ECE276A: Sensing & Estimation in Robotics
Extended and Uscented Kalman Filtering Example

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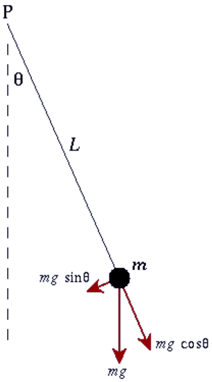
Noisy Pendulum Tracking

- Consider a simple pendulum consisting of a mass m hanging from a string of length L and fixed at a pivot point P
- The differential equation for the pendulum motion can be obtained using Newton's second law for rotational systems which relates the net external torque τ (position $\times$ force) to the product of the moment of inertia $I = mL^2$ and the angular acceleration $\ddot{\theta}(t)$:



$$\tau = -mgL\sin\theta(t) = mL^2\ddot{\theta}(t) \quad \Rightarrow \quad \ddot{\theta}(t) = -\frac{g}{L}\sin\theta(t) + \underbrace{w(t)}_{\text{noise}\sim\mathcal{N}(0,q)}$$

- The model can be converted into a state-space model with state $x(t) := (\theta(t), \omega(t))^T$, where $\omega(t) := \dot{\theta}(t)$ as follows:

$$\frac{d}{dt}\begin{pmatrix}\theta(t)\\ \omega(t)\end{pmatrix} = \begin{bmatrix}\omega(t)\\ -\frac{g}{L}\sin(\theta(t))\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} w(t)$$

Discrete-time Model

- **Motion model**: a simple discretization of the pendulum state-space model with sampling period τ leads to:

$$x_{t+1} = \begin{pmatrix} \theta_{t+1} \\ \omega_{t+1} \end{pmatrix} = \underbrace{\begin{bmatrix} \theta_t + \tau\omega_t \\ \omega_t - \tau\frac{g}{L}\sin\theta_t \end{bmatrix}}_{a(x_t)} + w_t, \quad w_t \sim \mathcal{N}\left(0, \underbrace{q\begin{bmatrix} \frac{\tau^3}{3} & \frac{\tau^2}{2} \\ \frac{\tau^2}{2} & \tau \end{bmatrix}}_{W}\right)$$

- **Observation model**: consider estimating the angle θ_t and the velocity ω_t of the pendulum using measurements of its deviation from rest position, i.e.,:

$$z_t = \underbrace{L\sin(\theta_t)}_{h(x_t)} + v_t, \qquad v_t \sim \mathcal{N}(0, V)$$

Extended Kalman Filter

- **Jacobians** of the motion and observation model for $x = (\theta, \omega)^T$:

$$A(x) := \begin{bmatrix} 1 & \tau \\ -\tau \frac{g}{L} \cos\theta & 1 \end{bmatrix} \qquad H(x) = \begin{bmatrix} \cos\theta & 0 \end{bmatrix}$$

- **Prior**: $x_t \mid z_{0:t} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$

- **Prediction**:

$$\mu_{t+1|t} = a(\mu_{t|t}) = \begin{bmatrix} \mu^{\theta}_{t|t} + \tau \mu^{\omega}_{t|t} \\ \mu^{\omega}_{t|t} - \tau \frac{g}{L} \sin \mu^{\theta}_{t|t} \end{bmatrix}$$

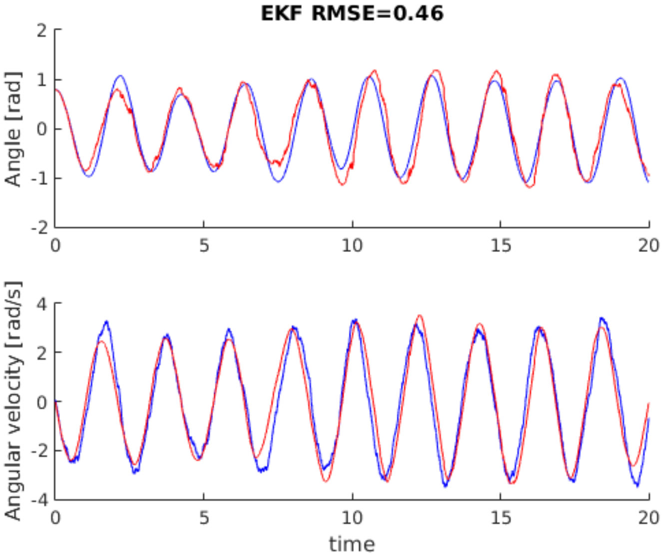
$$\Sigma_{t+1|t} = A(\mu_{t|t}) \Sigma_{t|t} A(\mu_{t|t})^T + W$$

