ECE276A: Sensing & Estimation in Robotics
Lecture 12: SE(3) Geometry and Kinematics

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Representation of Orientation

- **Rotation Matrix**: an element of the **Special Orthogonal Group**:

$$R \in SO(3) := \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; \underbrace{R^T R = I}_{\text{distances preserved}}, \underbrace{\det(R) = 1}_{\text{no reflection}} \right\}$$

- **Unit Quaternion**: $q = [q_s,\ \mathbf{q}_v] \in \left\{ q \in \mathbb{H} \mid q_s^2 + \mathbf{q}_v^T \mathbf{q}_v = 1 \right\}$:

$$R = E(q)G(q)^T \quad E(q) = [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \quad G(q) = [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v]$$

- **Euler Angles**: roll ϕ, pitch θ, roll ψ specifying a **rzyx** rotation:

$$R = R_z(\psi)R_y(\theta)R_x(\phi)$$

- **Rotation Vector**: $\boldsymbol{\theta} \in \mathbb{R}^3$ specifying a rotation about an axis $\frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$ through an angle $\|\boldsymbol{\theta}\|$:

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \hat{\boldsymbol{\theta}} + \frac{1}{2!}\hat{\boldsymbol{\theta}}^2 + \frac{1}{3!}\hat{\boldsymbol{\theta}}^3 + \ldots$$

Representation of Pose

- The pose of a rigid body is described by the **Special Euclidean Group**:

$$SE(3) := \left\{ T := \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \middle| \; R \in SO(3), p \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times 4}$$

- The pose T of a rigid body in the world frame specifies a transformation from the body frame to the world frame
- A point with body frame coordinates s_B, has world frame coordiantes:

$$\begin{bmatrix} s_W \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_B \\ 1 \end{bmatrix}$$

- A point with world frame coordinates s_W, has body frame coordiantes:

$$\begin{bmatrix} s_B \\ 1 \end{bmatrix} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_W \\ 1 \end{bmatrix}$$

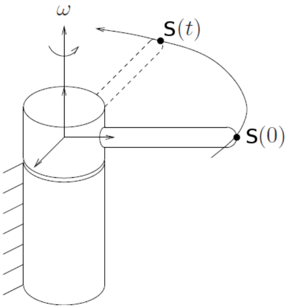
- The relative transformation from inertial frame $\{2\}$ with world-frame pose T_2 to an inertial frame $\{1\}$ with world-frame pose T_1 is:

$${}_1T_2 = T_1^{-1} T_2$$

Rotation Kinematics

- Suppose that a point s_0 is rotated about an axis $\eta := \frac{\omega}{\|\omega\|}$ through an angle $\theta := \|\omega\|$
- The rotation can be achieved by imagining that $s(t)$ rotates at a constant rate of 1 rad/s from time $t = 0$ to $t = \theta$:

$$\dot{s}(t) = \eta \times s(t) = \hat{\eta} s(t), \quad s(0) = s_0$$
$$\Rightarrow \ s(\theta) = e^{\hat{\eta}\theta} s_0 = R_{\eta,\theta} s_0$$



- If $\|\omega\| = 1$ and $R(t_0) = I$, then $R(t) = \exp(\hat{\omega} t)$ is simply a rotation around the axis $\omega \in \mathbb{R}^3$ by an angle of t radians.
- t can be absorbed into ω so that $R = \exp(\hat{\boldsymbol{\theta}})$ for $\boldsymbol{\theta}$ with arbitrary norm.
- **The matrix exponential defines a map from the space $\mathfrak{so}(3)$ of skew symmetric matrices to the space $SO(3)$ of rotation matrices.**

Rotation Kinematics

- The trajectory $R(t)$ of a continuous rotation motion should satisfy:

$$R(t)R^T(t) = I \quad \Rightarrow \quad \dot{R}(t)R^T(t) + R(t)\dot{R}^T(t) = 0.$$

- The matrix $\dot{R}(t)R^T(t)$ is **skew-symmetric** and there must exist some vector-valued function $\omega(t) \in \mathbb{R}^3$ such that:

$$\dot{R}(t)R^T(t) = \hat{\omega}(t) \quad \Rightarrow \quad \boxed{\dot{R}(t) = \hat{\omega}(t)R(t)}$$

- A skew-symmetric matrix gives a first order approximation to a rotation matrix: $R(t + dt) \approx R(t) + \hat{\omega}(t)R(t)dt$.

