ECE276A: Sensing & Estimation in Robotics
Lecture 7: Extended and Unscented Kalman Filtering

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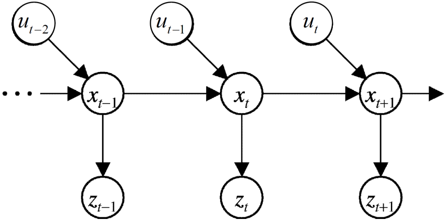
Bayes Filter



- **Motion model**:
  $x_{t+1} = a(x_t, u_t, w_t) \sim p_a(\cdot \mid x_t, u_t)$
- **Observation model**:
  $z_t = h(x_t, v_t) \sim p_h(\cdot \mid x_t)$
- **Filtering**: keeps track of
  $$p_{t|t}(x_t) := p(x_t \mid z_{0:t}, u_{0:t-1})$$
  $$p_{t+1|t}(x_{t+1}) := p(x_{t+1} \mid z_{0:t}, u_{0:t})$$
- **Bayes filter**:
  $$p_{t+1|t+1}(x_{t+1}) = \underbrace{\overbrace{\frac{1}{p(z_{t+1}|z_{0:t}, u_{0:t})}}^{\frac{1}{\eta_{t+1}}} p_h(z_{t+1} \mid x_{t+1}) \overbrace{\int p_a(x_{t+1} \mid x_t, u_t) p_{t|t}(x_t) dx_t}^{\textbf{Predict: } p_{t+1|t}(x_{t+1})}}_{\textbf{Update}}$$
- **Joint distribution**:
  $$p(x_{0:T}, z_{0:T}, u_{0:T-1}) = \underbrace{p_{0|0}(x_0)}_{\text{prior}} \prod_{t=0}^{T} \underbrace{p_h(z_t \mid x_t)}_{\text{observation model}} \prod_{t=0}^{T} \underbrace{p_a(x_t \mid x_{t-1}, u_{t-1})}_{\text{motion model}}$$

Nonlinear Kalman Filter

- A **nonlinear Kalman filter** is a Bayes filter for which:
  - The prior pdf $p_{0|0}$ is Gaussian
  - The motion model is ~~linear in the state~~ and affected by Gaussian noise
  - The observation model is ~~linear in the state~~ and affected by Gaussian noise
  - The process noise w_t and measurement noise v_t are independent of each other, of the state x_t and across time
  - The posterior pdf is **forced to be Gaussian via approximation**

- **Prior**: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$

- **Motion Model**: $x_{t+1} = a(x_t, u_t, w_t), \quad w_t \sim \mathcal{N}(0, W)$

- **Observation Model**: $z_t = h(x_t, v_t), \quad v_t \sim \mathcal{N}(0, V)$

- **Challenge**: the predicted and updated pdfs are not Gaussian and can no longer be evaluated in closed form

- **Moment matching**: we can force the predicted and updated pdfs to be Gaussian by evaluating their first and second moments and approximating them with Gaussians with the same moments

Nonlinear Kalman Filter Prediction

- Motion model with given prior:

$$x_{t+1} = a(x_t, u_t, w_t), \qquad w_t \sim \mathcal{N}(0, W), \quad x_t \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$$

- Force predicted pdf to be Gaussian: $\phi(\cdot; \mu_{t+1|t}, \Sigma_{t+1|t})$

$$\mu_{t+1|t} = \mathbb{E}\left[x_{t+1}\right] = \int\int a(x, u_t, w)\phi(x; \mu_{t|t}, \Sigma_{t|t})\phi(w; 0, W)dxdw$$

$$\Sigma_{t+1|t} = \int\int a(x, u_t, w)a(x, u_t, w)^T\phi(x; \mu_{t|t}, \Sigma_{t|t})\phi(w; 0, W)dxdw - \mu_{t+1|t}\mu_{t+1|t}^T$$

Nonlinear Kalman Filter Update

- Observation model with given prior:

$$z_{t+1} = h(x_{t+1}, v_{t+1}), \qquad v \sim \mathcal{N}(0, V), \quad x \sim \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$$

