

ECE276A: Sensing & Estimation in Robotics

Lecture 13: Localization from Point Features

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2-D Localization from Relative Position Measurements

- ▶ **Goal:** determine the robot position $p \in \mathbb{R}^2$ and orientation $\phi \in (-\pi, \pi]$
- ▶ **Given:** two landmark positions $l_1, l_2 \in \mathbb{R}^2$ (world frame) and **relative position** measurements (body frame):

$$z_i = R^T(\phi)(l_i - p) \in \mathbb{R}^2, \quad i = 1, 2$$

- ▶ Let $\delta z := z_1 - z_2$ and $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so that:

$$l_1 - l_2 = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} (z_1 - z_2) = \begin{bmatrix} \delta z & J\delta z \end{bmatrix} \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}$$

- ▶ As long as $\det \begin{bmatrix} \delta z & J\delta z \end{bmatrix} = \|\delta z\|_2^2 = \|l_1 - l_2\|_2^2 \neq 0$, we can compute:

$$\begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} = \frac{1}{\|\delta z\|_2^2} \begin{bmatrix} \delta z_x & \delta z_y \\ -\delta z_y & \delta z_x \end{bmatrix} (l_1 - l_2) \quad \boxed{\phi = \mathbf{atan2}(\sin \phi, \cos \phi)}$$

- ▶ Given the orientation ϕ , we can then obtain the robot position:

$$\boxed{p = \frac{1}{2}((l_1 + l_2) - R(\phi)(z_1 + z_2))}$$

3-D Localization from Relative Position Measurements

- ▶ **Goal:** determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- ▶ **Given:** three landmark positions $l_1, l_2, l_3 \in \mathbb{R}^3$ (world frame) and **relative position** measurements (body frame):

$$z_i = R^T(l_i - p) \in \mathbb{R}^3, \quad i = 1, 2, 3$$

- ▶ Let $l_{ij} := l_i - l_j$ and $z_{ij} = z_i - z_j$ and compute:

$$l_{12} \times l_{13} = (Rz_{12}) \times (Rz_{13}) = R(z_{12} \times z_{13})$$

- ▶ The vector $l_{12} \times l_{13}$ provides orthogonal information to l_1 and l_2 and can be used to estimate the orientation R **as long as the three features are not all on the same line:**

$$\begin{bmatrix} l_1 & l_2 & l_{12} \times l_{13} \end{bmatrix} = R \begin{bmatrix} z_1 & z_2 & z_{12} \times z_{13} \end{bmatrix}$$

$$R = \begin{bmatrix} l_1 & l_2 & l_{12} \times l_{13} \end{bmatrix} \begin{bmatrix} z_1 & z_2 & z_{12} \times z_{13} \end{bmatrix}^{-1}$$

- ▶ Given the orientation R , we can then obtain the robot position:

$$p = \frac{1}{3} \sum_{i=1}^3 (l_i - Rz_i)$$

3-D Localization from Relative Position Measurements

- ▶ **Goal:** determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- ▶ **Given:** n landmark positions $l_i \in \mathbb{R}^3$ (world frame) and **relative position** measurements (body frame):

$$z_i = R^T(l_i - p) \in \mathbb{R}^3, \quad i = 1, \dots, n$$

- ▶ Define the landmark centroids in the world and body frames:

$$\bar{l} := \frac{1}{n} \sum_{i=1}^n l_i \quad \bar{z} := \frac{1}{n} \sum_{i=1}^n z_i \quad \boxed{\bar{l} = p + R\bar{z}}$$

- ▶ Let $\delta l_i := l_i - \bar{l}$ and $\delta z_i := z_i - \bar{z}$ so that $\delta l_i = R\delta z_i$ for $i = 1, \dots, n$
- ▶ Estimate the orientation via least-squares:

$$\min_R \sum_{i=1}^n \|\delta l_i - R\delta z_i\|_2^2 = \min_R \sum_{i=1}^n \delta l_i^T \delta l_i - 2\delta l_i^T R\delta z_i - \delta z_i^T \underbrace{R^T R}_{I_{3 \times 3}} \delta z_i$$

Kabsch Algorithm

- ▶ Find transformation p, R to match two sets $\{l_i\}$ and $\{z_i\}$ of 3-D points
- ▶ Given the rotation R , the optimal translation is: $p = \bar{l} - R\bar{z}$
- ▶ Need to solve a least-squares problem in $SO(3)$ to determine R :

$$\max_R \sum_{i=1}^n \delta l_i^T R \delta z_i = \text{tr}(M^T R)$$

s.t. $R^T R = I_{3 \times 3}, \det(R) = 1$

where $M^T := \sum_{i=1}^n \delta z_i \delta l_i^T$

- ▶ Let $M = Z \Sigma L^T$ be a singular value decomposition with $\Sigma_{ii} \geq 0$, $\det L = \pm 1$, and $\det Z = \pm 1$
- ▶ Define a unitary matrix $U := Z^T R L \in \mathbb{R}^{n \times n}$.
- ▶ $\text{tr}(M^T R) = \text{tr}(\Sigma Z^T R L) = \text{tr}(\Sigma U) = \sum_{i=1}^n \Sigma_{ii} U_{ii}$ and since $\Sigma_{ii} \geq 0$ and $\det(U) = \pm 1$, the objective is maximized for:

$$U = Z^T R L = I_{n \times n} \quad \begin{array}{c} \text{avoids} \\ \Rightarrow \\ \text{reflection} \end{array} \quad R = Z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(ZL^T) \end{bmatrix} L^T$$

Iterative Closest Point (ICP)