- Locally, elements of $SO(3)$ depend only on three parameters $\boldsymbol{\theta} \in \mathbb{R}^3$

- The space of skew-symmetric matrices $\mathfrak{so}(3) := \{\hat{\boldsymbol{\theta}} \in \mathbb{R}^{3\times3} \mid \boldsymbol{\theta} \in \mathbb{R}^3\}$ is the **tangent space** at the identity of the rotation group $SO(3)$.

Special Orthogonal and Euclidean Groups

- $SO(3)$ and $SE(3)$ are **matrix Lie groups**
- A **group** is a set of elements with an operation that combines any two elements to form a third one also in the set. A group satisfies four axioms: closure, associativity, identity, and invertibility
- A **Lie group** is a group that is also a differentiable manifold with the property that the group operations are smooth
- A **matrix** Lie group further specifies that the group elements are matrices, the combination operation is matrix multiplication, and the inversion operation is matrix inversion
- The **exponential map** relates a matrix Lie group to its **Lie algebra**

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \qquad \log(A) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (A - I)^n$$

Lie Algebra

- A **Lie algebra** is associated with every matrix Lie group.
- A Lie algebra is a vector space $\mathbb{V}$ over some field $\mathbb{F}$ with a binary operation, $[\cdot, \cdot]$, called a **Lie bracket**
- The vector space of a Lie algebra is the **tangent space** of the associated Lie group at the identity element of the group
- For all $X, Y, Z \in \mathbb{V}$ and $a, b \in \mathbb{F}$, the Lie bracket satisfies:

$$
\begin{aligned}
\text{closure}: \quad & [X, Y] \in \mathbb{V} \\
\text{bilinearity}: \quad & [aX + bY, Z] = a[X, Z] + b[Y, Z] \\
& [Z, aX + bY] = a[Z, X] + b[Z, Y] \\
\text{alternating}: \quad & [X, X] = 0 \\
\text{Jacobi identity}: \quad & [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0
\end{aligned}
$$

Special Orthogonal Lie Algebra $\mathfrak{so}(3)$

- The Lie algebra of $SO(3)$ is the space of skew-symmetric matrices

$$\mathfrak{so}(3) := \{\hat{\boldsymbol{\theta}} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{\theta} \in \mathbb{R}^3\}$$

- The **Lie bracket** of $\mathfrak{so}(3)$ is:

$$[\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2] = \hat{\boldsymbol{\theta}}_1\hat{\boldsymbol{\theta}}_2 - \hat{\boldsymbol{\theta}}_2\hat{\boldsymbol{\theta}}_1 = \left(\hat{\boldsymbol{\theta}}_1\boldsymbol{\theta}_2\right)^{\wedge} \in \mathfrak{so}(3)$$

- **Generators of $\mathfrak{so}(3)$**: derivatives of rotations around each standard axis:

$$G_x = \frac{d}{d\phi}R_x(\phi)\bigg|_{\phi=0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad G_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad G_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The elements $\hat{\boldsymbol{\theta}} = \alpha_1 G_x + \alpha_2 G_y + \alpha_3 G_z \in \mathfrak{so}(3)$ are linear combinations of generators and can be mapped to $SO(3)$ via the **exponential map**:

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \hat{\boldsymbol{\theta}} + \frac{1}{2!}\hat{\boldsymbol{\theta}}^2 + \frac{1}{3!}\hat{\boldsymbol{\theta}}^3 + \ldots \qquad \boldsymbol{\theta} = \log(R)^{\vee}$$

Exponential Map from $\mathfrak{so}(3)$ to $SO(3)$

- The exponential map is **surjective** but **not injective**, i.e., every element of $SO(3)$ can be generated from multiple elements of $\mathfrak{so}(3)$
- **Rodrigues Formula**: the surjective property of the exponential map can be understood by obtaining a closed-from expression:

$$
\begin{aligned}
R = \exp(\hat{\boldsymbol{\theta}}) &= I + \sum_{n=1}^{\infty} \frac{1}{n!}\hat{\boldsymbol{\theta}}^n = I + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\hat{\boldsymbol{\theta}}^{2n+1} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!}\hat{\boldsymbol{\theta}}^{2n+2} \\
&= I + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \|\boldsymbol{\theta}\|^{2n}}{(2n+1)!}\right)\hat{\boldsymbol{\theta}} + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \|\boldsymbol{\theta}\|^{2n}}{(2n+2)!}\right)\hat{\boldsymbol{\theta}}^2 \\
&= I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2
\end{aligned}
$$