Extended Kalman Filter

- **Innovation**: $\nu_{t+1|t} := z_{t+1} - L\sin(\mu^{\theta}_{t+1|t})$
- **Measurement/innovation covariance**: $S_{t+1|t} := H(\mu_{t+1|t})\Sigma_{t+1|t}H(\mu_{t+1|t})^T + V$
- **State-measurement cross-covariance**: $\Sigma_{t+1|t}H(\mu_{t+1|t})^T$
- **Kalman gain**: $K_{t+1|t} = \Sigma_{t+1|t}H(\mu_{t+1|t})^T S^{-1}_{t+1|t}$
- **Update**: $$\begin{aligned} \mu_{t+1|t+1} &= \mu_{t+1|t} + K_{t+1|t}\nu_{t+1|t} \\ \Sigma_{t+1|t+1} &= \Sigma_{t+1|t} - K_{t+1|t}H(\mu_{t+1|t})\Sigma_{t+1|t} \end{aligned}$$

EKF Performance

- $\tau = 0.001$, $q = 0.3$, $g = 9.81$, $L = 1$, $V = 0.64$
- Prediction at 1000 Hz, update at 20 Hz



Uscented Kalman Filter Prediction

- **Prior**: $x_t \mid z_{0:t} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$
- **Sigma points** with parameter $\lambda = \alpha^2(d+k) - d$ determining the sigma point spread (usual choice: $\alpha \in [0.001, 1]$, $k = 0$, $\beta \in \{0, 2\}$)

$$\mathcal{X}_{t|t}^{(0)} = \mu_{t|t}, \qquad \mathcal{X}_{t|t}^{(i)} = \mu_{t|t} \pm \sqrt{(d+\lambda)} \left[\sqrt{\Sigma_{t|t}}\right]_i, \quad i = 1, \ldots, 2d$$

$$W_m^{(0)} = \frac{\lambda}{d+\lambda}, \qquad W_m^{(i)} = \frac{1}{2(d+\lambda)}, \quad i = 1, \ldots, 2d$$

$$W_c^{(0)} = \frac{\lambda}{d+\lambda} + (1 - \alpha^2 + \beta), \qquad W_c^{(i)} = \frac{1}{2(d+\lambda)}, \quad i = 1, \ldots, 2d$$

- **Prediction**:

$$\mu_{t+1|t} = \sum_{i=0}^{2d} W_m^{(i)} a\left(\mathcal{X}_{t|t}^{(i)}\right)$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2d} W_c^{(i)} \left(a\left(\mathcal{X}_{t|t}^{(i)}\right) - \mu_{t+1|t}\right)\left(a\left(\mathcal{X}_{t|t}^{(i)}\right) - \mu_{t+1|t}\right)^T + W$$

Uscented Kalman Filter Update

- **Sigma points**:

$$\mathcal{X}^{(0)}_{t+1|t} = \mu_{t+1|t}, \qquad \mathcal{X}^{(i)}_{t+1|t} = \mu_{t+1|t} \pm \sqrt{(d+\lambda)}\left[\sqrt{\Sigma_{t+1|t}}\right]_i, \quad i = 1, \ldots, 2d$$

