbb{R}^6$ satisfy:

$$\dot{T}(t) = T(t)\hat{\boldsymbol{\zeta}}(t)$$

Hence, at time 1:

$$\hat{\boldsymbol{\zeta}}(1) = T(1)^{-1}\dot{T}(1) = \begin{bmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} & \cos\frac{\pi}{3} + 2\sin\frac{\pi}{3} \\ 0 & 1 & 0 & 0 \\ -\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} & 2\cos\frac{\pi}{3} - \sin\frac{\pi}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\pi}{3}\sin\frac{\pi}{3} & 0 & -\frac{\pi}{3}\cos\frac{\pi}{3} & 1 \\ 0 & 0 & 0 & 0 \\ \frac{\pi}{3}\cos\frac{\pi}{3} & 0 & -\frac{\pi}{3}\sin\frac{\pi}{3} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\frac{\pi}{3} & \frac{1}{2} + \sqrt{3} \\ 0 & 0 & 0 & 0 \\ \frac{\pi}{3} & 0 & 0 & 1 - \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the body-frame twist is $\boldsymbol{\zeta} = \begin{bmatrix} \frac{1}{2} + \sqrt{3} & 0 & 1 - \frac{\sqrt{3}}{2} & 0 & -\frac{\pi}{3} & 0 \end{bmatrix}^\top \in \mathbb{R}^6$. The first three components of the body-frame twist represent the linear velocity of the robot body frame with respect to the world frame as viewed in the robot body frame. To verify this, note that the world-frame coordinates of the first-three components of the body-frame twist are:

$$R(1) \begin{bmatrix} \frac{1}{2} + \sqrt{3} \\ 0 \\ 1 - \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \dot{\mathbf{p}}(1)$$

which is equal to the linear velocity of the body frame origin in the world frame. Similarly,

$$R(1) \begin{bmatrix} 0 \\ -\frac{\pi}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\pi}{3} \\ 0 \end{bmatrix}$$

which is equal to the rotational velocity of the body frame origin in the world frame.

The above observation holds in general, so that the body-frame twist is the linear velocity and rotation velocity of the robot body frame origin with respect to the world frame as viewed in the robot body frame. The angular and linear velocities of the body frame with respect to the body frame are, of course, **zero** regardless of which frame they are viewed in.

4. Let $\mathbf{p}_W = (9, 0, 0)$ be a point with coordinates specified in the world frame. Compute the coordinates $\mathbf{p}_R$ of the point $\mathbf{p}_W$ in the robot frame at time $t = 1$.

$$\mathbf{p}_R = R(1)^\top(\mathbf{p}_W - \mathbf{p}(1)) = \begin{bmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 - \sqrt{3} \\ 0 \\ -4\sqrt{3} - 1 \end{bmatrix}$$

## Problem 6

Consider a small perturbation $\boldsymbol{\xi} \in \mathbb{R}^6$ to the camera pose $T \in SE(3)$ so that

$$\begin{aligned} \pi_s\left(\left(\exp(\hat{\boldsymbol{\xi}})T\right)^{-1} \underline{\mathbf{m}}\right) &= \pi_s\left(T^{-1}\exp(-\hat{\boldsymbol{\xi}})\underline{\mathbf{m}}\right) \approx \pi_s\left(T^{-1}\left(I - \hat{\boldsymbol{\xi}}\right)\underline{\mathbf{m}}\right) \\ &= \pi_s\left(T^{-1}\underline{\mathbf{m}} - T^{-1}\hat{\boldsymbol{\xi}}\underline{\mathbf{m}}\right) = \pi_s\left(T^{-1}\underline{\mathbf{m}} - T^{-1}\underline{\mathbf{m}}^{\odot}\boldsymbol{\xi}\right) \\ &\approx \pi_s\left(T^{-1}\underline{\mathbf{m}}\right) \underbrace{-\frac{d\pi_s}{d\mathbf{q}}\left(T^{-1}\underline{\mathbf{m}}\right)T^{-1}\underline{\mathbf{m}}^{\odot}}_{J}\boldsymbol{\xi} \end{aligned}$$