- Any vector $\boldsymbol{\theta} + 2\pi k$ for integer k leads to the same $R \in SO(3)$
- The exponential map is also **not commutative**:

$$e^{\hat{\boldsymbol{\theta}}_1} e^{\hat{\boldsymbol{\theta}}_2} \neq e^{\hat{\boldsymbol{\theta}}_2} e^{\hat{\boldsymbol{\theta}}_1} \neq e^{\hat{\boldsymbol{\theta}}_1 + \hat{\boldsymbol{\theta}}_2}$$

unless $\hat{\boldsymbol{\theta}}_1\hat{\boldsymbol{\theta}}_2 = \hat{\boldsymbol{\theta}}_2\hat{\boldsymbol{\theta}}_1$, i.e., the **Lie bracket** on $\mathfrak{so}(3)$, $[\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2] = 0$.

Logarithm Map from $SO(3)$ to $\mathfrak{so}(3)$

- For any $R \in SO(3)$, there exists a (not unique) $\boldsymbol{\theta} \in \mathbb{R}^3$ such that $R = \exp(\hat{\boldsymbol{\theta}})$.

- The **logarithm map** $\log : SO(3) \to \mathfrak{so}(3)$ is the inverse of $\exp(\hat{\boldsymbol{\theta}})$:

$$\theta = \|\boldsymbol{\theta}\| = \arccos\left(\frac{\mathrm{tr}(R) - 1}{2}\right)$$

$$\eta = \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} = \frac{1}{2\sin(\|\boldsymbol{\theta}\|)}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

$$\hat{\boldsymbol{\theta}} = \log(R) = \frac{\|\boldsymbol{\theta}\|}{2\sin\|\boldsymbol{\theta}\|}(R - R^T)$$

  - If $R = I$, then $\theta = 0$ and η is undefined

  - If $\mathrm{tr}(R) = -1$, then $\theta = \pi$ and for any $i \in \{1, 2, 3\}$:

$$\eta = \frac{1}{\sqrt{2(1 + e_i^T R e_i)}}(I+R)e_i$$

- The log map has a singularity at $\theta = 0$ because there is an infinite choice of rotation axes or equivalently the exponential map is many-to-one.

- The matrix exponential "integrates" $\hat{\boldsymbol{\theta}} \in \mathfrak{se}(3)$ for one second; the matrix logarithm "differentiates" $R \in SO(3)$ to obtain $\hat{\boldsymbol{\theta}} \in \mathfrak{se}(3)$

Pose Kinematics

- **Angular velocity**: $R(t)R^T(t) = I \quad \Rightarrow \quad \dot{R}(t)R^T(t) = \hat{\omega}(t) \in \mathfrak{so}(3)$
- **Twist**: Similarly for $T(t) \in SE(3)$ consider:

$$\dot{T}(t)T^{-1}(t) = \begin{bmatrix} \dot{R}(t)R^T(t) & \dot{p}(t) - \dot{R}(t)R^T(t)p(t) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}(t) & v(t) \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$$

  where $\hat{\omega}(t) := \dot{R}(t)R^T(t)$ and $v(t) := \dot{p}(t) - \hat{\omega}(t)p(t)$ are the **world angular** and **linear** velocities of the point in the body that corresponds with the origin of the world frame.

- $\omega(t)$ is also equal to the **angular velocity of the body frame measured in the world frame**
- The **linear velocity** of a fixed body point s_B, measured in the world frame is:

$$\begin{aligned} s_W(t) &= R(t)s_B + p(t) \\ \dot{s}_W(t) &= \hat{\omega}(t)R(t)s_B + \dot{p}(t) = \hat{\omega}(t)s_W(t) + v(t) \end{aligned}$$

Special Euclidean Lie Algebra $\mathfrak{se}(3)$

- The Lie algebra of $SE(3)$ is the space of twist matrices:

$$\mathfrak{se}(3) := \left\{ \hat{\xi} := \begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \;\middle|\; \xi = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6 \right\}$$