- The Gaussian distribution which approximates the joint distribution of x_{t+1} and z_{t+1} via moment matching is:

$$\begin{pmatrix} x_{t+1} \\ z_{t+1} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{t+1|t} \\ m_{t+1|t} \end{pmatrix}, \begin{bmatrix} \Sigma_{t+1|t} & C_{t+1|t} \\ C_{t+1|t}^T & S_{t+1|t} \end{bmatrix} \right)$$

$$m_{t+1|t} := \int \int h(x, v)\phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V) dx dv$$

$$S_{t+1|t} := \int \int (h(x, v) - m_{t+1|t})(h(x, v) - m_{t+1|t})^T \phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V) dx dv$$

$$C_{t+1|t} := \int \int (x - \mu_{t+1|t})(h(x, v) - m_{t+1|t})^T \phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V) dx dv$$

- The conditional Gaussian distribution of $x_{t+1} \mid z_{t+1}$ is then:

$$\begin{aligned} \mu_{t+1|t+1} &:= \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t}) \\ \Sigma_{t+1|t+1} &:= \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K_{t+1|t}^T \\ K_{t+1|t} &:= C_{t+1|t} S_{t+1|t}^{-1} \end{aligned}$$

Extended and Unscented Kalman Filters

- The **EKF** and **UKF** use different methods to approximate the five integrals required to implement the nonlinear Kalman filter

- The **EKF** uses a first-order Taylor series approximation to the motion and observation models:

$$a(x_t, u_t, w_t) \approx a(\mu_{t|t}, u_t, 0) + \left[\frac{da}{dx}(\mu_{t|t}, u_t, 0)\right](x_t - \mu_{t|t}) + \left[\frac{da}{dw}(\mu_{t|t}, u_t, 0)\right](w_t - 0)$$

$$h(x_{t+1}, v_{t+1}) \approx h(\mu_{t+1|t}, 0) + \left[\frac{dh}{dx}(\mu_{t+1|t}, 0)\right](x_{t+1} - \mu_{t+1|t}) + \left[\frac{dh}{dv}(\mu_{t+1|t}, 0)\right](v_{t+1} - 0)$$

- The **UKF** uses a set of **sigma points** that capture the mean and covariance of the prior Gaussian pdfs to approximate the integrals via a sum. This resembles Monte Carlo approximation but the sigma points are selected **deterministically**.

Extended Kalman Filter Prediction

- Let $A_t := \frac{da}{dx}(\mu_{t|t}, u_t, 0)$ and $Q_t := \frac{da}{dw}(\mu_{t|t}, u_t, 0)$ so that:

$$a(x_t, u_t, w_t) \approx a(\mu_{t|t}, u_t, 0) + A_t(x_t - \mu_{t|t}) + Q_t w_t$$

- Then, the predicted mean and cov can be computed in closed form:

$$\begin{aligned}
\mu_{t+1|t} &\approx \iint \left(a(\mu_{t|t}, u_t, 0) + A_t(x - \mu_{t|t}) + Q_t w\right) \phi(x; \mu_{t|t}, \Sigma_{t|t})\phi(w; 0, W)dxdw \\
&= a(\mu_{t|t}, u_t, 0) + A_t\left(\int x\phi(x; \mu_{t|t}, \Sigma_{t|t})dx - \mu_{t|t}\right) + Q_t \int w\phi(w; 0, W)dw \\
&= \boxed{a(\mu_{t|t}, u_t, 0)}
\end{aligned}$$

$$\begin{aligned}
\Sigma_{t+1|t} \approx& \iint \left(a(\mu_{t|t}, u_t, 0) + A_t(x - \mu_{t|t}) + Q_t w\right)\left(a(\mu_{t|t}, u_t, 0) + A_t(x - \mu_{t|t}) + Q_t w\right)^T \phi(x; \mu_{t|t}, \Sigma_{t|t})\phi(w; 0, W)dxdw \\
&- \mu_{t+1|t}\mu_{t+1|t}^T \\
=& a(\mu_{t|t}, u_t, 0)\left(\int (x - \mu_{t|t})^T \phi(x; \mu_{t|t}, \Sigma_{t|t})dx\right) A_t^T + A_t\left(\int (x - \mu_{t|t})\phi(x; \mu_{t|t}, \Sigma_{t|t})dx\right) a(\mu_{t|t}, u_t, 0)^T \\
&+ A_t\left(\int (x - \mu_{t|t})(x - \mu_{t|t})^T \phi(x; \mu_{t|t}, \Sigma_{t|t})dx\right) A_t^T + Q_t\left(\int ww^T\phi(w; 0, W)dw\right) Q_t^T \\
=& \boxed{A_t\Sigma_{t|t}A_t^T + Q_t W Q_t^T}
\end{aligned}$$