- **Expected measurement**: $m_{t+1|t} = \sum_{i=0}^{2d} W_m^{(i)} h\left(\mathcal{X}^{(i)}_{t+1|t}\right)$
- **Innovation**: $\nu_{t+1|t} := z_{t+1} - m_{t+1|t}$
- **Measurement/innovation covariance**:

$$S_{t+1|t} = \sum_{i=0}^{2d} W_c^{(i)} \left(h\left(\mathcal{X}^{(i)}_{t+1|t}\right) - m_{t+1|t}\right)\left(h\left(\mathcal{X}^{(i)}_{t+1|t}\right) - m_{t+1|t}\right)^T + V$$

- **State-measurement cross-covariance**:

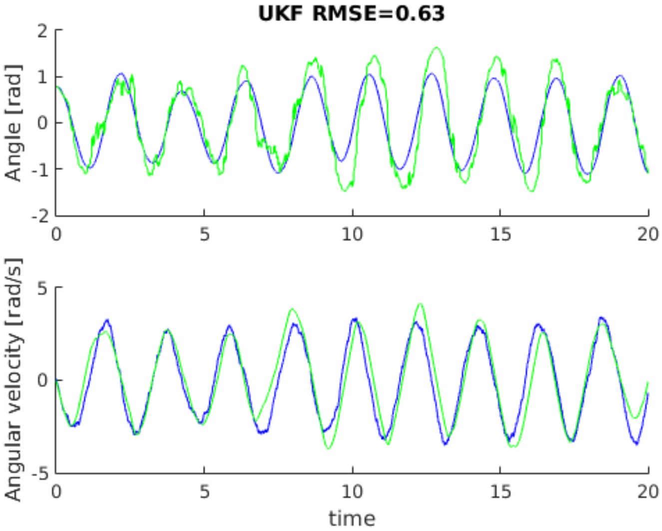
$$C_{t+1|t} = \sum_{i=0}^{2d} W_c^{(i)} \left(\mathcal{X}^{(i)}_{t+1|t} - \mu_{t+1|t}\right)\left(h\left(\mathcal{X}^{(i)}_{t+1|t}\right) - m_{t+1|t}\right)^T$$

- **Kalman gain**: $K_{t+1|t} = C_{t+1|t} S^{-1}_{t+1|t}$
- **Update**:

$$\begin{aligned} \mu_{t+1|t+1} &= \mu_{t+1|t} + K_{t+1|t}\nu_{t+1|t} \\ \Sigma_{t+1|t+1} &= \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K^T_{t+1|t} \end{aligned}$$

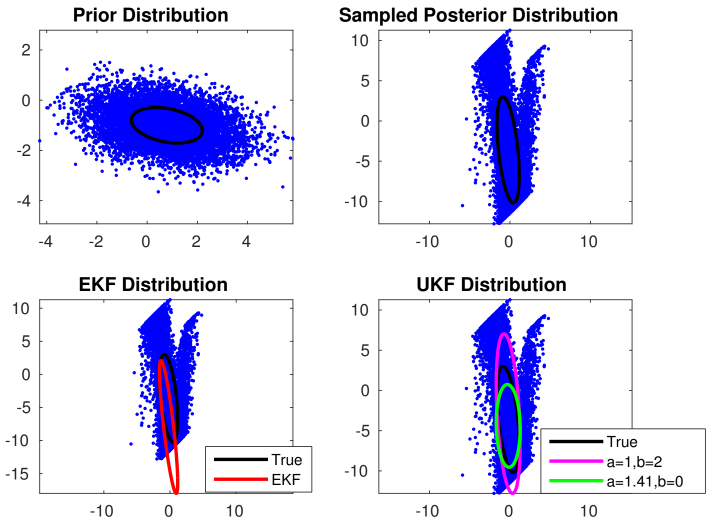
UKF Performance

- $\tau = 0.001$, $q = 0.3$, $g = 9.81$, $L = 1$, $V = 0.64$
- Prediction at 1000 Hz, update at 20 Hz



UKF vs EKF Predicted Covariance

- Prior: $\mathcal{N}\left(\begin{pmatrix}\frac{\pi}{4} \\ -1\end{pmatrix}, \begin{bmatrix}2 & -0.3 \\ -0.3 & 0.5\end{bmatrix}\right)$
- One prediction step with parameters $\tau = 1$, $g = 9.81$, $L = 1$