- The **Lie bracket** of $\mathfrak{se}(3)$ is:

$$[\hat{\xi}_1, \hat{\xi}_2] = \hat{\xi}_1\hat{\xi}_2 - \hat{\xi}_2\hat{\xi}_1 = \left(\overset{\curlywedge}{\xi}_1 \xi_2\right)^{\wedge} \in \mathfrak{se}(3) \qquad \overset{\curlywedge}{\xi} := \begin{bmatrix} \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\rho}} \\ 0 & \hat{\boldsymbol{\theta}} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- The elements $T \in SE(3)$ are related to the elements $\hat{\xi} \in \mathfrak{se}(3)$ through the exponential map:

$$T = \exp(\hat{\xi}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\hat{\xi})^n \qquad \xi = \log(T)^{\vee}$$

Exponential Map from $\mathfrak{se}(3)$ to $SE(3)$

- The exponential map is **surjective** but **not injective**, i.e., every element of $SE(3)$ can be generated from multiple elements of $\mathfrak{se}(3)$
- **Rodrigues Formula**: obtained using $\hat{\xi}^4 + \|\boldsymbol{\theta}\|^2 \hat{\xi}^2 = 0$:

$$
\begin{aligned}
T = \exp(\hat{\xi}) &= \begin{bmatrix} \exp(\hat{\boldsymbol{\theta}}) & J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= I + \hat{\xi} + \left(\frac{1 - \cos \|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2} \right) \hat{\xi}^2 + \left(\frac{\|\boldsymbol{\theta}\| - \sin \|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3} \right) \hat{\xi}^3
\end{aligned}
$$

- The matrix $J_L(\boldsymbol{\theta})$ is the **left Jacobian** of $SO(3)$

$$
J_L(\boldsymbol{\theta}) := \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\hat{\boldsymbol{\theta}} \right)^n \qquad R = I + \hat{\boldsymbol{\theta}} J_L(\boldsymbol{\theta})
$$

Logarithm Map from $SE(3)$ to $\mathfrak{se}(3)$

- **Logarithm map** $\log : SE(3) \to \mathfrak{se}(3)$: for any $T \in SE(3)$, there exists a (not unique) $\xi \in \mathbb{R}^6$ such that:

$$\xi = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} = \log(T)^\vee := \begin{cases} \boldsymbol{\theta} = \log(R)^\vee, \boldsymbol{\rho} = J_L^{-1}(\boldsymbol{\theta})p, & \text{if } R \neq I, \\ \boldsymbol{\theta} = 0, \boldsymbol{\rho} = p, & \text{if } R = I. \end{cases}$$

$SO(3)$ Jacobians

- **Left Jacobian of** $SO(3)$:

$$J_L(\boldsymbol{\theta}) = I + \left(\frac{1 - \cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\|\boldsymbol{\theta}\| - \sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\right)\hat{\boldsymbol{\theta}}^2$$

$$J_L(\boldsymbol{\theta})^{-1} = I - \frac{1}{2}\hat{\boldsymbol{\theta}} + \left(\frac{1}{\|\boldsymbol{\theta}\|^2} - \frac{1 + \cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}}^2$$

$$J_L(\boldsymbol{\theta})J_L(\boldsymbol{\theta})^T = I + \left(1 - 2\frac{1 - \cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2 \succ 0$$

$$\left(J_L(\boldsymbol{\theta})J_L(\boldsymbol{\theta})^T\right)^{-1} = I + \left(1 - 2\frac{\|\boldsymbol{\theta}\|^2}{1 - \cos\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}}^2$$

- **Right Jacobian of** $SO(3)$:

$$J_L(\boldsymbol{\theta}) = RJ_R(\boldsymbol{\theta}) = J_R(-\boldsymbol{\theta})$$

$$J_R(\boldsymbol{\theta}) = I - \left(\frac{1 - \cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\|\boldsymbol{\theta}\| - \sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\right)\hat{\boldsymbol{\theta}}^2$$

$$J_R(\boldsymbol{\theta})^{-1} = I + \frac{1}{2}\hat{\boldsymbol{\theta}} + \left(\frac{1}{\|\boldsymbol{\theta}\|^2} - \frac{1 + \cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}}^2$$

Baker-Campbell-Hausdorff Formulas

- Rotations:

$$\log(\exp(\hat{\boldsymbol{\theta}}_1)\exp(\hat{\boldsymbol{\theta}}_2))^\vee \approx \begin{cases} J_L(\boldsymbol{\theta}_2)^{-1}\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 & \text{if } \boldsymbol{\theta}_1 \text{ is small} \\ \boldsymbol{\theta}_1 + J_R(\boldsymbol{\theta}_1)^{-1}\boldsymbol{\theta}_2 & \text{if } \boldsymbol{\theta}_2 \text{ is small} \end{cases}$$

$$\begin{aligned} \exp\left((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^\wedge\right) &\approx \exp(\hat{\boldsymbol{\theta}})\exp\left((J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta})^\wedge\right) \\ &\approx \exp\left((J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta})^\wedge\right)\exp(\hat{\boldsymbol{\theta}}) \end{aligned}$$

