ECE276A: Sensing & Estimation in Robotics
Lecture 5: Rigid Body Motion

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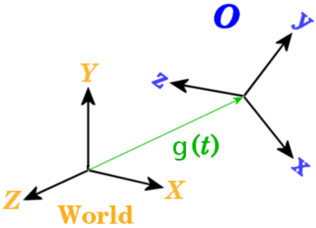
Rigid Body Motion

- Consider a moving object in a fixed **world reference frame** W.
- **Rigid object**: it is sufficient to specify the motion of one point $p(t) \in \mathbb{R}^3$ and 3 coordinate axes attached to that point (**body reference frame** B)



- A **rigid body motion** is a family of transformations $g_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that describes how the coordinates of points on the object change in time
- Rigid body motion preserves both distances (vector norms) and orientation (vector cross products)
- **Euclidean Group** $E(3)$: a set of maps $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that preserve the norm of any two vectors
- **Special Euclidean Group** $SE(3)$: a set of maps $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that preserve the norm and cross product of any two vectors

Special Euclidean Group

- A **group** is a set G with an associated operator $\odot$ (group law of G) that satisfies:
  - **Closure**: $a \odot b \in G, \forall a, b \in G$
  - **Identity element**: $\exists! e \in G$ (unique) such that $e \odot a = a \odot e = a$
  - **Inverse element**: for $a \in G$, $\exists b \in G$ such that $a \odot b = b \odot a = e$
  - **Associativity**: $(a \odot b) \odot c = a \odot (b \odot c), \forall a, b, c, \in G$

- $SE(3)$ is a group of maps $g : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve:
  1. Norm: $\|g(u) - g(v)\| = \|v - u\|, \forall u, v \in \mathbb{R}^3$
  2. Cross product: $g(u) \times g(v) = g(u \times v), \forall u, v \in \mathbb{R}^3$

- **Corollary**: $SE(3)$ elements also preserve:
  1. Angle: $u^T v = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) \Rightarrow u^T v = g(u)^T g(v), \forall u, v \in \mathbb{R}^3$
  2. Volume: $\forall u, v, w \in \mathbb{R}^3$, $g(u)^T(g(v) \times g(w)) = u^T(v \times w)$
     (volume of parallelepiped spanned by u, v, w)

Special Euclidean Group

- The configuration g_t of a moving rigid object at time t is determined by
  1. The position $p(t) \in \mathbb{R}^3$ of the body frame B relative to the world frame W
  2. The orientation $R(t) \in SO(3)$ of B relative to W

- The set of rigid body motions forms a group because:
  - We can combine several motions to generate a new one (**closure**)
  - We can execute a motion that leaves the object at the same state (**identity element**)
  - We can move rigid objects from one place to another and then reverse the action (**inverse element**)

- The space $\mathbb{R}^3$ of translations/positions is familiar

- How do we describe orientation?

Cross product

- The **cross product** of two vectors $\omega, \beta \in \mathbb{R}^3$ is also a vector in $\mathbb{R}^3$:

$$\omega \times \beta := \begin{bmatrix} \omega_2\beta_3 - \omega_3\beta_2 \\ \omega_3\beta_1 - \omega_1\beta_3 \\ \omega_1\beta_2 - \omega_2\beta_1 \end{bmatrix}$$

- For fixed ω, the cross product can be represented by a *linear* map $\omega \times \beta = \hat{\omega}\beta$ for $\hat{\omega} \in \mathbb{R}^{3\times 3}$

- The **hat map** $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms an $\mathbb{R}^3$ vector to a skew-symmetric matrix:

$$\hat{\omega} := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- The vector space $\mathbb{R}^3$ and the space of skew-symmetric 3×3 matrices $\mathfrak{so}(3)$ are isomorphic, i.e., there exists a one-to-one map (the hat map) that preserves their structure.