- ▶ Kabsch assumes **known point correspondences** (data association)
- ▶ The ICP Algorithm finds a rigid body transformation to match two sets $\{l_i\}$ and $\{z_i\}$ of 3-D points with **unknown correspondences**
- ▶ Start with (p_0, R_0) (**sensitive to initial guess**) and iterate
 1. Given (p, R) , find correspondences $l_i \leftrightarrow z_j$ based on **closest points**:

$$j^* = \arg \min_j \|l_i - (Rz_j + p)\|_2^2$$

2. Given correspondences, find (p, R) using the **Kabsch** algorithm



Probabilistic ICP

- ▶ Place a small probabilistic ball around each l_i to define a mixture distribution for the data:

$$p(x) = \sum_i \alpha_i \phi(x; l_i, \sigma_i^2 I_{3 \times 3})$$

- ▶ Find parameters (p, R) to max the likelihood of $\{Rz_j + p\}$ under $p(x)$:

$$\max_{p, R} \sum_j \log \sum_i \alpha_i \phi(Rz_j + p; l_i, \sigma_i^2 I_{3 \times 3})$$

- ▶ Use **EM!**
- ▶ ICP is a special case with $\sigma_i^2 \rightarrow 0$
- ▶ **Robustness:** use $\exp\left(-\frac{|x-l_i|^\beta}{2\sigma_i^2}\right)$ with $\beta \in (0, 2)$ instead of $\exp\left(-\frac{|x-l_i|^2}{2\sigma_i^2}\right)$

2-D Odometry from Relative Position Measurements

- ▶ **Goal:** determine the relative transformation ${}^t p_{t+1} \in \mathbb{R}^2$ and ${}^t \phi_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t + 1$ and t
- ▶ **Given:** relative position measurements $z_{t,1}, z_{t,2} \in \mathbb{R}^2$ and $z_{t+1,1}, z_{t+1,2} \in \mathbb{R}^2$ at consecutive time steps to two **unknown** landmarks
- ▶ If we consider the robot frame at time t to be the “world frame”, **this problem is the same as 2-D localization from relative position measurements** with $l_i := z_{t,i}$, $z_i := z_{t+1,i}$, $p := {}^t p_{t+1}$, $\phi := {}^t \phi_{t+1}$

3-D Odometry from Relative Position Measurements

- ▶ **Goal:** determine the relative transformation ${}^t p_{t+1} \in \mathbb{R}^3$ and ${}^t R_{t+1} \in SO(3)$ between two robot frames at time $t + 1$ and t
- ▶ **Given:** relative position measurements $z_{t,i} \in \mathbb{R}^3$ and $z_{t+1,i} \in \mathbb{R}^3$ at consecutive time steps to n **unknown** landmarks
- ▶ If we consider the robot frame at time t to be the “world frame”, **this problem is the same as 3-D localization from relative position measurements** with $l_i := z_{t,i}$, $z_i := z_{t+1,i}$, $p := {}^t p_{t+1}$, $R := {}^t R_{t+1}$

Summary: Rel. Position Measurements $z_i = R^T(l_i - p)$

► Localization

$l_1, l_2, z_1, z_2 \in \mathbb{R}^2$	$(l_1 - l_2) = R(\phi)(z_1 - z_2)$ $p = \frac{1}{2} \sum_{i=1}^2 (l_i - Rz_i)$
$l_1, z_i \in \mathbb{R}^3, i = 1, 2, 3$ $l_{ij} := l_i - l_j, z_{ij} := z_i - z_j$	$[l_1 \ l_2 \ l_{12} \times l_{13}] = R [z_1 \ z_2 \ z_{12} \times z_{13}]$ $p = \frac{1}{3} \sum_{i=1}^3 (l_i - Rz_i)$
$l_i, z_i \in \mathbb{R}^3, i = 1, \dots, n$ $\delta l_i := l_i - \frac{1}{n} \sum_{j=1}^n l_j,$ $\delta z_i := z_i - \frac{1}{n} \sum_{j=1}^n z_j$	$R = \arg \max_{R \in SO(3)} \sum_{i=1}^n \delta l_i^T R \delta z_i$ <p style="text-align: center;">Kabsch algorithm</p> $\frac{\text{SVD}(\sum_{i=1}^n \delta l_i \delta z_i^T) = Z \Sigma L^T}{Z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(ZL^T) \end{bmatrix} L^T}$ $p = \frac{1}{n} \sum_{i=1}^n (l_i - Rz_i)$

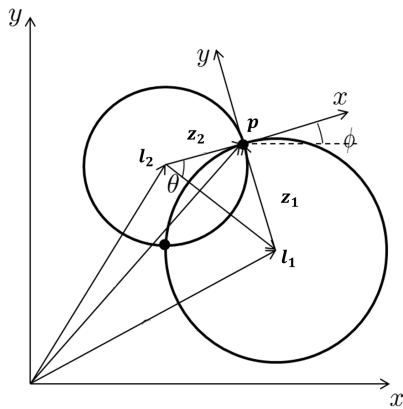
► **Odometry:** same with $l_i = z_{t,i}, z_i := z_{t+1,i}, p := {}_t p_{t+1}, R := {}_t R_{t+1}$

2-D Localization from Range Measurements

- ▶ **Goal:** determine the robot position $p \in \mathbb{R}^2$ and orientation $\phi \in (-\pi, \pi]$
- ▶ **Given:** two landmark positions $l_1, l_2 \in \mathbb{R}^2$ (world frame) and **range** measurements (body frame):

$$z_i = \|l_i - p\|_2 \in \mathbb{R}, \quad i = 1, 2$$

- ▶ Because all possible positions whose distance to l_1 is z_1 is a circle, the possible robot positions are given by the intersection of two circles