- Poses:

$$\log(\exp(\hat{\xi}_1)\exp(\hat{\xi}_2))^\vee \approx \begin{cases} \mathcal{J}_L(\xi_2)^{-1}\xi_1 + \xi_2 & \text{if } \xi_1 \text{ is small} \\ \xi_1 + \mathcal{J}_R(\xi_1)^{-1}\xi_2 & \text{if } \xi_2 \text{ is small} \end{cases}$$

$$\begin{aligned} \exp\left((\xi + \delta\xi)^\wedge\right) &\approx \exp(\hat{\xi})\exp\left((\mathcal{J}_R(\xi)\delta\xi)^\wedge\right) \\ &\approx \exp\left((\mathcal{J}_L(\xi)\delta\xi)^\wedge\right)\exp(\hat{\xi}) \end{aligned}$$

$SE(3)$ Jacobians

- **Left Jacobian of $SE(3)$:** $\mathcal{J}_L(\xi) = \begin{bmatrix} J_L(\boldsymbol{\theta}) & Q_L(\xi) \\ 0 & J_L(\boldsymbol{\theta}) \end{bmatrix}$
- **Right Jacobian of $SE(3)$:** $\mathcal{J}_R(\xi) = \begin{bmatrix} J_R(\boldsymbol{\theta}) & Q_R(\xi) \\ 0 & J_R(\boldsymbol{\theta}) \end{bmatrix}$

$$
\begin{aligned}
Q_L(\xi) = \frac{1}{2}\hat{\boldsymbol{\rho}} &+ \left(\frac{\|\boldsymbol{\theta}\| - \sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\right)\left(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\theta}}\right) \\
&+ \left(\frac{\|\boldsymbol{\theta}\|^2 + 2\cos\|\boldsymbol{\theta}\| - 2}{2\|\boldsymbol{\theta}\|^4}\right)\left(\hat{\boldsymbol{\theta}}^2\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\theta}}^2 - 3\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\theta}}\right) \\
&+ \left(\frac{2\|\boldsymbol{\theta}\| - 3\sin\|\boldsymbol{\theta}\| + \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^5}\right)\left(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\theta}}^2 + \hat{\boldsymbol{\theta}}^2\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\theta}}\right)
\end{aligned}
$$

- $Q_R(\xi) = Q_L(-\xi) = RQ_L(\xi) + (J_L(\boldsymbol{\theta})\boldsymbol{\rho})^\wedge RJ_L(\boldsymbol{\theta})$

Distances in $SO(3)$

- There are two ways to define the difference between two rotations:

$$\boldsymbol{\theta}_{12} = \log\left(R_1^T R_2\right)^\vee \qquad \boldsymbol{\theta}_{21} = \log\left(R_2 R_1^T\right)^\vee \qquad R_1, R_2 \in SO(3)$$

- Inner product on $\mathfrak{so}(3)$:

$$\langle \hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2 \rangle = \frac{1}{2}\,\mathrm{tr}\left(\hat{\boldsymbol{\theta}}_1 \hat{\boldsymbol{\theta}}_2^T\right) = \boldsymbol{\theta}_1^T \boldsymbol{\theta}_2$$

- The metric distance between two rotations is the magnitude of the rotation difference:

$$\sqrt{\langle \log\left(R_1^T R_2\right), \log\left(R_1^T R_2\right)\rangle} = \|\boldsymbol{\theta}_{12}\| \qquad \sqrt{\langle \log\left(R_2 R_1^T\right), \log\left(R_2 R_1^T\right)\rangle} = \|\boldsymbol{\theta}_{21}\|$$

Integration in $SO(3)$

- The distance between a rotation $R = \exp(\hat{\boldsymbol{\theta}})$ and a small perturbation $\exp((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^\wedge)$ can be approximated using the BCH formulas:

$$\log\left(\exp(\hat{\boldsymbol{\theta}})^T \exp((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^\wedge)\right)^\vee \approx \log\left(R^T R \exp\left((J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta})^\wedge\right)\right)^\vee = J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta}$$

$$\log\left(\exp((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^\wedge) \exp(\hat{\boldsymbol{\theta}})^T\right)^\vee \approx \log\left(\exp\left((J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta})^\wedge\right) R^T R\right)^\vee = J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta}$$