Hat Map Properties

- **Lemma**: A matrix $M \in \mathbb{R}^{3\times 3}$ is skew-symmetric iff $M = \hat{\omega}$ for some $\omega \in \mathbb{R}^3$.
- The inverse of the hat map is the **vee operator**, $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$, that extracts the components of the vector $\omega = \hat{\omega}^\vee$ from the matrix $\hat{\omega}$.
- For any $x, y \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$, the hat map satisfies:
  - $\hat{x}y = x \times y = -y \times x = -\hat{y}x$
  - $\hat{x}^2 = xx^T - x^T x\, I_{3\times 3}$
  - $\hat{x}^{2k+1} = (-x^T x)^k \hat{x}$
  - $-\frac{1}{2}\operatorname{tr}(\hat{x}\hat{y}) = x^T y$
  - $\hat{x}A + A^T\hat{x} = \left((\operatorname{tr}(A)I_{3\times 3} - A)x\right)^\wedge$
  - $\operatorname{tr}(\hat{x}A) = \frac{1}{2}\operatorname{tr}(\hat{x}(A - A^T)) = -x^T(A - A^T)^\vee$
  - $\widehat{Ax} = \det(A)A^{-T}\hat{x}A^{-1}$

3-D Orientation

- The orientation of a body frame B is determined by the coordinates of the three orthogonal vectors $r_1 = g(e_1)$, $r_2 = g(e_2)$, $r_3 = g(e_3)$ relative to the world frame W, i.e., by the 3×3 matrix:

$$R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

- Since r_1, r_2, r_3 form an orthonormal basis:
  - $r_i^T r_j = \delta_{ij}$
  - R is an **orthogonal matrix** $R^T R = RR^T = I$
  - R's inverse is its transpose: $R^{-1} = R^T$
  - $\det(R) = r_1^T(r_2 \times r_3) = 1$
  - R belongs to the **special orthogonal group**:

$$SO(3) := \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det(R) = 1\}$$

Special Orthogonal Lie Group $SO(n)$

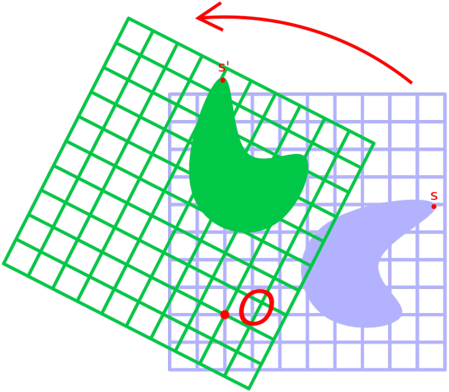
- $SO(n) := \{R \in \mathbb{R}^{n \times n} \mid R^T R = I, \det(R) = 1\}$
- Closed under multiplication: $R_1 R_2 \in SO(n)$
- Identity: $I \in SO(n)$
- Inverse: $R^{-1} = R^T \in SO(n)$
- Associative property: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
- **Manifold structure**: n^2 parameters with $n(n+1)/2$ constraints (due to $R^T R = I$) and hence $n(n-1)/2$ degrees of freedom
- Distances are preserved:
  $\|x - y\|_2^2 = \|R(x - y)\|_2^2 = (x - y)^T R^T R (x - y) \Rightarrow R^T R = I$
- No reflections allowed, i.e., a right-handed coordinate system is kept:
  $R(x \times y) = (Rx) \times (Ry) = \widehat{Rx} Ry = \det(R) R \hat{x} R^T R y \Rightarrow \det(R) = 1$

2-D Rotation

- A 2-D rotation of point $s \in \mathbb{R}^2$ through an angle θ can be described by a rotation matrix $R(\theta) \in SO(2)$:

$$s' = R(\theta)s := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} s$$

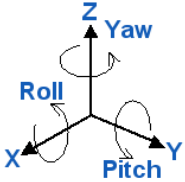
- $\theta > 0$: counterclockwise rotation



- There is a one-to-one correspondence between 2-D rotation matrices and unit-norm complex numbers:

$$e^{i\theta}(s_x + is_y) = (s_x \cos\theta - s_y \sin\theta) + i(s_x \sin\theta + s_y \cos\theta)$$