Extended Kalman Filter Update

- Let $H_{t+1} := \frac{dh}{dx}(\mu_{t+1|t}, 0)$ and $R_{t+1} := \frac{dh}{dv}(\mu_{t+1|t}, 0)$ so that:

$$h(x_{t+1}, v_{t+1}) \approx h(\mu_{t+1|t}, 0) + H_{t+1}(x_{t+1} - \mu_{t+1|t}) + R_{t+1} v_{t+1}$$

- The joint distribution of x_{t+1} and z_{t+1} can be computed in closed form:

$$m_{t+1|t} := \int\int h(x, v)\phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V)dxdv \approx \boxed{h(\mu_{t+1|t}, 0)}$$

$$S_{t+1|t} := \int\int (h(x, v) - m_{t+1|t})(h(x, v) - m_{t+1|t})^T \phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V)dxdv$$

$$\approx \boxed{H_{t+1}\Sigma_{t+1|t}H_{t+1}^T + R_{t+1}VR_{t+1}^T}$$

$$C_{t+1|t} := \int\int (x - \mu_{t+1|t})(h(x, v) - m_{t+1|t})^T \phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V)dxdv$$

$$\approx \boxed{\Sigma_{t+1|t}H_{t+1}^T}$$

- The conditional Gaussian distribution of $x_{t+1} \mid z_{t+1}$ is then:

$$\mu_{t+1|t+1} := \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t})$$

$$\Sigma_{t+1|t+1} := \Sigma_{t+1|t} - K_{t+1|t}S_{t+1|t}K_{t+1|t}^T$$

$$K_{t+1|t} := C_{t+1|t}S_{t+1|t}^{-1}$$

Extended Kalman Filter

Prior: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$

Motion model:

$$x_{t+1} = a(x_t, u_t, w_t), \quad w_t \sim \mathcal{N}(0, W)$$

$$A_t := \frac{da}{dx}(\mu_{t|t}, u_t, 0), \quad Q_t := \frac{da}{dw}(\mu_{t|t}, u_t, 0)$$

Obs. model:

$$z_t = h(x_t, v_t), \quad v_t \sim \mathcal{N}(0, V)$$

$$H_t := \frac{dh}{dx}(\mu_{t|t-1}, 0), \quad R_t := \frac{dh}{dv}(\mu_{t|t-1}, 0)$$

Prediction:

$$\mu_{t+1|t} = a(\mu_{t|t}, u_t, 0)$$

$$\Sigma_{t+1|t} = A_t \Sigma_{t|t} A_t^T + Q_t W Q_t^T$$

Update:

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - h(\mu_{t+1|t}, 0))$$

$$\Sigma_{t+1|t+1} = (I - K_{t+1|t} H_{t+1}) \Sigma_{t+1|t}$$

Kalman Gain:

$$K_{t+1|t} := \Sigma_{t+1|t} H_{t+1}^T \left(H_{t+1} \Sigma_{t+1|t} H_{t+1}^T + R_{t+1} V R_{t+1}^T \right)^{-1}$$

Unscented Transform

- The **unscented transform** (Julier et al., 1995, 2000) is a numerical method for approximating the joint distribution of a Gaussian random variable $y \in \mathbb{R}^d$ and a nonlinear transformation g of it:

$$s = g(y), \qquad \begin{pmatrix} y \\ s \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ m_U \end{pmatrix}, \begin{bmatrix} \Sigma & C_U \\ C_U^T & S_U \end{bmatrix} \right)$$