Project 2 Recap

- **State**: quaternion $q_t \in \mathbb{S}^3$ (body-to-world transformation)
- **Control**: angular velocity $\omega_t \in \mathbb{R}^3$ (body frame)
- **Observation**: linear acceleration $a_t \in \mathbb{R}^3$ (body frame)
- **Motion model**: comes from discretizing $\dot{q} = q \circ [0, \frac{1}{2}\omega]$:

$$q_{t+1} = q_t \circ \exp\left(\left[0, \frac{1}{2}w_t\right]\right) \circ \exp\left(\left[0, \frac{1}{2}\omega_t \tau\right]\right), \qquad w_t \sim \mathcal{N}(0, W)$$

- **Observation model**: comes from comparing the linear acceleration (body frame) to the gravity vector $e_3 \in \mathbb{R}^3$ (world frame) rotated to the body frame:

$$a_t = \bar{q}_t \circ [0, e_3] \circ q_t + v_t, \quad v_t \sim \mathcal{N}(0, V)$$

Project 2 UKF Prediction

- **Prior**: $x_t \mid a_{0:t}, \omega_{0:t-1} \sim \mathcal{N}(q_{t|t}, \Sigma_{t|t})$
- **Sigma Point Parameters**: $\alpha = 1$, $k = 0$, $\beta = 2$, $\lambda = 0$, $d = 3$
- **Sigma points**: zero-centered in the space of axis-angle vectors:

$$\mathcal{X}_{t|t}^{(0)} = 0, \qquad \mathcal{X}_{t|t}^{(i)} = \pm\sqrt{(d+\lambda)}\left[\sqrt{\Sigma_{t|t} + W}\right]_i, \; i = 1, \ldots, 2d$$

$$W_m^{(0)} = \frac{\lambda}{d+\lambda}, \qquad W_m^{(i)} = \frac{1}{2(d+\lambda)}, \quad i = 1, \ldots, 2d$$

$$W_c^{(0)} = \frac{\lambda}{d+\lambda} + (1 - \alpha^2 + \beta), \qquad W_c^{(i)} = \frac{1}{2(d+\lambda)}, \quad i = 1, \ldots, 2d$$

- Convert sigma points to quaternions: $q_{t|t}^{(i)} := q_{t|t} \circ \exp\left(\left[0, \frac{1}{2}\mathcal{X}_{t|t}^{(i)}\right]\right)$
- **Prediction**:

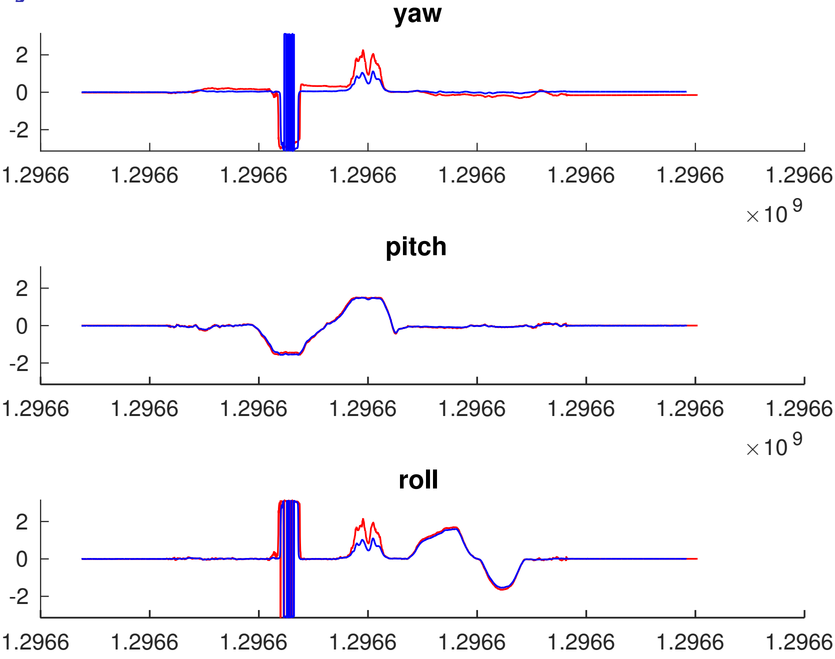
$$q_{t+1|t}, e_{t+1|t}^{(i)} = \textbf{quat\_avg}\left(\left\{q_{t|t}^{(i)} \circ \exp\left(\left[0, \frac{1}{2}\omega_t \tau\right]\right)\right\}, \left\{W_m^{(i)}\right\}\right)$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2d} W_c^{(i)} e_{t+1|t}^{(i)} \left[e_{t+1|t}^{(i)}\right]^T$$