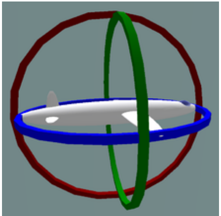
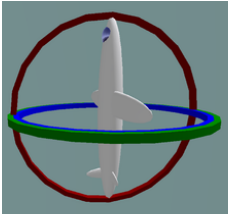
3-D Rotation (Euler Angles)



- **Euler Angles**: parameterize a 3-D rotation via three angles: **roll** (ϕ), **pitch** (θ), and **yaw** (ψ).
- The conventional XYZ extrinsic (fixed) angle representation, equivalent to the ZYX intrinsic (rotating) angle representation, is:

$$R = R_z(\psi)R_y(\theta)R_x(\phi)$$
$$= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

- Angle parameterizations have **singularities** (not one-to-one), which can result in **gimbal lock**, e.g., if $\theta = 90^\circ$, the roll and yaw are degenerate.

Quaternions

- **Quaternions**: $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ generalize complex numbers $\mathbb{C} = \mathbb{R} + \mathbb{R}i$

  $q = q_s + q_1 i + q_2 j + q_3 k = [q_s,\ \mathbf{q}_v] \qquad ij = -ji = k,\ i^2 = j^2 = k^2 = -1$

- Just as in 2-D, a 3-D rotation matrix R can be mapped onto **unit-norm** quaternions $\mathbb{S}^3 := \{q \in \mathbb{H} \mid q_s^2 + \mathbf{q}_v^T \mathbf{q}_v = 1\}$.

- To represent rotations, $\mathbb{S}^3$ embeds a 3-D space into a 4-D space (**no singularities**) and introduces a constraint.

- A rotation around a unit axis $\xi := \frac{\omega}{\|\omega\|} \in \mathbb{R}^3$ by angle $\theta := \|\omega\|$ can be represented by a unit quaternion:

$$q = \left[\cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)\xi\right]$$

- A rotation around a unit axis $\xi \in \mathbb{R}^3$ by angle θ can be recovered from a unit quaternion $q \in \mathbb{S}^3$:

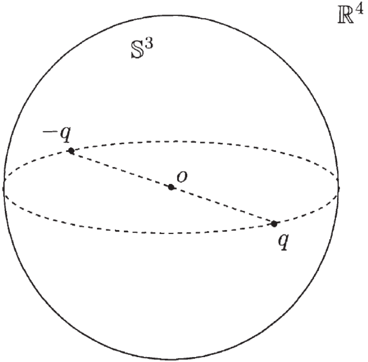
$$\theta = 2\arccos(q_s) \qquad \xi = \begin{cases} \frac{1}{\sin(\theta/2)}\mathbf{q}_v, & \text{if } \theta \neq 0 \\ 0, & \text{if } \theta = 0 \end{cases}$$

Quaternions

- A rotation matrix $R \in SO(3)$ can be obtained from a quaternion $q \in \mathbb{S}^3$:

  $$R(q) = E(q)G(q)^T \quad E(q) = [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \quad G(q) = [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v]$$

- $\mathbb{S}^3$ is a **double covering** of $SO(3)$ because two unit quaternions correspond to the same rotation matrix: $R(q) = R(-q)$:



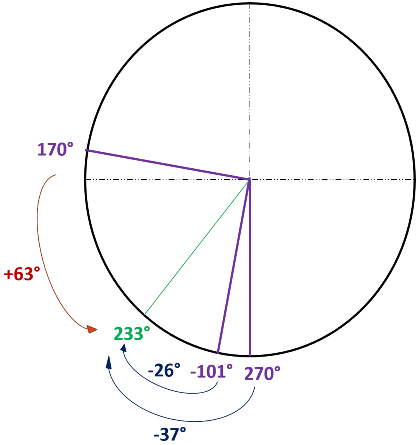
Quaternion Properties

Addition	$q + p = [q_s + p_s,\ \mathbf{q}_v + \mathbf{p}_v]$
Multiplication	$q \circ p = \left[q_s p_s - \mathbf{q}_v^T \mathbf{p}_v,\ q_s \mathbf{p}_v + p_s \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v\right]$
Conjugate	$\bar{q} = [q_s,\ -\mathbf{q}_v]$
Norm	$\lvert q \rvert := \sqrt{q_s^2 + \mathbf{q}_v^T \mathbf{q}_v} \quad \lvert q \circ p \rvert = \lvert q \rvert \lvert p \rvert$
Inverse	$q^{-1} = \frac{\bar{q}}{\lvert q \rvert^2}$
Rotation	$[0,\ \mathbf{x}'] = q \circ [0,\ \mathbf{x}] \circ q^{-1} = [0,\ R(q)\mathbf{x}]$
Rot. Velocity	$\dot{q} = \frac{1}{2}[0,\ \omega] \circ q = \frac{1}{2} E(q)^T \omega = \frac{1}{2} q \circ [0,\ \omega_B] = \frac{1}{2} G(q)^T \omega_B$
Exp	$\exp(q) := e^{q_s} \left[\cos \lVert \mathbf{q}_v \rVert,\ \frac{\mathbf{q}_v}{\lVert \mathbf{q}_v \rVert} \sin \lVert \mathbf{q}_v \rVert\right]$
Log	$\log(q) := \left[\log \lvert q \rvert,\ \frac{\mathbf{q}_v}{\lVert \mathbf{q}_v \rVert} \arccos \frac{q_s}{\lvert q \rvert}\right]$

- **Exp**: constructs $q \in \mathbb{S}^3$ from rotation vector $\omega \in \mathbb{R}^3$: $q = \exp\left(\left[0,\ \frac{\omega}{2}\right]\right)$
- **Log**: recovers a rotation vector $\omega \in \mathbb{R}^3$ from $q \in \mathbb{S}^3$: $[0,\ \omega] = 2\log(q)$

Angle Averaging

- What is the average of $\{170^\circ, -101^\circ, 270^\circ\}$?
- The average $\tilde{\beta}$ satisfies: $\frac{1}{3}\sum_{j=1}^{3} r(\tilde{\beta} - \beta_j) = 0$
- **Restrict angle** to $[-\pi, \pi)$: $r(\theta) := -\pi + \textbf{mod}(\theta + \pi, 2\pi)$



Quaternion Averaging

- Given a collection of quaternions $\{q_i\}_{i=1}^n$ with associated weights $\{\alpha_i\}_{i=1}^n$, we can obtain a weighted average quaternion $\tilde{q}$, which is not unique and depends on the initialization point as follows.

Algorithm 1 Quaternion average

Input: $\{q_i\}_{i=1}^n$, $\{\alpha_i\}_{i=1}^n$, initial guess $\tilde{q}_0$
for $t = 0, \ldots, T$ **do**
  $q_i^e = \left[q_{s,i}^e,\ \mathbf{q}_{v,i}^e\right] = \tilde{q}_t^{-1} \circ q_i$
  $[0,\ \mathbf{e}_{v,i}] = 2\log(q_i^e)$ ▷ Error rot. vector from quaternion
  $\mathbf{e}_{v,i} = (-\pi + \textbf{mod}(\|\mathbf{e}_{v,i}\| + \pi, 2\pi))\frac{\mathbf{e}_{v,i}}{\|\mathbf{e}_{v,i}\|}$ ▷ Restrict angles to $[-\pi, \pi)$
  $\mathbf{e}_v = \sum_{i=1}^n \alpha_i \mathbf{e}_{v,i}$
  $\tilde{q}_{t+1} = \tilde{q}_t \circ \exp\left(\left[0,\ \frac{\mathbf{e}_v}{2}\right]\right)$ ▷ Error rot. vector to quaternion
  if $\|\mathbf{e}_v\| < \epsilon$ **then return** $\tilde{q}_{t+1}$

Rotation Dynamics

- The trajectory $R(t)$ of a continuous rotation motion should satisfy:

$$R(t)R^T(t) = I \quad \Rightarrow \quad \dot{R}(t)R^T(t) + R(t)\dot{R}^T(t) = 0.$$

- The matrix $\dot{R}(t)R^T(t)$ is **skew-symmetric** and there must exist some vector-valued function $\omega(t) \in \mathbb{R}^3$ such that:

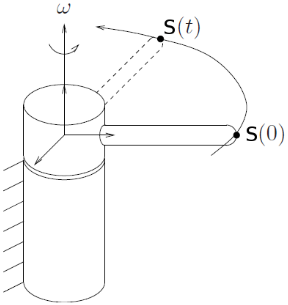
$$\dot{R}(t)R^T(t) = \hat{\omega}(t) \quad \Rightarrow \quad \boxed{\dot{R}(t) = \hat{\omega}(t)R(t)}$$

- A skew-symmetric matrix gives a first order approximation to a rotation matrix: $R(t + dt) \approx R(t) + \hat{\omega}(t)R(t)dt$.

- Locally, elements of $SO(3)$ depend only on three parameters $\omega \in \mathbb{R}^3$

- The space of skew-symmetric matrices $\mathfrak{so}(3) := \{\hat{\omega} \in \mathbb{R}^{3\times 3} \mid \omega \in \mathbb{R}\}$ is the **tangent space** at the identity of the rotation group $SO(3)$.

Rotation Dynamics



- **Rotation vector** $\omega \in \mathbb{R}^3$: every rotation is a rotation about an axis $\xi := \frac{\omega}{\|\omega\|_2}$ through an angle $\theta := \|\omega\|_2$.
- Consider a point s rotating about an axis ξ at constant unit velocity:
  $\dot{s}(t) = \xi \times s(t) = \hat{\xi} s(t), \quad s(0) = s_0$

  $\Rightarrow \; s(\theta) = e^{\hat{\xi}\theta} s_0 = R_{\xi,\theta} s_0$
- **Rotation matrix representation**: if $\hat{\omega}$ is constant:

$$\dot{R}(t) = \hat{\omega} R(t) \quad \Rightarrow \quad R(t) = \exp(\hat{\omega} t) R(t_0)$$

- If $\|\omega\| = 1$ and $R(t_0) = I$, then $R(t) = \exp(\hat{\omega} t)$ is simply a rotation around the axis $\omega \in \mathbb{R}^3$ by an angle of t radians.
- t can be absorbed into ω so that $R = \exp(\hat{\omega})$ for ω with arbitrary norm.
- **The matrix exponential defines a map from $\mathfrak{so}(3)$ to $SO(3)$.**

Special Orthogonal Lie Algebra $\mathfrak{so}(3)$

- Associated with every Lie group is its **Lie algebra** – a linear space of the same dimension, closed under a bi-linear alternating product called the **Lie bracket**: $[\hat{\omega}_1, \hat{\omega}_2] = \hat{\omega}_1\hat{\omega}_2 - \hat{\omega}_2\hat{\omega}_1$
- The Lie algebra of $SO(3)$ is the space of skew-symmetric matrices $\mathfrak{so}(3) := \{\hat{\omega} \in \mathbb{R}^{3\times 3} \mid \omega \in \mathbb{R}^3\}$
- **Generators of $\mathfrak{so}(3)$**: derivatives of rotations around each standard axis:

$$G_x = \frac{d}{d\phi}R_x(\phi)\Big|_{\phi=0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad G_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad G_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The elements $\hat{\omega} = \alpha_1 G_x + \alpha_2 G_y + \alpha_3 G_z \in \mathfrak{so}(3)$ are linear combinations of the generators and can be mapped to $SO(3)$ via **exponential map**:

$$R = \exp(\hat{\omega}) = I + \hat{\omega} + \frac{1}{2!}\hat{\omega}^2 + \frac{1}{3!}\hat{\omega}^3 + \dots$$

Infinitesimal Rotation

- Infinitesimal rotation $\delta\theta$ about the z-axis ξ_z:

$$s' = s + \delta s = s + \delta\theta \xi_z \times s = R_z(\delta\theta)s = (I + \delta\theta G_z)s$$

- Infinitesimal rotation $\delta\theta$ about an arbitrary axis ξ:

$$R_{\delta\theta} = (I + \delta\theta\xi_x G_x + \delta\theta\xi_y G_y + \delta\theta\xi_z G_z)$$