- Regardless of which distance metric we use, the infinitesimal volume element is the same:

$$\det(J_L(\boldsymbol{\theta})) = \det(J_R(\boldsymbol{\theta})) \qquad dR = |det(J(\boldsymbol{\theta}))| d\boldsymbol{\theta} = 2\left(\frac{1 - \cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right) d\boldsymbol{\theta}$$

- Integrating functions of rotations can then be carried out as follows:

$$\int_{SO(3)} f(R)dR = \int_{\|\boldsymbol{\theta}\|<\pi} f(\boldsymbol{\theta})|det(J(\boldsymbol{\theta}))| d\boldsymbol{\theta}$$

Integration in $SE(3)$

- The distance between a pose $T = \exp(\hat{\xi})$ and a small perturbation $\exp((\xi + \delta\xi)^{\wedge})$ can be approximated using the BCH formulas:

$$\log\left(\exp(\hat{\xi})^{-1}\exp((\xi+\delta\xi)^{\wedge})\right)^{\vee} \approx \mathcal{J}_R(\xi)\delta\xi$$
$$\log\left(\exp((\xi+\delta\xi)^{\wedge})\exp(\hat{\xi})^{-1}\right)^{\vee} \approx \mathcal{J}_L(\xi)\delta\xi$$

- $|\det(\mathcal{J}(\xi))| = |\det(J(\boldsymbol{\theta}))|^2 = 4\left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)^2$

- Integrating functions of poses can then be carried out as follows:

$$\int_{SE(3)} f(T)dT = \int_{\|\boldsymbol{\theta}\|<\pi} f(\xi)|det(\mathcal{J}(\xi))|d\xi$$

Derivatives in $SO(3)$

- Using the BCH formula with the right Jacobian of $SO(3)$:

$$
\begin{aligned}
\exp\left((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^\wedge\right) s &\approx \exp(\hat{\boldsymbol{\theta}}) \exp\left((J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta})^\wedge\right) s \\
&\approx \exp(\hat{\boldsymbol{\theta}}) \left(I + (J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta})^\wedge\right) s \\
&= \exp(\hat{\boldsymbol{\theta}}) s - \exp(\hat{\boldsymbol{\theta}}) \hat{s} J_R(\boldsymbol{\theta}) \delta\boldsymbol{\theta} \\
&= Rs - R\hat{s} J_R(\boldsymbol{\theta}) \delta\boldsymbol{\theta}
\end{aligned}
$$

- The derivative of a rotated point Rs with respect to the Lie algebra vector $\boldsymbol{\theta}$ representing the rotation is:

$$
\frac{d(Rs)}{d\boldsymbol{\theta}} = -R\hat{s}J_R(\boldsymbol{\theta}) = -R\hat{s}R^T J_L(\boldsymbol{\theta}) = -(Rs)^\wedge J_L(\boldsymbol{\theta})
$$

- Chain rule for a function $u(\mathbf{x})$ of $\mathbf{x} = Rs$:

$$
\frac{\partial u(\mathbf{x})}{\partial \boldsymbol{\theta}} = \frac{\partial u(\mathbf{x})}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} = -\frac{\partial u(\mathbf{x})}{\partial \mathbf{x}} R\hat{s}J_R(\boldsymbol{\theta})
$$

Gradient Descent in $SO(3)$

- An even simpler way to think about optimization over rotation matrices is to skip the derivatives altogether and think in terms of small perturbations $\psi := J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta}$ applied to an initial guess $R^{(k)}$:

$$
\begin{aligned}
u(R^{(k+1)}s) = u(R^{(k)}\exp(\hat{\psi})s) &\approx u\left(R^{(k)}(I+\hat{\psi})s\right) \\
&\approx u(R^{(k)}s) \underbrace{-\frac{du}{d\mathbf{x}}(R^{(k)}s)R^{(k)}\hat{s}}_{\delta^T}\psi = u(R^{(k)}s) + \delta^T\psi
\end{aligned}
$$