2-D Localization from Range Measurements

- ▶ Applying the law of cosines to the triangle gives:

$$z_2^2 = z_1^2 + \|l_2 - l_1\|_2^2 - 2z_1\|l_2 - l_1\|_2 \cos \theta$$

- ▶ Solving for θ and then the circle intersection points provides the possible robot positions:

$$p = l_2 + z_2 R(\pm\theta) \frac{l_1 - l_2}{\|l_1 - l_2\|_2}$$

- ▶ The orientation of the robot ϕ is **not identifiable**

2-D Localization from Range Measurements

- ▶ **Pose disambiguation:** the robot can make a move with known translation p_Δ (measured in the frame at time t and take two new range measurements
- ▶ There are 2 possible robot positions at each time frame for a total of 4 combinations but comparing $\|p_{t+1} - p_t\|_2$ to the known $\|p_\Delta\|_2$ leaves only two valid options (and we cannot distinguish between them)
- ▶ To obtain the orientation, we use geometric constraints:

$$p_{t+1} - p_t = R(\phi_t)p_\Delta = \begin{bmatrix} p_{\Delta,x} & -p_{\Delta,y} \\ p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} \begin{bmatrix} \cos \phi_t \\ \sin \phi_t \end{bmatrix}$$

- ▶ As long as $\det \begin{bmatrix} p_{\Delta,x} & -p_{\Delta,y} \\ p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} = \|p_\Delta\|_2^2 \neq 0$, we can compute:

$$\begin{bmatrix} \cos \phi_t \\ \sin \phi_t \end{bmatrix} = \frac{1}{\|p_\Delta\|_2^2} \begin{bmatrix} p_{\Delta,x} & p_{\Delta,y} \\ -p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} (p_{t+1} - p_t)$$

$$\phi_t = \mathbf{atan2}(\sin \phi_t, \cos \phi_t)$$

3-D Localization from Range Measurements

- ▶ **Goal:** determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- ▶ **Given:** three landmark positions $l_1, l_2, l_3 \in \mathbb{R}^3$ (world frame) and **range** measurements (body frame):

$$z_i = \|l_i - p\|_2 \in \mathbb{R}, \quad i = 1, 2, 3$$

- ▶ All possible positions whose distance to l_1 is z_1 is a sphere
- ▶ The possible robot positions are the intersections of three spheres
- ▶ To find the intersection of 3 spheres, we first find the intersection of sphere one and two (a circle) and of sphere two and three (a circle). The intersection of these two circles gives the possible robot positions.
- ▶ **Degenerate case:** all landmarks are on the same line – the intersection of the spheres is a circle with infinitely many possible robot positions

3-D Localization from Range Measurements

- ▶ **Intersecting circle of spheres with radii z_1 and z_2 :** center O_{12} , radius r_{12} , normal vector n_{12} (perpendicular to the circle plane)

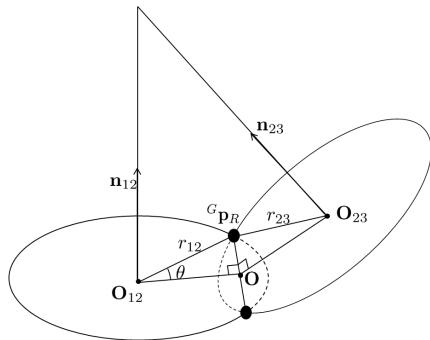
- ▶ Law of Cosines: $z_2^2 = z_1^2 + \|l_2 - l_1\|_2^2 - 2z_1\|l_2 - l_1\|_2 \cos \theta_{12}$

- ▶ Geometric relationships:

$$O_{12} = l_1 + z_1 \cos \theta_{12} n_{12}$$

$$r_{12} = z_1 |\sin(\theta_{12})|$$

$$n_{12} = \frac{l_2 - l_1}{\|l_2 - l_1\|_2}$$



- ▶ **Intersecting circle of spheres with radii z_2 and z_3 :** center O_{23} , radius r_{23} , normal vector n_{23} (perpendicular to the circle plane):

$$O_{23} = l_2 + z_2 \cos \theta_{23} n_{23} \quad r_{23} = z_2 |\sin(\theta_{23})| \quad n_{23} = \frac{l_3 - l_2}{\|l_3 - l_2\|_2} \quad 15$$

3-D Localization from Range Measurements

- ▶ The intersecting points of the two circles can be obtained from the geometric relationships:

$$\begin{aligned} n_{12}^T(O_{12} - O) &= 0 \\ n_{23}^T(O_{23} - O) &= 0 \\ (n_{12} \times n_{23})^T(O_{12} - O) &= 0 \end{aligned} \quad \begin{bmatrix} n_{12}^T \\ n_{23}^T \\ (n_{12} \times n_{23})^T \end{bmatrix} O = \begin{bmatrix} n_{12}^T O_{12} \\ n_{23}^T O_{23} \\ (n_{12} \times n_{23})^T O_{12} \end{bmatrix}$$

- ▶ As long as the three landmarks are not on the same line, we can uniquely solve for O :

$$\det \begin{bmatrix} n_{12}^T \\ n_{23}^T \\ (n_{12} \times n_{23})^T \end{bmatrix} \neq 0 \quad \Leftrightarrow \quad n_{12} \text{ and } n_{23} \text{ not colinear}$$

- ▶ The two possible robot positions are:

$$\cos \theta = \frac{\|O - O_{12}\|_2}{r_{12}} \quad p = O_{12} + r_{12}R(n_{12}, \pm\theta) \frac{O - O_{12}}{\|O - O_{12}\|_2}$$