Thus, the Jacobian of the pixel observation $\mathbf{z}$ with respect to the six degrees of freedom of the camera pose $T$ is:

$$J = -\frac{d\pi_s}{d\mathbf{q}}\left(T^{-1}\underline{\mathbf{m}}\right)T^{-1}\underline{\mathbf{m}}^{\odot} \in \mathbb{R}^{3\times 6}$$

where $\frac{d\pi_s}{d\mathbf{q}}(\mathbf{q}) = \frac{1}{\|\mathbf{q}\|_2^3}\left(\|\mathbf{q}\|_2^2 I - \mathbf{q}\mathbf{q}^\top\right) \in \mathbb{R}^{4\times 4}$

## Problem 7

Consider a Gaussian random vector $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$. The distribution of an affine transformation $\bar{\mathbf{y}} := A\mathbf{y}+\mathbf{b}$ of $\mathbf{y}$ is $\mathcal{N}\left(A\boldsymbol{\mu}+\mathbf{b}, A\Sigma A^\top\right)$. This can be shown by computing the mean and covariance:

$$\begin{aligned}
\mathbb{E}[\bar{\mathbf{y}}] &= A\mathbb{E}[\mathbf{y}] + \mathbf{b} = A\boldsymbol{\mu} + \mathbf{b} \\
\mathbf{Var}[\bar{\mathbf{y}}] &= \mathbf{Var}[A\mathbf{y}] + \mathbf{Var}[\mathbf{b}] = \mathbb{E}\left[\left(A(\mathbf{y}-\boldsymbol{\mu})\right)\left(A(\mathbf{y}-\boldsymbol{\mu})\right)^\top\right] + 0 \\
&= A\mathbb{E}\left[(\mathbf{y}-\boldsymbol{\mu})(\mathbf{y}-\boldsymbol{\mu})^\top\right]A^\top = A\Sigma A^\top
\end{aligned}$$

Based on the above derivation, the distribution of $z_t \mid x$ is $\mathcal{N}(0, x)$ with associated pdf $\phi(z_t; 0, x)$. The prior of $x$ is Inv-Gamma$(\alpha, \beta)$ with associated pdf $\gamma^{-1}(x; \alpha, \beta)$. The Bayes filter update step is:

$$\begin{aligned}
p(x \mid z_0) \propto p(z_0 \mid x)p(x) &= \phi(z_0; 0, x)\gamma^{-1}(x; \alpha, \beta) \\
&= \frac{1}{\sqrt{2\pi x}}\exp\left(-\frac{z_0^2}{2x}\right)\frac{\beta^\alpha}{\Gamma(\alpha)}x^{-\alpha-1}\exp\left(-\frac{\beta}{x}\right) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}}x^{-\alpha-3/2}\exp\left(-\frac{1}{x}\left(\beta + \frac{1}{2}z_0^2\right)\right)
\end{aligned}$$

We can see that after normalization, this will look like another Inverse Gamma distribution, $x \mid z_0 \sim$ Inv-Gamma$\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}z_0^2\right)$. Applying the update step repeatedly preserves the posterior distribution as Inverse Gamma:

$$x \mid z_{0:T} \sim \text{Inv-Gamma}\left(\alpha + \frac{T+1}{2}, \beta + \frac{1}{2}\sum_{t=0}^{\top} z_t^2\right)$$

## Problem 8

1. Gaussian Naïve Bayes models the joint distribution $p(y, \mathbf{x})$ of an example $\mathbf{x} \in \mathbb{R}^d$ and its label $y \in \{1, \ldots, K\}$ as: $p(y, \mathbf{x}) = p(y; \boldsymbol{\theta})p(\mathbf{x}|y; \boldsymbol{\omega})$, where $p(y = k; \boldsymbol{\theta}) = \boldsymbol{\theta}_k$ and $p(\mathbf{x}|y = k; \boldsymbol{\omega}) = \prod_{l=1}^d p(x_l|y = k; \boldsymbol{\omega}) = \prod_{l=1}^d \phi(x_l; \mu_{kl}, \sigma_{kl}^2)$.

2. The space of $3 \times 3$ skew-symmetric matrices is defined as: $\mathfrak{so}(3) = \left\{\hat{\mathbf{x}} \in \mathbb{R}^{3\times 3} | \mathbf{x} \in \mathbb{R}^3\right\}$.