- Rotation θ about an arbitrary axis ξ:

$$R_\theta = \lim_{N\to\infty} \left(I + \frac{1}{N}\theta\xi_x G_x + \frac{1}{N}\theta\xi_y G_y + \frac{1}{N}\theta\xi_z G_z \right)^N = \sum_{n=0}^{\infty} \frac{1}{n!}(\theta\hat{\xi})^n = \exp(\theta\hat{\xi})$$

- Properties:
  - All eigenvalues of a skew-symmetric matrix $\hat{\omega} \in \mathfrak{so}(3)$ are imaginary
  - All eigenvalues of a rotation matrix $R \in SO(3)$ fit on the unit sphere and their product is 1 ($\det(R) = 1$)
  - The eigenvalues of $\hat{\omega}$ come in conjugate pairs – in odd dimensions, one eigenvalue of $\hat{\omega}$ is 0 and hence of R is 1, i.e., there is an invariant direction under the rotation.

Rodrigues Formula

- Any rotation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $\frac{\omega}{\|\omega\|_2}$ through an angle $\theta := \|\omega\|_2$ and can be obtained via:

$$R = \exp(\hat{\omega}) = I + \sum_{n=1}^{\infty} \frac{1}{n!}\hat{\omega}^n = I + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\hat{\omega}^{2n+1} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!}\hat{\omega}^{2n+2}$$
$$= I + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n+1)!}\right)\hat{\omega} + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n+2)!}\right)\hat{\omega}^2$$
$$= I + \left(\frac{\sin\theta}{\theta}\right)\hat{\omega} + \left(\frac{1-\cos\theta}{\theta^2}\right)\hat{\omega}^2$$

- The exponential map from $\mathfrak{so}(3)$ to $SO(3)$ is <u>not one-to-one</u> since any vector of the form $2k\pi\omega$ with integer k will result in the same R.

- The exponential map is also <u>not commutative</u>:

$$e^{\hat{\omega}_1} e^{\hat{\omega}_2} \neq e^{\hat{\omega}_2} e^{\hat{\omega}_1} \neq e^{\hat{\omega}_1 + \hat{\omega}_2}$$

unless $\hat{\omega}_1\hat{\omega}_2 = \hat{\omega}_2\hat{\omega}_1$, i.e., the **Lie bracket** on $\mathfrak{so}(3)$, $[\hat{\omega}_1, \hat{\omega}_2] = 0$.

Logarithm Map

- For any $R \in SO(3)$, there exists a (not necessarily unique) $\omega \in \mathbb{R}^3$ such that $R = \exp(\hat{\omega})$.

- The **logarithm map** $\log : SO(3) \to \mathfrak{so}(3)$ is the inverse of the exponential map:

$$\hat{\omega} = \log(R) = \frac{\theta}{2 \sin \theta}(R - R^T)$$
$$\theta = \|\omega\| = \arccos\left(\frac{\mathrm{tr}(R) - 1}{2}\right)$$
$$\xi = \frac{\omega}{\|\omega\|} = \frac{1}{2 \sin(\|\omega\|)} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

- The log map has a singularity at $\theta = 0$ because there is an infinite choice of rotation axes or equivalently the exponential map is many-to-one.

Rigid Body Motion

- Let B be a body frame whose position and orientation with respect to the world frame W are $p \in \mathbb{R}^3$ and $R \in SO(3)$, respectively.
- The coordinates of a point $s_B \in \mathbb{R}^3$ in the body frame B can be converted to the world frame by first rotating the point and then translating it to the world frame: $s_W = R s_B + p$.
- **Homogeneous coordinates**: the rigid-body motion transformation is not linear but **affine**. It can be converted to linear by appending 1 to the coordinates of a point s:

$$\begin{bmatrix} s_W \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_B \\ 1 \end{bmatrix}$$

- **Rigid body motion** can be described by a matrix parameterization:

$$SE(3) := \left\{ g := \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \middle| \; R \in SO(3), p \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times 4}$$

Special Euclidean Group $SE(3)$

- Using homogeneous coordinates, it can be verified that $SE(3)$ satisfies all requirements of a group:
  - $g_1 g_2 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix} \in SE(3)$
  - $\begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} \in SE(3)$
  - $\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \in SE(3)$
  - $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ for all $g_1, g_2, g_3 \in SE(3)$