- ▶ As in the 2-D case, the robot orientation R is **not identifiable**

3-D Localization from Range Measurements

- ▶ **Pose disambiguation:** the robot can make a move with known translation $p_\Delta \in \mathbb{R}^3$ and rotation $R_\Delta \in SO(3)$ and take three new range measurements
- ▶ As in the 2-D case, after eliminating the impossible robot positions, we should be left with only two options for p_t and p_{t+1}
- ▶ Given p_t , p_{t+1} , p_Δ , and R_Δ , we can now obtain R_t

$$p_{t+1} = p_t + R_t p_\Delta$$

- ▶ This is not sufficient because the rotation about p_Δ is not identifiable
- ▶ The robot needs to **move a second time** to a third pose p_{t+2} , R_{t+2} with known translation $p_{\Delta,2} \in \mathbb{R}^3$ and take three more range measurements to the three landmarks:

$$p_{t+2} = p_{t+1} + R_{t+1} p_{\Delta,2} = p_{t+1} + R_t R_\Delta p_{\Delta,2}$$

3-D Localization from Range Measurements

- ▶ Putting the previous two equations together:

$$\begin{aligned}p_{t+1} - p_t &= R_t p_\Delta \\ p_{t+2} - p_{t+1} &= R_t R_\Delta p_{\Delta,2}\end{aligned}$$

- ▶ Taking a cross product between the two:

$$(p_{t+1} - p_t) \times (p_{t+2} - p_{t+1}) = R_t (p_\Delta \times R_\Delta p_{\Delta,2})$$

- ▶ As long as $U := [p_\Delta, R_\Delta p_{\Delta,2}, p_\Delta \times R_\Delta p_{\Delta,2}]$ is nonsingular, i.e., p_Δ and $R_\Delta p_{\Delta,2}$ are not co-linear or equivalently **the three robot positions are not on the same line**, we can determine the robot orientation:

$$R_t = [(p_{t+1} - p_t), (p_{t+2} - p_{t+1}), (p_{t+1} - p_t) \times (p_{t+2} - p_{t+1})] U^{-1}$$

2-D Odometry from Range Measurements

- ▶ **Goal:** determine the relative transformation ${}_t p_{t+1} \in \mathbb{R}^2$ and ${}_t \phi_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t + 1$ and t
- ▶ **Given:** range measurements $z_{t,i} \in \mathbb{R}$ and $z_{t+1,i} \in \mathbb{R}$ at consecutive time steps to n **unknown** landmarks
- ▶ Let $l_{t+1,i}$ be the relative position to the i -th landmark at $t + 1$ so that:

$$\begin{aligned}z_{t+1,i} &= \|l_{t+1,i}\|_2 \\z_{t,i} &= \|{}_t p_{t+1} + R({}_t \phi_{t+1})l_{t+1,i}\|_2\end{aligned}$$

- ▶ Squaring and combining these equations, we get:

$$[{}_t p_{t+1}]^T {}_t p_{t+1} + 2l_{t+1,i}^T R^T({}_t \phi_{t+1}) {}_t p_{t+1} = z_{t,i}^2 - z_{t+1,i}^2, \quad i = 1, \dots, n$$

- ▶ We have n equations with $n + 3$ unknowns (3 for the relative pose and n for the unknown directions to the landmarks at $t + 1$), which is **not solvable**.

3-D Odometry from Range Measurements

- ▶ **Goal:** determine the relative transformation ${}^t p_{t+1} \in \mathbb{R}^3$ and ${}^t R_{t+1} \in SO(3)$ between two robot frames at time $t + 1$ and t
- ▶ **Given:** range measurements $z_{t,i} \in \mathbb{R}$ and $z_{t+1,i} \in \mathbb{R}$ at consecutive time steps to n **unknown** landmarks
- ▶ Following the same derivation as in the 2-D case, we obtain:

$$[{}^t p_{t+1}]^T {}^t p_{t+1} + 2l_{t+1,i}^T [{}^t R_{t+1}]^T {}^t p_{t+1} = z_{t,i}^2 - z_{t+1,i}^2, \quad i = 1, \dots, n$$

- ▶ We have n equations with $2n + 6$ unknowns (6 for the relative pose and $2n$ for the unknown directions to the landmarks at $t + 1$), which is **not solvable**.

Summary: Range Measurements $z_i = \|l_i - p\|_2$

- ▶ **2-D Localization:** given $l_1, l_2 \in \mathbb{R}^2$ and $z_1, z_2 \in \mathbb{R}$
 1. Law of Cosines: $z_2^2 = z_1^2 + \|l_2 - l_1\|_2^2 - 2z_1\|l_2 - l_1\|_2 \cos \theta$
 2. Position: $p = l_2 + z_2 R(\pm\theta) \frac{l_1 - l_2}{\|l_1 - l_2\|_2}$
 3. Move with known p_Δ, ϕ_Δ (in frame t)
 4. Orientation: $(p_{t+1} - p_t) = R(\phi_t) p_\Delta$

- ▶ **3-D Localization:** given $l_1, l_2, l_3 \in \mathbb{R}^3$ and $z_1, z_2, z_3 \in \mathbb{R}$
 1. Intersection of 2 circles with centers O_{12}, O_{23} , radii r_{12}, r_{23} , normals n_{12}, n_{23} obtained via Law of Cosines and point O on intersecting line:

$$\begin{bmatrix} n_{12}^T \\ n_{23}^T \\ (n_{12} \times n_{23})^T \end{bmatrix} O = \begin{bmatrix} n_{12}^T O_{12} \\ n_{23}^T O_{23} \\ (n_{12} \times n_{23})^T O_{12} \end{bmatrix}$$