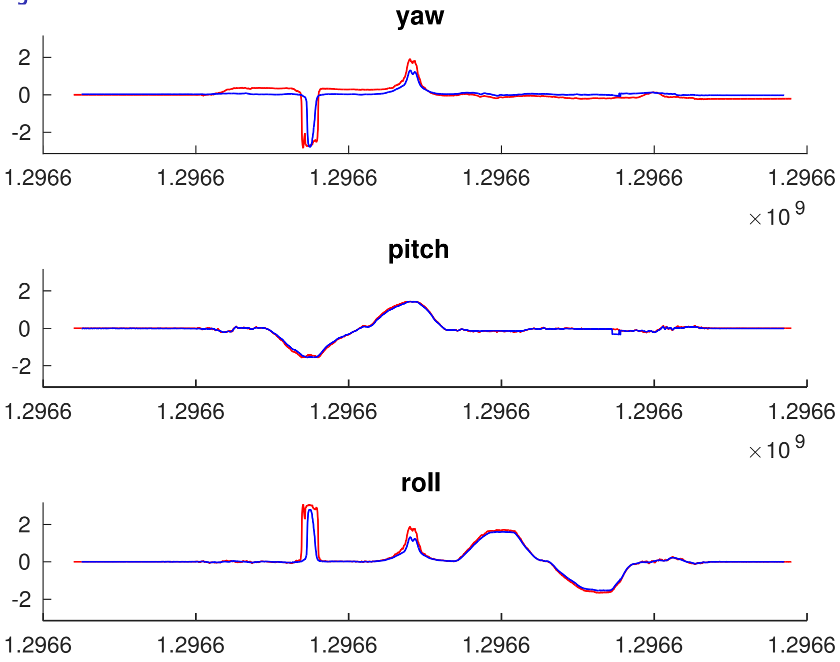
Project 2 UKF Update

- We should generate new sigma points from $\mathcal{N}(q_{t+1|t}, \Sigma_{t+1|t})$
- Instead of generating new sigma points, E. Kraft simplifies this step and uses the predicted sigma points:
  $q^{(i)}_{t+1|t} := q_{t|t} \circ \exp\left(\left[0, \frac{1}{2}\mathcal{X}^{(i)}_{t|t}\right]\right) \circ \exp\left(\left[0, \frac{1}{2}\omega_t \tau\right]\right)$
- **Expected measurement**: $m_{t+1|t} = \sum_{i=0}^{2d} W_m^{(i)} \left[\bar{q}^{(i)}_{t+1|t} \circ [0, e_3] \circ q^{(i)}_{t+1|t}\right]$
- **Innovation**: $\nu_{t+1|t} := a_{t+1} - m_{t+1|t}$
- **Measurement/innovation covariance**:
  $$S_{t+1|t} = \sum_{i=0}^{2d} W_c^{(i)} \left(\left[\bar{q}^{(i)}_{t+1|t} \circ [0, e_3] \circ q^{(i)}_{t+1|t}\right] - m_{t+1|t}\right)\left(\left[\bar{q}^{(i)}_{t+1|t} \circ [0, e_3] \circ q^{(i)}_{t+1|t}\right] - m_{t+1|t}\right)^T + V$$
- **State-measurement cross-covariance**:
  $C_{t+1|t} = \sum_{i=0}^{2d} W_c^{(i)} \left(e^{(i)}_{t+1|t}\right)\left(\left[\bar{q}^{(i)}_{t+1|t} \circ [0, e_3] \circ q^{(i)}_{t+1|t}\right] - m_{t+1|t}\right)^T$
- **Kalman gain**: $K_{t+1|t} = C_{t+1|t} S^{-1}_{t+1|t}$
- **Update**:
  $$q_{t+1|t+1} = q_{t+1|t} \circ \exp\left(\left[0, \frac{1}{2} K_{t+1|t} \nu_{t+1|t}\right]\right)$$
  $$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K^T_{t+1|t}$$