- Choose a set of $2d + 1$ **sigma points** using the i-th columns of the square root $\sqrt{\Sigma}$ of the covariance $\Sigma = \sqrt{\Sigma}\sqrt{\Sigma}^T$ (**note**: $\sqrt{\Sigma}$ is lower-triangular and can be obtained, e.g., via **Cholesky factorization**):

$$\mathcal{Y}^{(0)} = \mu, \qquad \mathcal{Y}^{(i)} = \mu \pm \sqrt{d + \lambda} \left[\sqrt{\Sigma} \right]_i, \quad i = 1, \ldots, d$$

$$\lambda := \alpha^2(d + k) - d, \qquad (\alpha, k) : \text{determine sigma points spread, e.g.,}$$

$$\alpha \in [0.0001, 1], k = 0$$

Unscented Transform

- The sigma points capture the shape of the original distribution of y well and can be propagated through the nonlinear function g to estimate the mean and covariance of the transformed variable

- The mean and covariance of $s = g(y)$ are estimated using the sigma points $\mathcal{Y}^{(0)} = \mu$ and $\mathcal{Y}^{(i)} = \mu \pm \sqrt{d+\lambda}\left[\sqrt{\Sigma}\right]_i$, $i = 1, \ldots, d$:

$$m_U = \sum_{i=0}^{2d} W_i^{(m)} g\left(\mathcal{Y}^{(i)}\right), \qquad W_0^{(m)} = \frac{\lambda}{d+\lambda}, \qquad W_i^{(m)} = \frac{1}{2(d+\lambda)}, \quad i = 1, \ldots, 2d$$

$$S_U = \sum_{i=0}^{2d} W_i^{(c)} \left(g\left(\mathcal{Y}^{(i)}\right) - m_U\right)\left(g\left(\mathcal{Y}^{(i)}\right) - m_U\right)^T, \qquad W_0^{(c)} = \frac{\lambda}{d+\lambda} + (1 - \alpha^2 + \beta)$$

$$\text{e.g., } \alpha \in [0.0001, 1],\ \beta = 2$$

$$C_U = \sum_{i=0}^{2d} W_i^{(c)} \left(\mathcal{Y}^{(i)} - \mu\right)\left(g\left(\mathcal{Y}^{(i)}\right) - m_U\right)^T, \qquad W_i^{(c)} = \frac{1}{2(d+\lambda)}, \quad i = 1, \ldots, 2d$$

Unscented Kalman Filter Prediction

- **Prior**: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$
- **Motion Model**: $x_{t+1} = a(x_t, u_t, w_t), \quad w_t \sim \mathcal{N}(0, W)$

$$\begin{pmatrix} \mathcal{X}_{t|t}^{(0)} \\ \mathcal{W}^{(0)} \end{pmatrix} = \begin{pmatrix} \mu_{t|t} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{X}_{t|t}^{(i)} \\ \mathcal{W}^{(i)} \end{pmatrix} = \begin{pmatrix} \mu_{t|t} \\ 0 \end{pmatrix} \pm \sqrt{(d+d_w)} \begin{bmatrix} \sqrt{\Sigma_{t|t}} & 0 \\ 0 & \sqrt{W} \end{bmatrix}_i$$

- **Prediction**: for $\alpha = 1$, $k = 0$, $\lambda = 0$, and $\beta = 2$

$$\mu_{t+1|t} = \sum_{i=0}^{2(d+d_w)} W_i^{(m)} a\left(\mathcal{X}_{t|t}^{(i)}, u_t, \mathcal{W}^{(i)}\right)$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2(d+d_w)} W_i^{(c)} \left(a\left(\mathcal{X}_{t|t}^{(i)}, u_t, \mathcal{W}^{(i)}\right) - \mu_{t+1|t}\right)\left(a\left(\mathcal{X}_{t|t}^{(i)}, u_t, \mathcal{W}^{(i)}\right) - \mu_{t+1|t}\right)^T$$

- **Weights**:

$$W_0^{(m)} = 0, \qquad W_i^{(m)} = \frac{1}{2(d+d_w)}, \quad i = 1, \ldots, 2(d+d_w)$$

$$W_0^{(c)} = 2, \qquad W_i^{(c)} = \frac{1}{2(d+d_w)}, \quad i = 1, \ldots, 2(d+d_w)$$