2. Position: $p = O_{12} + r_{12} R(n_{12}, \pm\theta) \frac{O - O_{12}}{\|O - O_{12}\|_2}$, where $\cos \theta = \frac{\|O - O_{12}\|_2}{r_{12}}$
3. Move twice with known $p_\Delta, R_\Delta, p_{\Delta,2}, R_{\Delta,2}$
4. Orientation: as long as $U := [p_\Delta, R_\Delta p_{\Delta,2}, p_\Delta \times R_\Delta p_{\Delta,2}]$ is nonsingular:

$$R_t = [(p_{t+1} - p_t), (p_{t+2} - p_{t+1}), (p_{t+1} - p_t) \times (p_{t+2} - p_{t+1})] U^{-1}$$

- ▶ **Odometry:** not solvable

2-D Localization from Bearing Measurements

- ▶ **Goal:** determine the robot position $p \in \mathbb{R}^2$ and orientation $\phi \in (-\pi, \pi]$
- ▶ **Given:** two landmark positions $l_1, l_2 \in \mathbb{R}^2$ (world frame) and **bearing** measurements (body frame):

$$z_i = \arctan\left(\frac{l_{i,y} - p_y}{l_{i,x} - p_x}\right) - \phi \in \mathbb{R}, \quad i = 1, 2$$

- ▶ The bearing constraints are equivalent to:

$$\frac{l_i - p}{\|l_i - p\|_2} = \begin{bmatrix} \cos(z_i + \phi) \\ \sin(z_i + \phi) \end{bmatrix} = R(z_i + \phi)e_1 \quad \Rightarrow \quad R^T(z_i)(l_i - p) = \|l_i - p\|_2 \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$

- ▶ To eliminate ϕ , the two constraints can be combined via:

$$\begin{aligned} 0 &= \|l_1 - p\|_2 \begin{bmatrix} \sin \phi & -\cos \phi \end{bmatrix} \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} \|l_2 - p\|_2 \\ &= \|l_1 - p\|_2 \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}^T R\left(\frac{\pi}{2}\right) \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} \|l_2 - p\|_2 \end{aligned}$$

2-D Localization from Bearing Measurements

- ▶ The previous equation is quadratic in p :

$$(l_1 - p)^T R(z_1) R\left(\frac{\pi}{2}\right) R^T(z_2)(l_2 - p) = 0$$

- ▶ Let $\eta := z_1 - z_2 + \pi/2$, so that:

$$p^T R(\eta)p - \left(l_1^T R(\eta) + l_2^T R^T(\eta)\right)p + l_1^T R(\eta)l_2 = 0$$

- ▶ Use the following to solve the quadratic equation:

- ▶ $p^T R(\eta)p = \cos \eta p^T p$

- ▶ $p^T p + 2b^T p + c = (p + b)^T (p + b) + c - b^T b$

- ▶ As long as $\cos \eta \neq 0$, i.e., **the robot and the two landmarks are not on the same line**:

$$(p - p_0)^T (p - p_0) = \left(p_0^T p_0 - \frac{1}{\cos \eta} l_1^T R(\eta) l_2\right) \quad p_0 := \frac{1}{2 \cos \eta} \left(R^T(\eta) l_1 + R(\eta) l_2\right)$$

- ▶ The position p lies on one of the two possible circles containing l_1 and l_2

2-D Localization from Bearing Measurements

- **Pose disambiguation:** obtain a third bearing measurement:

$$R^T(z_i)(l_i - p) = \|l_i - p\|_2 \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}, \quad i = 1, 2, 3$$

- Find β and γ such that $R^T(z_1) + \beta R^T(z_2) + \gamma R^T(z_3) = 0$. Then:

$$\underbrace{R^T(z_1)l_1 + \beta R^T(z_2)l_2 + \gamma R^T(z_3)l_3}_{:=u} - \underbrace{\left[R^T(z_1) + \beta R^T(z_2) + \gamma R^T(z_3) \right]}_0 p$$
$$= (\|l_1 - p\|_2 + \beta\|l_2 - p\|_2 + \gamma\|l_3 - p\|_2) \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$

- We can compute ϕ as $\begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} = \frac{u}{\|u\|_2}$ and recover p from:

$$R^T(z_i)(l_i - p) = \|l_i - p\|_2 \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}, \quad i = 1, 2, 3$$

3-D Localization from Bearing Measurements (P3P)

- ▶ **Goal:** determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- ▶ **Given:** three landmark positions $l_i \in \mathbb{R}^3$ (world frame) and pixel measurements $z_i \in \mathbb{R}^3$ (homogeneous coordinates, body frame) obtained from a (calibrated pinhole) camera:

$$z_i = \frac{1}{\lambda_i} R^T (l_i - p) \qquad \lambda_i = \|R^T (l_i - p)\|_2 = \text{unknown scale}$$

- ▶ If we determine λ_i , we can transform the P3P problem to 3-D localization from relative position measurements

Find the depths λ_i via Grunert's method (1841)

- ▶ Cosines of the angles among the bearing vectors z_1, z_2, z_3 :

$$\cos(\gamma_{ij}) = \frac{z_i^T z_j}{\|z_i\|_2 \|z_j\|_2} \Rightarrow \cos(\gamma_{ij}) = z_i^T z_j$$