3. Let $(x, y, z) \in \mathbb{R}^3$ be a point in the optical frame of a monocular camera. The 3D-to-2D perspective projection operation transforms $(x, y, z)$ to: $(fx/z, fy/z)$, where $f$ is the focal length.

4. The prediction step of the Bayes filter is: $p_{t+1|t}(\mathbf{x}) = \int p_f(\mathbf{x}|\mathbf{s}, \mathbf{u}_t)p_{t|t}(\mathbf{s})d\mathbf{s}$, where $p_f(\mathbf{x}_{t+1}|\mathbf{x}_t, \mathbf{u}_t)$ is the motion model.

5. Consider a joint Gaussian distribution of the form:

$$\begin{pmatrix}\mathbf{x} \\ \mathbf{z}\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}\boldsymbol{\mu} \\ \boldsymbol{\eta}\end{pmatrix}, \begin{bmatrix}\boldsymbol{\Sigma} & \boldsymbol{\Sigma}\mathbf{H}^\top \\ \mathbf{H}\boldsymbol{\Sigma} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}^\top + \mathbf{V}\end{bmatrix}\right).$$

The distribution of $\mathbf{x}$ conditioned on $\mathbf{z}$ is: $\mathcal{N}(\boldsymbol{\mu} + \mathbf{K}(\mathbf{z} - \boldsymbol{\eta}), \boldsymbol{\Sigma} - \mathbf{K}\mathbf{H}\boldsymbol{\Sigma})$, where $\mathbf{K} = \boldsymbol{\Sigma}\mathbf{H}^\top(\mathbf{H}\boldsymbol{\Sigma}\mathbf{H}^\top + \mathbf{V})^{-1}$.

## Problem 9

1. The body position and orientation at time $t$ are $\mathbf{p}$ and $R\exp(t\hat{\boldsymbol{\omega}})$. The coordinates of $\mathbf{m}$ in the world frame are $\bar{\mathbf{m}}(t) = R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} + \mathbf{p}$.

2. The observation model of the range sensor is:

$$z(t) = \|\bar{\mathbf{m}}(t) - \mathbf{a}\|_2^2 = (\bar{\mathbf{m}}(t) - \mathbf{a})^\top(\bar{\mathbf{m}}(t) - \mathbf{a}) \tag{5}$$

The observation model can be simplified to:

$$z(t) = \mathbf{m}^\top\mathbf{m} + 2(\mathbf{p} - \mathbf{a})^\top R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} + (\mathbf{p} - \mathbf{a})^\top(\mathbf{p} - \mathbf{a}) \tag{6}$$

3. The derivative of $z(t)$ with respect to $t$ is:

$$\begin{aligned} \frac{d}{dt}z(t) &= 2(\mathbf{p} - \mathbf{a})^\top R\left[\frac{d}{dt}\exp(t\hat{\boldsymbol{\omega}})\right]\mathbf{m} \\ &= 2(\mathbf{p} - \mathbf{a})^\top R\exp(t\hat{\boldsymbol{\omega}})\hat{\boldsymbol{\omega}}\mathbf{m} \end{aligned} \tag{7}$$

4. Using a left perturbation $\boldsymbol{\psi} = J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta}$ with respect to $R = \exp(\hat{\boldsymbol{\theta}})$, we have:

$$\begin{aligned} 2(\mathbf{p} - \mathbf{a})^\top \exp(\hat{\boldsymbol{\psi}})R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} &\approx 2(\mathbf{p} - \mathbf{a})^\top R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} + 2(\mathbf{p} - \mathbf{a})^\top\hat{\boldsymbol{\psi}}R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} \\ &= 2(\mathbf{p} - \mathbf{a})^\top R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} - 2(\mathbf{p} - \mathbf{a})^\top\left[R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m}\right]^\wedge\boldsymbol{\psi}. \end{aligned} \tag{8}$$

Thus, the derivative of the range measurement $z(t)$ at time $t$ with respect to $\boldsymbol{\theta}$ is:

$$-2(\mathbf{p} - \mathbf{a})^\top\left[R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m}\right]^\wedge J_L(\boldsymbol{\theta}) \tag{9}$$