- **Gradient descent**: $\psi = -\alpha D\delta$ for a small step size $\alpha > 0$ and any positive-definite matrix $D \succ 0$ leads to:

$$
\begin{aligned}
\delta^{(k)} &= -\frac{du}{d\mathbf{x}}(R^{(k)}s)R^{(k)}\hat{s} \\
R^{(k+1)} &= R^{(k)}\exp\left(-\alpha D\hat{\delta}^{(k)}\right)
\end{aligned}
$$

Gauss-Newton Optimization in $SO(3)$

- Optimization problem:

$$\min_R J(R) := \frac{1}{2}\sum_m (u_m(Rv_m))^2$$

- Linearize $J(R)$ using $\beta_m^{(k)} = u_m(R^{(k)}v_m)$ and $\delta_m^{(k)} = -\frac{du_m}{d\mathbf{x}}(R^{(k)}v_m)R^{(k)}\hat{v}_m$

$$J(R) \approx \frac{1}{2}\sum_m (\delta_m^T\psi + \beta_m)^2$$

- The cost is quadratic in ψ and setting its gradient to zero leads to:

$$\left(\sum_m \delta_m^{(k)}\left(\delta_m^{(k)}\right)^T\right)\psi^{(k)} = -\sum_m \beta_m^{(k)}\delta_m^{(k)}$$

- Apply the optimal perturbation $\psi^{(k)}$ to the initial guess $R^{(k)}$ according to our perturbation scheme:

$$R^{(k+1)} = R^{(k)}\exp(\hat{\psi}^{(k)}) \in SO(3)$$

Rotation Kinematics

- Let $R \in SO(3)$ be the orientation of a rigid body rotating with angular velocity $\omega \in \mathbb{R}^3$ with respect to the world frame. Then, the kinematic equations of motion of R are:

$$\dot{R} = R\hat{\omega}_B = \hat{\omega}_W R$$

  where ω_B and $\omega_W = R\omega_B$ are the body-frame and world-frame coordinates of ω, respectively.

- The relationship between the body-frame and world-frame coordinates is:

$$\hat{\omega}_W = \widehat{R\omega_B} = R\hat{\omega}_B R^T$$

- Interestingly, ω_B does not depend on the choice of world frame and ω_W does not depend on the choice of body frame

Rotation Kinematics

- The kinematics in the Lie algebra lead to the pleasing result:

$$\omega_W = J_L(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \omega_B = J_R(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

- Note that $J_L^{-1}(\boldsymbol{\theta})$ does not exist at $\|\boldsymbol{\theta}\| = 2\pi m$ due to singularities of the 3×1 representation of rotation but we do not have to worry about constraints and can use numerical integration

- Assuming ω is constant over a short period τ. Then:

$$R(t+\tau) = \exp(\tau\hat{\omega}_W)R(t) = R(t)\exp(\tau\hat{\omega}_B)$$

Pose Kinematics

- Consider a moving body frame B with pose $T(t) \in SE(3)$. The velocity of a point $s_B \in \mathbb{R}^3$ in the body frame with respect to the world frame W can be determined as follows:

$$
\begin{aligned}
s_W(t) &= T(t)s_B \\
\dot{s}_W(t) &= \dot{T}(t)s_B = \dot{T}(t)T(t)^{-1}s_W(t) = \hat{\zeta}(t)s_W(t) = \hat{\omega}(t)s_W(t) + v(t)
\end{aligned}
$$

- $\hat{\zeta}(t)$ is the velocity of the body frame moving relative to the world frame, as viewed in the world frame.
- The world frame and body frame twists are related via: $\hat{\zeta}_W = T\hat{\zeta}_B T^{-1}$

Pose Kinematics

- A transformation matrix ${}_W T_B \in SE(3)$ can be related to the corresponding Lie algebra element $\xi \in \mathfrak{se}(3)$:

$$T = \begin{bmatrix} R & p \\ \mathbf{0}^T & 1 \end{bmatrix} = \exp(\hat{\xi}) = \begin{bmatrix} \exp(\hat{\boldsymbol{\theta}}) & J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- Pose kinematics for velocity $v \in \mathbb{R}^3$ and rotational velocity $\omega \in \mathbb{R}^3$:

$$\begin{aligned} \dot{p} &= \hat{\omega}_W p + v_W \\ \dot{R} &= \hat{\omega}_W R \end{aligned}$$

- The pose kinematics can be written in combined form for a **twist** ζ:

$$\dot{T} = \hat{\zeta}_W T = T\hat{\zeta}_B \qquad \zeta = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6$$

- **Hybrid kinematics** keeping the rotation in the Lie algebra:

$$\begin{bmatrix} \dot{p} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} I & -\hat{p} \\ 0 & J_L(\boldsymbol{\theta})^{-1} \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Pose Integration

- The kinematics in the Lie algebra are:

$$\zeta_W = \mathcal{J}_L(\xi)\dot{\xi} \qquad \zeta_B = \mathcal{J}_R(\xi)\dot{\xi}$$

- We can integrate the $\mathfrak{se}(3)$ kinematics without worrying about constraints.