- ▶ Let $\epsilon_{ij} := \|l_i - l_j\|_2$ be the lengths of the triangle formed in the world frame by l_1, l_2, l_3 . Applying the law of cosines gives:

$$\lambda_i^2 + \lambda_j^2 - 2\lambda_i \lambda_j \cos(\gamma_{ij}) = \epsilon_{ij}^2 \quad \text{for } \lambda_i := \|l_i - p\|_2$$

- ▶ Let $\lambda_2 = u\lambda_1$ and $\lambda_3 = v\lambda_1$ so that:

$$\lambda_1^2(u^2 + v^2 - 2uv \cos(\gamma_{23})) = \epsilon_{23}^2$$

$$\lambda_1^2(1 + v^2 - 2v \cos(\gamma_{13})) = \epsilon_{13}^2$$

$$\lambda_1^2(u^2 + 1 - 2u \cos(\gamma_{12})) = \epsilon_{12}^2$$

- ▶ Equivalently

$$\lambda_1^2 = \frac{\epsilon_{23}^2}{u^2 + v^2 - 2uv \cos(\gamma_{23})} = \frac{\epsilon_{13}^2}{1 + v^2 - 2v \cos(\gamma_{13})} = \frac{\epsilon_{12}^2}{u^2 + 1 - 2u \cos(\gamma_{12})}$$

Find the depths λ_j via Grunert's method (1841)

- ▶ Cross-multiplying the second fraction, with the first and the third:

$$u^2 + \frac{\epsilon_{13}^2 - \epsilon_{23}^2}{\epsilon_{13}^2} v^2 - 2uv \cos(\gamma_{23}) + \frac{2\epsilon_{23}^2}{\epsilon_{13}^2} v \cos(\gamma_{13}) - \frac{\epsilon_{23}^2}{\epsilon_{13}^2} = 0 \quad (1)$$

$$u^2 - \frac{\epsilon_{12}^2}{\epsilon_{13}^2} v^2 + 2v \frac{\epsilon_{12}^2}{\epsilon_{13}^2} \cos(\gamma_{13}) - 2u \cos(\gamma_{12}) + \frac{\epsilon_{13}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} = 0 \quad (2)$$

- ▶ Substituting (1) into (2):

$$u = \frac{\left(-1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right) v^2 - 2\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right) \cos(\gamma_{13}) v + 1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}}{2(\cos(\gamma_{12}) - v \cos(\gamma_{23}))} \quad (3)$$

- ▶ Substituting (3) into (1), we get a fourth-order polynomial in v :

$$A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0$$

Polynomial Coefficients

$$A_4 = \left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} - 1 \right)^2 - 4 \frac{\epsilon_{12}^2}{\epsilon_{13}^2} \cos^2(\gamma_{23})$$

$$A_3 = 4 \left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \left(1 - \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \right) \cos(\gamma_{13}) - \left(1 - \frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2} \right) \cos(\gamma_{23}) \cos(\gamma_{12}) + 2 \frac{\epsilon_{12}^2}{\epsilon_{13}^2} \cos^2(\gamma_{23}) \cos(\gamma_{13}) \right)$$

$$A_2 = 2 \left(\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \right)^2 - 1 + 2 \left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \right)^2 \cos^2(\gamma_{13}) + 2 \left(\frac{\epsilon_{13}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \right) \cos^2(\gamma_{23}) + 2 \left(\frac{\epsilon_{13}^2 - \epsilon_{23}^2}{\epsilon_{13}^2} \right) \cos^2(\gamma_{12}) - 4 \left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \right) \cos(\gamma_{23}) \cos(\gamma_{13}) \cos(\gamma_{12}) \right)$$

$$A_1 = 4 \left(- \left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \right) \left(1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \right) \cos(\gamma_{13}) - \left(1 - \frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2} \right) \cos(\gamma_{23}) \cos(\gamma_{12}) + 2 \frac{\epsilon_{23}^2}{\epsilon_{13}^2} \cos^2(\gamma_{12}) \cos(\gamma_{13}) \right)$$

$$A_0 = \left(1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} \right)^2 - \frac{4\epsilon_{23}^2}{\epsilon_{13}^2} \cos^2(\gamma_{12})$$

- ▶ We can obtain up to 4 real solutions for v , which we can substitute in (3) to obtain u .
- ▶ We can recover λ_1 from u and v via the fractions relationship
- ▶ Having $\lambda_1, \lambda_2 := u\lambda_1$, and $\lambda_3 := v\lambda_1$ we have converted the P3P problem into 3-D localization from relative position measurements

3-D Localization from Bearing Measurements (PnP)

- ▶ **Goal:** determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- ▶ **Given:** landmark positions $l_i \in \mathbb{R}^3$ (world frame) and pixel measurements $z_i \in \mathbb{R}^3$ (homogeneous coordinates) obtained from a (calibrated pinhole) camera for $i = 1, \dots, n$:

$$z_i = \frac{1}{\lambda_i} R^T (l_i - p) \quad \lambda_i = \|R^T (l_i - p)\|_2 = \text{unknown scale}$$

- ▶ The PnP can be formulated as a **constrained nonlinear least-squares** minimization:

$$\begin{aligned} \min_{\lambda_i, R, p} \quad & \sum_{i=1}^n \left\| z_i - \frac{1}{\lambda_i} R^T (l_i - p) \right\|_2^2 \\ \text{s.t.} \quad & R^T R = I, \quad \det R = 1, \quad \lambda_i = \|R^T (l_i - x)\|_2 \end{aligned}$$

Reformulation into a Polynomial System

- ▶ The constraints $\lambda_i z_i = R^T(l_i - p)$ can be re-written in matrix form as:

$$\underbrace{\begin{bmatrix} z_1 & & & -I \\ & \ddots & & \vdots \\ & & z_n & -I \end{bmatrix}}_A \underbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ -R^T p \end{bmatrix}}_x = \underbrace{\begin{bmatrix} R^T & & \\ & \ddots & \\ & & R^T \end{bmatrix}}_W \underbrace{\begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix}}_d$$

where A and d are known or measured, x are the unknowns we wish to eliminate, and W is a block diagonal matrix of the unknown rotation R

- ▶ We can express p and λ_i in terms of the other quantities as follows:

$$x = (A^T A)^{-1} A^T W d = \begin{bmatrix} U \\ V \end{bmatrix} W d$$

where $(A^T A)^{-1} A^T$ is partitioned so that the scale parameters are a function of U and the translation $-R^T p$ is a function of V .