“Smart” Plus and Minus

- For $x := (p, \theta) \in SE(2)$ with position $p \in \mathbb{R}^2$, orientation $\theta \in [-\pi, \pi)$, and inverse $x^{-1} = (-R(\theta)^T p, -\theta)$, define:

$$x_t \oplus x_{t+1} = \begin{bmatrix} p_t + R(\theta_t) p_{t+1} \\ \theta_t + \theta_{t+1} \end{bmatrix} \quad x_{t+1} \ominus x_t = x_t^{-1} \oplus x_{t+1} = \begin{bmatrix} R(\theta_t)^T (p_{t+1} - p_t) \\ \theta_{t+1} - \theta_t \end{bmatrix}$$

- For $x := (p, q) \in SE(3)$ with position $p \in \mathbb{R}^3$, orientation $q \in \mathbb{S}^3$, and inverse $x^{-1} = (-R(q)^T p, q^{-1})$ define:

$$x_t \oplus x_{t+1} = \begin{bmatrix} p_t + R(q_t) p_{t+1} \\ q_t q_{t+1} \end{bmatrix} \quad x_{t+1} \ominus x_t = x_t^{-1} \oplus x_{t+1} = \begin{bmatrix} R(q_t)^T (p_{t+1} - p_t) \\ q_t^{-1} q_{t+1} \end{bmatrix}$$

- For $x := (p, R) \in SE(3)$ with position $p \in \mathbb{R}^3$, orientation $R \in SO(3)$, and inverse $x^{-1} = (-R^T p, R^T)$ define:

$$x_t \oplus x_{t+1} = \begin{bmatrix} p_t + R_t p_{t+1} \\ R_t R_{t+1} \end{bmatrix} \quad x_{t+1} \ominus x_t = x_t^{-1} \oplus x_{t+1} = \begin{bmatrix} R_t^T (p_{t+1} - p_t) \\ R_t^T R_{t+1} \end{bmatrix}$$

Special Euclidean Lie Algebra $\mathfrak{se}(3)$

- **Angular velocity**: $R(t)R^T(t) = I \quad \Rightarrow \quad \dot{R}(t)R^T(t) = \hat{\omega}(t) \in \mathfrak{so}(3)$
- **Twist**: Similarly for $g(t) \in SE(3)$ consider:

$$\dot{g}(t)g^{-1}(t) = \begin{bmatrix} \dot{R}(t)R^T(t) & \dot{p}(t) - \dot{R}(t)R^T(t)p(t) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}(t) & v(t) \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$$

where $\boxed{\hat{\omega}(t) := \dot{R}(t)R^T(t)}$ and $v(t) := \dot{p}(t) - \hat{\omega}(t)p(t)$ are the **world angular** and **linear** velocities of the point in the body that corresponds with the origin of the world frame.

- $\omega(t)$ is also equal to the **angular velocity of the body frame measured in the world frame**
- The **linear velocity** of a fixed body point s_B, measured in the world frame is:

$$\boxed{\dot{s}_W(t) = \hat{\omega}(t)R(t)s_B + \dot{p}(t) = \hat{\omega}(t)s_W(t) + v(t)}$$

Special Euclidean Lie Algebra $\mathfrak{se}(3)$

- The set of all twists is the **Lie algebra** (or tangent space) of the Lie group $SE(3)$:

$$\mathfrak{se}(3) := \left\{ \begin{bmatrix} \hat{\omega}(t) & v(t) \\ 0 & 0 \end{bmatrix} \middle| \hat{\omega} \in \mathfrak{so}(3), v \in \mathbb{R}^3 \right\}$$

- A tangent vector $\hat{\zeta} \in \mathfrak{se}(3)$ approximates $g(t) \in SE(3)$ locally:

$$g(t + dt) \approx (I + \hat{\zeta}(t)dt)g(t)$$

Canonical Exponential Coordinates for $SE(3)$

- For a constant twist $\hat{\zeta} \in \mathfrak{se}(3)$ and $g(t) \in SE(3)$:

$$\dot{g}(t) = \hat{\zeta} g(t), \quad g(0) = I \qquad \Rightarrow \qquad g(t) = \exp(\hat{\zeta} t)$$