Unscented Kalman Filter Update

- **Prior**: $x_{t+1} \mid z_{0:t}, u_{0:t} \sim \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$
- **Observation Model**: $z_{t+1} = h(x_{t+1}, v_{t+1}), \quad v_{t+1} \sim \mathcal{N}(0, V)$

$$\begin{pmatrix} \mathcal{X}^{(0)}_{t+1|t} \\ \mathcal{V}^{(0)} \end{pmatrix} = \begin{pmatrix} \mu_{t+1|t} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{X}^{(i)}_{t+1|t} \\ \mathcal{V}^{(i)} \end{pmatrix} = \begin{pmatrix} \mu_{t+1|t} \\ 0 \end{pmatrix} \pm \sqrt{(d+d_v)} \begin{bmatrix} \sqrt{\Sigma_{t+1|t}} & 0 \\ 0 & \sqrt{V} \end{bmatrix}_i$$

- **Update**:
$$\begin{aligned} \mu_{t+1|t+1} &= \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t})) \\ \Sigma_{t+1|t+1} &= \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K^T_{t+1|t} \end{aligned}$$
- **Kalman Gain**: $K_{t+1|t} = C_{t+1|t} S^{-1}_{t+1|t}$

$$m_{t+1|t} = \sum_{i=0}^{2(d+d_v)} W_i^{(m)} h\left(\mathcal{X}^{(i)}_{t+1|t}, \mathcal{V}^{(i)}\right)$$

$$S_{t+1|t} = \sum_{i=0}^{2(d+d_v)} W_i^{(c)} \left(h\left(\mathcal{X}^{(i)}_{t+1|t}, \mathcal{V}^{(i)}\right) - m_{t+1|t}\right)\left(h\left(\mathcal{X}^{(i)}_{t+1|t}, \mathcal{V}^{(i)}\right) - m_{t+1|t}\right)^T$$

$$C_{t+1|t} = \sum_{i=0}^{2(d+d_v)} W_i^{(c)} \left(\mathcal{X}^{(i)}_{t+1|t} - \mu_{t+1|t}\right)\left(h\left(\mathcal{X}^{(i)}_{t+1|t}, \mathcal{V}^{(i)}\right) - m_{t+1|t}\right)^T$$

Unscented Kalman Filter (additive noise)

Prior $\quad x_{t+1} \mid z_{0:t}, u_{0:t} \sim \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$

Motion model $\quad x_{t+1} = a(x_t, u_t) + w_t, \quad w_t \sim \mathcal{N}(0, W)$

Obs. model $\quad z_{t+1} = h(x_{t+1}) + v_{t+1}, \quad v_{t+1} \sim \mathcal{N}(0, V)$

Predict

$$\mu_{t+1|t} = \sum_{i=0}^{2d} W_i^{(m)} a\left(\mathcal{X}_{t|t}^{(i)}, u_t\right), \quad \mathcal{X}_{t|t}^{(0)} = \mu_{t|t}, \quad \mathcal{X}_{t|t}^{(i)} = \mu_{t|t} \pm \sqrt{d+\lambda}\left[\sqrt{\Sigma_{t|t}}\right]_i$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2d} W_i^{(c)} \left(a\left(\mathcal{X}_{t|t}^{(i)}, u_t\right) - \mu_{t+1|t}\right)\left(a\left(\mathcal{X}_{t|t}^{(i)}, u_t\right) - \mu_{t+1|t}\right)^T + W$$

Update

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t})$$

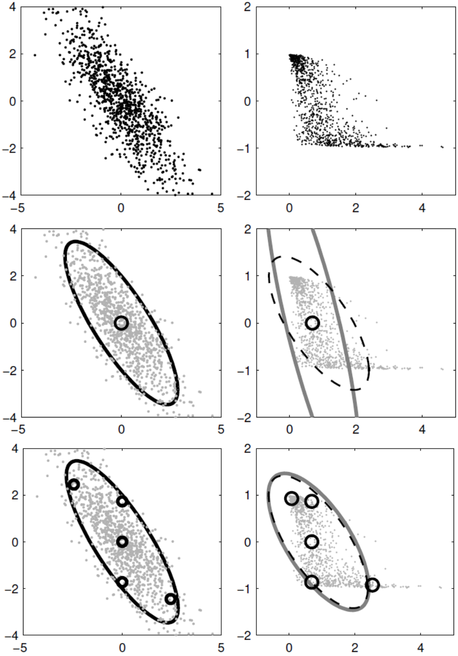
$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K_{t+1|t}^T$$