Reformulation into a Polynomial System

$$x = (A^T A)^{-1} A^T Wd = \begin{bmatrix} U \\ V \end{bmatrix} Wd$$

- ▶ Exploiting the sparse structure of A , the matrices U and V can be computed in closed form
- ▶ Both λ_i and $-R^T p$ are linear functions of the unknown R^T :

$$\lambda_i = u_i^T Wd \quad -R^T p = VWd, \quad i = 1, \dots, n$$

where u_i^T is the i -th row of U .

- ▶ We can rewrite the constraints $\lambda_i z_i = R^T (l_i - p)$ as:

$$\underbrace{u_i^T Wd}_{\lambda_i} z_i = R^T l_i + \underbrace{VWd}_{-R^T p}$$

- ▶ We have reduced the number of unknowns from $6 + n$ to 3

Reformulation into a Polynomial System

► Cayley-Gibbs-Rodrigues Rotation Parameterization

$$R^T = \frac{\bar{C}}{1 + s^T s} \quad \bar{C} = ((1 - s^T s)I_3 + 2\hat{s} + 2ss^T)$$

- The CGR parameters automatically satisfy the rotation matrix constraints, i.e., $R^T R = I$ and $\det(R) = 1$ and allow us to formulate an unconstrained least-squares minimization in s .
- Since R^T appears linearly in the equations, we can cancel the denominator $1 + s^T s$. This leads to the following formulation of the PnP problem:

$$\min_s J(s) = \sum_{i=1}^n \left\| u_i^T \begin{bmatrix} \bar{C} & & \\ & \ddots & \\ & & \bar{C} \end{bmatrix} dz_i - \bar{C}l_i - V \begin{bmatrix} \bar{C} & & \\ & \ddots & \\ & & \bar{C} \end{bmatrix} d \right\|^2$$

which contains all monomials up to degree four, i.e., $\{1, s_1, s_2, s_3, s_1 s_2, s_1 s_3, s_2 s_3, \dots, s_1^4, s_2^4, s_3^4\}$.

Macaulay Matrix

- ▶ Since $J(s)$ is a fourth-order polynomial, the optimality conditions form a system of three third-order polynomials (derivatives with respect to s_1 , s_2 and s_3).
- ▶ We use a **Macaulay resultant matrix** (matrix of polynomial coefficients) to find the roots of the third-order polynomials and hence compute all critical points of $J(s)$
- ▶ Since the polynomial system is of constant degree (independent of n), it is only necessary to compute the Macaulay matrix symbolically once.
- ▶ Online, the elements of the Macaulay matrix are formed from the data (linear operation in n) and the roots are determined via an eigen-decomposition of the Schur complement (dense 27×27 matrix) of the top block of the Macaulay matrix (sparse 120×120 matrix)

2-D Odometry from Bearing Measurements

- ▶ **Goal:** determine the relative transformation ${}^t p_{t+1} \in \mathbb{R}^2$ and ${}^t \phi_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t + 1$ and t
- ▶ **Given:** bearing measurements $z_{t,i} \in \mathbb{R}^2$ (unit vector) and $z_{t+1,i} \in \mathbb{R}^2$ at consecutive time steps to n **unknown** landmarks
- ▶ The measurements are related as follows:

$$d_{t,i} b_{t,i} = {}^t p_{t+1} + R({}^t \phi_{t+1}) d_{t+1,i} b_{t+1,i}, \quad i = 1, \dots, n$$

where $d_{t,i}, d_{t+1,i}$ are the unknown distances to l_i .

- ▶ There are $2n$ equations and $2n + 3$ unknowns, which means that this problem is **not solvable**.

3-D Odometry from Bearing Measurements (5-Pt Alg)

- ▶ **Goal:** determine the relative transformation ${}_t p_{t+1} \in \mathbb{R}^3$ and ${}_t R_{t+1} \in SO(3)$ between two robot frames at time $t + 1$ and t
- ▶ **Given:** bearing measurements $z_{t,i} \in \mathbb{R}^3$ (unit vector) and $z_{t+1,i} \in \mathbb{R}^3$ at consecutive time steps to n **unknown** landmarks ($n \geq 5$)
- ▶ **Essential matrix:** $E := [{}_t \hat{p}_{t+1}] [{}_t R_{t+1}]$
- ▶ **Epipolar constraint:** $0 = z_{t,i}^T E z_{t+1,i}$, for $i = 1, \dots, n$
- ▶ **Idea:** recover the essential matrix between the two views first

3-D Odometry from Bearing Measurements (5-Pt Alg)

- ▶ The epipolar constraint is linear in the elements of E :

$$0 = \bar{z}_i^T \bar{E}$$

where $\bar{E} := [E_{11} \ E_{12} \ E_{13} \ E_{21} \ E_{22} \ E_{23} \ E_{31} \ E_{32} \ E_{33}]^T$ and $\bar{z}_i := \mathbf{vec}(z_{t+1,i} z_{t,i}^T) \in \mathbb{R}^9$ where $\mathbf{vec}(\cdot)$ is a row-wise vectorization.