Alternatively, using a right perturbation $\boldsymbol{\psi} = J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta}$ with respect to $R = \exp(\hat{\boldsymbol{\theta}})$, we have:

$$\begin{aligned} 2(\mathbf{p} - \mathbf{a})^\top R\exp(\hat{\boldsymbol{\psi}})\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} &\approx 2(\mathbf{p} - \mathbf{a})^\top R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} + 2(\mathbf{p} - \mathbf{a})^\top R\hat{\boldsymbol{\psi}}\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} \\ &= 2(\mathbf{p} - \mathbf{a})^\top R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m} - 2(\mathbf{p} - \mathbf{a})^\top R\left[\exp(t\hat{\boldsymbol{\omega}})\mathbf{m}\right]^\wedge\boldsymbol{\psi}. \end{aligned} \tag{10}$$

The derivative is the same:

$$\begin{aligned} -2(\mathbf{p} - \mathbf{a})^\top R\left[\exp(t\hat{\boldsymbol{\omega}})\mathbf{m}\right]^\wedge J_R(\boldsymbol{\theta}) &= -2(\mathbf{p} - \mathbf{a})^\top\left[R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m}\right]^\wedge RJ_R(\boldsymbol{\theta}) \\ &= -2(\mathbf{p} - \mathbf{a})^\top\left[R\exp(t\hat{\boldsymbol{\omega}})\mathbf{m}\right]^\wedge J_L(\boldsymbol{\theta}) \end{aligned} \tag{11}$$

## Problem 10

1. The update step for the variance of the Kalman filter with $H = 1$ and $V = b^2$ is:

$$\sigma_{t|t}^2 = \sigma_{t|t-1}^2 - \sigma_{t|t-1}^2(\sigma_{t|t-1}^2 + b^2)^{-1}\sigma_{t|t-1}^2 \tag{12}$$

The prediction step for the variance of the Kalman filter with $F = \frac{1}{2}$ and $W = a^2$ is:

$$\sigma_{t+1|t}^2 = \frac{1}{4}\sigma_{t|t}^2 + a^2 \tag{13}$$

Thus, the function $f$ that relates $\sigma_{t+1|t}^2$ to $\sigma_{t|t-1}^2$ is:

$$f(\sigma) = \frac{1}{4}\left(\sigma^2 - \frac{\sigma^4}{\sigma^2 + b^2}\right) + a^2 = \frac{1}{4}\frac{\sigma^2 b^2}{\sigma^2 + b^2} + a^2 \tag{14}$$

2. The solution $\sigma_\infty^2$ to $\sigma = f(\sigma)$ satisfies:

$$\begin{aligned}
4(\sigma_\infty^2 - a^2)(\sigma_\infty^2 + b^2) &= b^2\sigma_\infty^2 \\
4\sigma_\infty^4 + 4b^2\sigma_\infty^2 - 4a^2\sigma_\infty^2 - 4a^2b^2 - b^2\sigma_\infty^2 &= 0 \\
4\sigma_\infty^4 + (3b^2 - 4a^2)\sigma_\infty^2 - 4a^2b^2 &= 0 \\
\sigma_\infty^2 = \frac{1}{8}\left((4a^2 - 3b^2) \pm \sqrt{(4a^2 - 3b^2)^2 + 64a^2b^2}\right)
\end{aligned} \tag{15}$$

3. When $a = b$:

$$\sigma_\infty^2 = \frac{a^2}{8} + \frac{1}{8}\sqrt{65b^4} = \frac{(1 + \sqrt{65})}{8}b^2 \approx 1.13b^2 \tag{16}$$

The Kalman gain is $K_\infty = \frac{\sigma_\infty^2}{\sigma_\infty^2 + b^2} \approx \frac{1.13}{2.13} \approx 0.53$.

When $a = 2b$:

$$\sigma_\infty^2 = \frac{13b^2}{8} + \frac{5\sqrt{17}}{8}b^2 \approx 4.20b^2 \tag{17}$$

The Kalman gain is $K_\infty = \frac{\sigma_\infty^2}{\sigma_\infty^2 + b^2} \approx \frac{4.20}{5.20} \approx 0.81$.

As the motion noise increases relative to the measurement noise, the Kalman gain $K$ increases and the filter puts more emphasis on the measurements rather than the motion predictions.