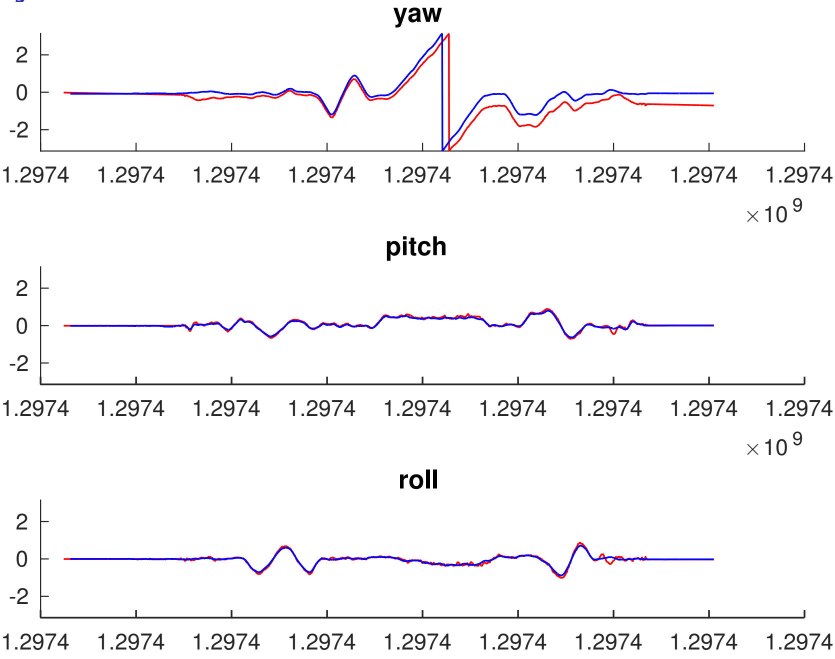
Project 2 UKF Performance on Dataset 1



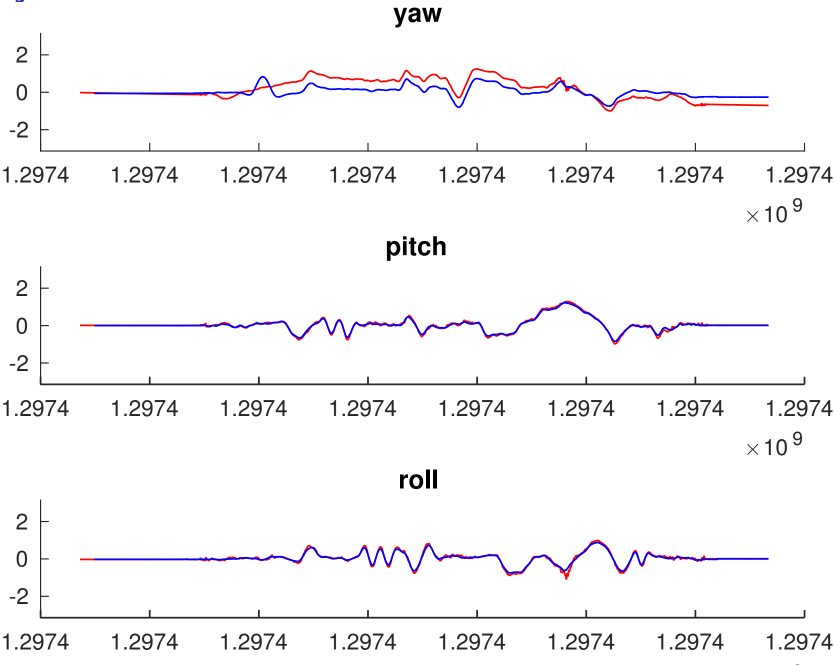
Project 2 UKF Performance on Dataset 2



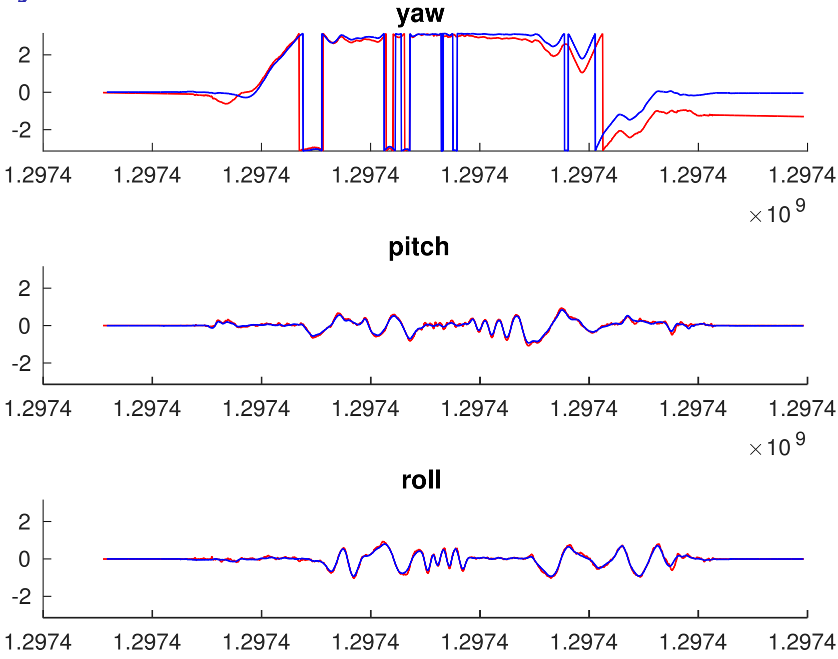