- Assume ζ is constant over a short period τ. Then:

$$T(t+\tau) = \exp(\tau\hat{\zeta}_W)\,T(t) = T(t)\exp(\tau\hat{\zeta}_B)$$

- To construct the relative transformation $\Delta T := \exp(\tau\hat{\zeta}_W)$:
    - Let $\xi = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} = \tau \begin{bmatrix} v \\ \omega \end{bmatrix}$
    - Let $R = \exp(\hat{\omega})$ computed via Rodrigues formula
    - Let $p = J_L(\boldsymbol{\theta})\boldsymbol{\rho}$
    - Update: $T(t+\tau) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} T(t)$

Lie Group Probability and Statistics

- **Lie Group**: needed to transform points in the real world; free of singularities but have constraints
- **Lie Algebra**: can be treated as a vector space; free of constraints but have singularities
- The elements of matrix Lie groups do not satisfy some basic operations that we normally take for granted
- We need a different way to define random variables because matrix Lie groups are not closed under the usual addition operation:

$$x = \mu + \epsilon \qquad \epsilon \sim \mathcal{N}(0, \Sigma)$$

- **Idea**: define random variables over the Lie algebra, exploiting its vector space characteristics:

	$SO(3)$	$\mathfrak{so}(3)$
left	$R = \exp(\hat{\epsilon}_L)\bar{R}$	$\boldsymbol{\theta} \approx \mu + J_L^{-1}(\mu)\epsilon_L$
right	$R = \bar{R}\exp(\hat{\epsilon}_R)$	$\boldsymbol{\theta} \approx \mu + J_R^{-1}(\mu)\epsilon_R$

Lie Group Probability and Statistics

- $SO(3)$ Random Variable: $R = \exp(\hat{\epsilon})\bar{R}$, where $\bar{R}$ is a 'large' noise-free nominal rotation and $\epsilon \in \mathbb{R}^3$ is a 'small' noisy component

- Note that $\epsilon = \log\left(R\bar{R}^T\right)^\vee$

- Assuming $\epsilon \sim \mathcal{N}(0, \Sigma)$ with most mass on $\|\epsilon\| < \pi$ and using that $dR = |det(J_L(\epsilon))|d\epsilon$, we can obtain the pdf of R using the **Change of Density** formula:

$$p(R) = \frac{1}{\sqrt{(2\pi)^3 \det(\Sigma)}} \exp\left(-\frac{1}{2}\left(\log\left(R\bar{R}^T\right)^\vee\right)^T \Sigma^{-1} \log\left(R\bar{R}^T\right)^\vee\right) \frac{1}{|det(J_L(\epsilon))|}$$

- The choice of $\bar{R}$ and Σ as the mean and variance of R are justified because:

$$\int \log\left(R\bar{R}^T\right)^\vee p(R)dR = 0$$

$$\int \log\left(R\bar{R}^T\right)^\vee \left(\log\left(R\bar{R}^T\right)^\vee\right)^\vee p(R)dR = \mathbb{E}[\epsilon\epsilon^T] = \Sigma$$

Rotation of a Rotation Random Variable

- Let $Q \in SO(3)$ and $\boldsymbol{\theta} \in \mathbb{R}^3$. Then:

$$Q \exp(\hat{\boldsymbol{\theta}}) Q^T = \exp\left(Q\hat{\boldsymbol{\theta}}Q^T\right) = \exp\left((Q\boldsymbol{\theta})^\wedge\right)$$

- Let R be a random rotation with mean $\bar{R}$ and covariance Σ. Then, the random variable $Y = QR$ satisfies:

$$\begin{aligned} Y &= QR = Q\exp(\hat{\epsilon})\bar{R} = \exp\left((Q\epsilon)^\wedge\right) Q\bar{R} \\ \mathbb{E}[Y] &= Q\bar{R} \\ \mathbf{Var}[Y] &= \mathbf{Var}[Q\epsilon] = Q\Sigma Q^T \end{aligned}$$