Kalman gain $\quad K_{t+1|t} = C_{t+1|t} S_{t+1|t}^{-1}$

$$m_{t+1|t} = \sum_{i=0}^{2d} W_i^{(m)} h\left(\mathcal{X}_{t+1|t}^{(i)}\right), \quad \mathcal{X}_{t+1|t}^{(0)} = \mu_{t+1|t}, \quad \mathcal{X}_{t+1|t}^{(i)} = \mu_{t+1|t} \pm \sqrt{d+\lambda}\left[\sqrt{\Sigma_{t+1|t}}\right]_i$$

$$S_{t+1|t} = \sum_{i=0}^{2d} W_i^{(c)} \left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)\left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)^T + V$$

$$C_{t+1|t} = \sum_{i=0}^{2d} W_i^{(c)} \left(\mathcal{X}_{t+1|t}^{(i)} - \mu_{t+1|t}\right)\left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)^T$$

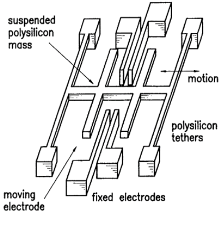


Applying a nonlinear transformation to the random variable on the left results in a new random variable (right)

EKF: Linearization-based approximation to the transformation formed by calculating the curvature at the mean (solid) and true covariance (dashed).

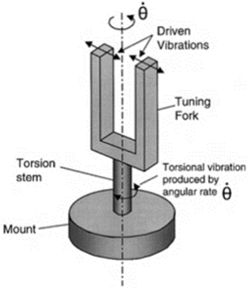
UKF: Unscented transform approximation to the transformation formed by propagating sigma points through the nonlinearity and estimating the covariance from them (solid) and true covariance (dashed).

MEMS Strapdown IMU



Surface Micromachined Accelerometer

- **MEMS**: micro-electro-mechanical system
- **IMU**: inertial measurement unit:
  - triaxial accelerometer (measures linear acceleration)
  - triaxial gyroscope (measures angular velocity)
  - **Strapdown**: the IMU and the object/vehicle inertial frames are joined together/identical
- **Accelerometer**:
  - A mass m on a spring with constant k. The spring displacement is prop. to the system acceleration: $F = ma = kd \quad \Rightarrow \quad d = \frac{ma}{k}$
  - VLSI Fabrication: the displacement of a metal plate with mass m is measured with respect to another plate using capacitance
  - Used for car airbags (if the acceleration goes above $2g$, the car is hitting something!)



- **Gyroscope**: uses Coriolis force to detect rotational velocity from the changing mechanical resonsance of a tuning fork

Project 2 Overview

- **State**: orientation $q_t \in \mathbb{S}^3$ of the body frame relative to the world frame (maps points from body to world frame)
- **Control**: rotational velocity $\omega_t \in \mathbb{R}^3$ obtained from gyroscope measurements in rad/sec during a time interval Δt
- **Noise**: $w_t \sim \mathcal{N}(0, W)$ with $W \in \mathbb{S}^{3\times 3}_{\succeq 0}$ (symmetric positive semidefinite)
- **Motion model**:
  $q_{t+1} = a(q_t, \omega_t, w_t) := q_t \circ \exp([0, \ \frac{w_t}{2}]) \circ \exp([0, \ \frac{\omega_t \Delta t}{2}])$
  (angular velocity in the body frame multiplies the rotation on the right!)
- **Obs. model**: global frame gravitational acceleration $g = e_3$ ("down"):

$$z_t = h(q_t) + v_t := \bar{q}_t \circ [0, \ g] \circ q_t + v_t, \qquad v_t \sim \mathcal{N}(0, V), \quad V \in \mathbb{S}^{3\times 3}_{\succeq 0}$$

- **Bias**: use stationary portion of the training data to estimate the bias
- **Scale**: set to datasheet value
- Correct both by comparing to the Vicon ground truth data