Project 2 UKF Performance on Dataset 3



Project 2 UKF Performance on Dataset 4



Project 2 UKF Performance on Dataset 5



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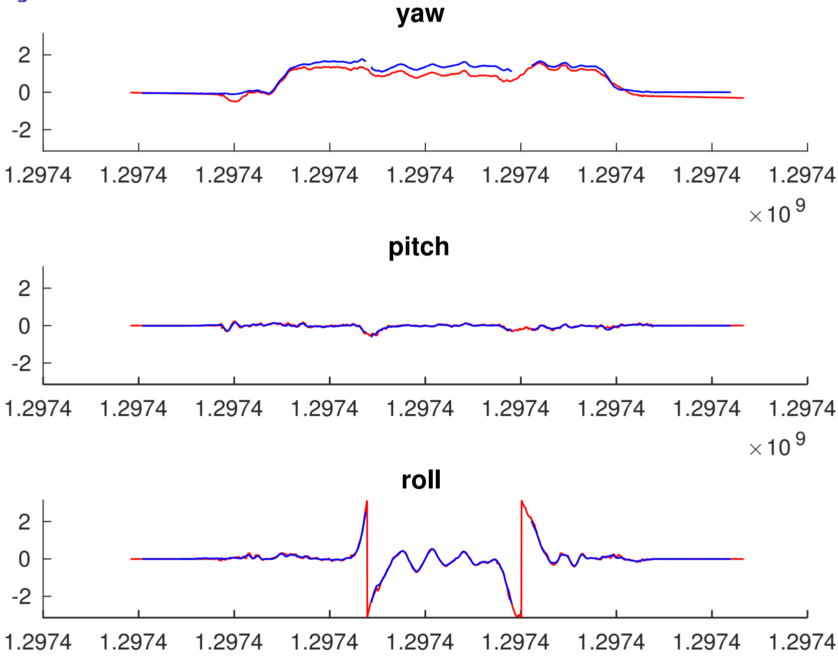