- ▶ Stacking \bar{z}_i 's together, we obtain a 5×9 matrix $\bar{Z} := [\bar{z}_1 \ \dots \ \bar{z}_5]^T$
- ▶ The right nullspace of \bar{Z} has dimension 4 and the vectors that span the nullspace (obtained from SVD or QR decomposition) correspond to a 3×3 matrices N_i , $i = 1, \dots, 4$ so that

$$E = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + \alpha_4 N_4, \quad \alpha_i \in \mathbb{R}$$

- ▶ Since the measurements are scale-invariant, we can arbitrarily fix $\alpha_4 = 1$
- ▶ The essential matrix E always satisfies $0 = EE^T E - \frac{1}{2} \text{tr}(EE^T)E$ and substituting $E = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + N_4$, we obtain 9 cubic-in- α_i equations and can recover up to 10 solutions for E

3-D Odometry from Bearing Measurements (5-Pt Alg)

- ▶ Once E is recovered, ${}^t p_{t+1}$ and ${}^t R_{t+1}$ can be computed from the singular value decomposition (SVD) of E
- ▶ **Pose recovery from the Essential matrix:** There are exactly two relative poses corresponding to a non-zero essential matrix $E = U \mathbf{diag}(\sigma, \sigma, 0) V^T$:

$$({}^t \hat{p}_{t+1}, {}^t R_{t+1}) = \left(UR_z \left(\frac{\pi}{2} \right) \mathbf{diag}(\sigma, \sigma, 0) U^T, UR_z^T \left(\frac{\pi}{2} \right) V^T \right)$$

$$({}^t \hat{p}_{t+1}, {}^t R_{t+1}) = \left(UR_z \left(-\frac{\pi}{2} \right) \mathbf{diag}(\sigma, \sigma, 0) U^T, UR_z^T \left(-\frac{\pi}{2} \right) V^T \right)$$

- ▶ Only one of these will place the points in front of both cameras
- ▶ The ambiguity can be resolved by intersecting the measurements of a single point and verifying which solution places it on the positive optical z-axis of both cameras

Summary: Bearing Measurements $z_i = \frac{1}{\lambda_i} R^T(l_i - p)$

- ▶ **2-D Localization:** given $l_1, l_2 \in \mathbb{R}^2$ and $z_1, z_2 \in [-\pi, \pi]$

1. 2-D bearing: $\frac{1}{\lambda_i} R^T(\phi)(l_i - p) = R(z_i)e_1$
2. Eliminate ϕ :

$$0 = \lambda_1 e_1^T R(\phi) R\left(\frac{\pi}{2}\right) R(\phi) e_1 \lambda_2 = (l_1 - p)^T R(z_1) R\left(\frac{\pi}{2}\right) R^T(z_2) (l_2 - p)$$

3. The position p is on one of two circles containing l_1 and l_2 and we need a third bearing measurement z_3 to disambiguate it
4. Find β, γ such that $R^T(z_1) + \beta R^T(z_2) + \gamma R^T(z_3) = 0$ and combine

$$R^T(z_i)(l_i - p) = \lambda_i \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} \text{ to solve for } \phi$$

5. Orientation: $\begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} = \frac{u}{\|u\|_2}$ for $u := R^T(z_1)l_1 + \beta R^T(z_2)l_2 + \gamma R^T(z_3)l_3$

- ▶ **3-D Localization (P3P):** $l_i \in \mathbb{R}^3$, $z_i \in \mathbb{R}^3$ (homogeneous), $i = 1, 2, 3$

1. Convert P3P to relative position localization by determining the depths $\lambda_1, \lambda_2, \lambda_3$ via Grunert's method
2. Define the angles γ_{ij} among z_1, z_2, z_3 and apply the law of cosines:
 $\lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j \cos(\gamma_{ij}) = \|l_i - l_j\|_2^2$
3. Let $\lambda_2 = u\lambda_1$ and $\lambda_3 = v\lambda_1$ and combine the 3 equations to get a fourth order polynomial: $A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0$

Summary: Bearing Measurements $z_i = \frac{1}{\lambda_i} R^T (l_i - p)$

► 3-D Localization (PnP)

1. Rewrite $\lambda_i z_i = R^T (l_i - p)$ in matrix form and solve for $x := (\lambda_1, \dots, \lambda_n, -R^T p)^T$ in terms of R
2. The equations for λ_i and $-R^T p$ turn out to be linear in R so we are left with one equation with 3 unknowns (the 3 degrees of freedom of R)
3. Obtain a fourth order polynomial $J(s)$ in terms of the Cayley-Gibbs-Rodrigues rotation parameterization s
4. Compute a Macaulay matrix of the coefficients of $J(s)$ symbolically once. Online, determine the roots of $J(s)$ via an eigen-decomposition of the Schur complement of the Macaulay matrix.

► 2-D Odometry: not solvable

► 3-D Odometry: 5-point or 8-point algorithm:

1. Obtain E from the epipolar constraint: $0 = \mathbf{vec}(z_{t+1,i} z_{t,i}^T)^T \mathbf{vec}(E)$, $i = 1, \dots, 5$ and the Essential matrix property $0 = EE^T E - \frac{1}{2} \text{tr}(EE^T) E$
2. Recover two possible camera poses based on $SVD(E) = U \mathbf{diag}(\sigma, \sigma, 0) V^T$ and choose the one that places the measurements in front of both cameras