- **Exponential map**: $\exp : \mathfrak{se}(3) \to SE(3)$ with $\zeta := (v, \omega) \in \mathbb{R}^6$:

$$\exp(\hat{\zeta}) = \begin{cases} \begin{bmatrix} e^{\hat{\omega}} & \frac{(I - e^{\hat{\omega}})\hat{\omega} v + \omega\omega^T v}{\|\omega\|} \\ 0 & 1 \end{bmatrix} & \text{if } \omega \neq 0 \\ \begin{bmatrix} I & v \\ 0 & 1 \end{bmatrix} & \text{if } \omega = 0 \end{cases}$$

- The exponential map from $\mathfrak{se}(3)$ to $SE(3)$ is <u>not one-to-one</u>

- Two rigid-body motions $g_1 = \exp(\hat{\zeta}_1)$ and $g_2 = \exp(\hat{\zeta}_2)$ commute, $g_1 g_2 = g_2 g_1$, iff $[\hat{\zeta}_1, \hat{\zeta}_2] = 0$.

- The **Lie bracket** of $\mathfrak{se}(3)$ is:

$$[\hat{\zeta}_1, \hat{\zeta}_2] = \hat{\zeta}_1 \hat{\zeta}_2 - \hat{\zeta}_2 \hat{\zeta}_1 = \begin{bmatrix} \widehat{\omega_1 \times \omega_2} & \omega_1 \times v_2 - \omega_2 \times v_1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$$

Logarithm Map

- **Logarithm map** $\log : SE(3) \to \mathfrak{se}(3)$: for any $g := (p, R) \in SE(3)$, there exists a (not necessarily unique) twist $\zeta := (v, \omega) \in \mathbb{R}^6$ such that $\hat{\zeta} = \log(g)$:

$$\zeta = \begin{cases} \omega = \log(R)^{\vee}, v = \|\omega\| \left((I - e^{\hat{\omega}})\hat{\omega} + \omega\omega^T\right)^{-1} p, & \text{if } R \neq I, \\ \omega = 0, v = p, & \text{if } R = I. \end{cases}$$

Velocity Transformations

- Consider a moving body frame B with pose $g(t) \in SE(3)$. The velocity of a point $s_B \in \mathbb{R}^3$ in the body frame with respect to the world frame W can be determined as follows:

$$s_W(t) = g(t)s_B$$
$$\dot{s}_W(t) = \dot{g}(t)s_B = \dot{g}(t)g(t)^{-1}s_W(t) = \hat{\zeta}(t)s_W(t) = \hat{\omega}(t)s_W(t) + v(t)$$

- $\hat{\zeta}(t)$ is the velocity of the body frame moving relative to the world frame, as viewed in the world frame.

- The **adjoint map** $ad_g : \mathfrak{se}(3) \to \mathfrak{se}(3)$ transforms velocities from one frame to another via the transformation g and is defined as $\hat{\zeta} \to g\hat{\zeta}g^{-1}$.

Summary

	Rotation $SO(3)$	**Rigid-body motion $SE(3)$**
Matrix representations	$R : \begin{cases} R^T R = I \\ \det(R) = 1 \end{cases}$	$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$
Transformation	$s_W = R s_B$	$s_W = R s_B + p$
Inverse	$R^{-1} = R^T$	$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$
Exponential	$R = \exp(\hat{\omega})$	$g = \exp(\hat{\zeta})$
Velocity	$\dot{R} = \hat{\omega} R = R \hat{\omega}_B$	$\dot{g} = \hat{\zeta} g = g \hat{\zeta}_B$
Velocity	$\dot{s}_W = \hat{\omega} s_W$	$\dot{s}_W = \hat{\omega} s_W + v$
Adjoint map	$\hat{\omega} \to R \hat{\omega} R^T$	$\hat{\zeta} \to g \hat{\zeta} g^{-1}$