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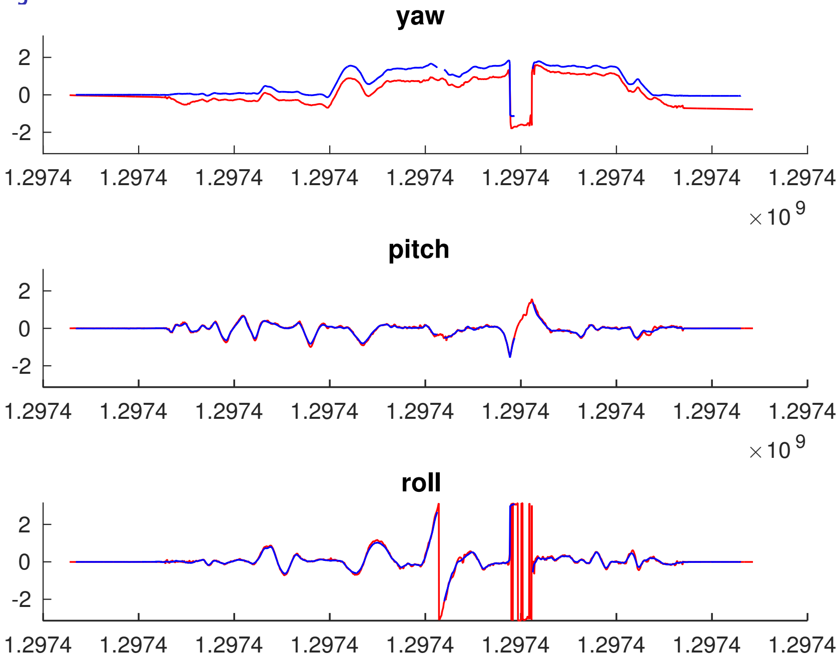
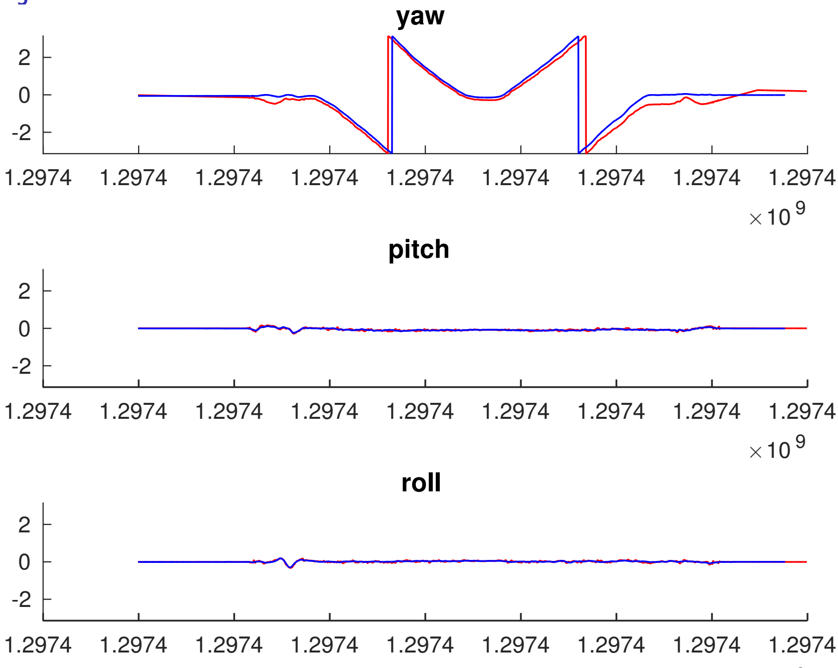


Project 2 UKF Performance on Dataset 8



Project 2 UKF Performance on Dataset 9