ECE276A: Sensing & Estimation in Robotics
Lecture 14: Extended and Unscented Kalman Filtering

Instructor:

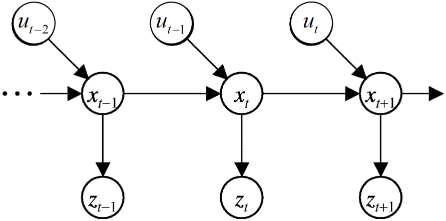
Nikolay Atanasov: natanasov@ucsd.edu

Teaching Assistants:

Qiaojun Feng: qif007@eng.ucsd.edu
Tianyu Wang: tiw161@eng.ucsd.edu
Ibrahim Akbar: iakbar@eng.ucsd.edu
You-Yi Jau: yjau@eng.ucsd.edu
Harshini Rajachander: hrajacha@eng.ucsd.edu

UC San Diego
JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

Bayes Filter



- **Motion model**: $x_{t+1} = f(x_t, u_t, w_t) \sim p_f(\cdot \mid x_t, u_t)$
- **Observation model**: $z_t = h(x_t, v_t) \sim p_h(\cdot \mid x_t)$
- **Filtering**: keeps track of

$$p_{t|t}(x_t) := p(x_t \mid z_{0:t}, u_{0:t-1})$$
$$p_{t+1|t}(x_{t+1}) := p(x_{t+1} \mid z_{0:t}, u_{0:t})$$

- **Bayes filter**:

$$p_{t+1|t+1}(x_{t+1}) = \underbrace{\overbrace{\frac{1}{p(z_{t+1}|z_{0:t}, u_{0:t})}}^{\frac{1}{\eta_{t+1}}} p_h(z_{t+1} \mid x_{t+1}) \overbrace{\int p_f(x_{t+1} \mid x_t, u_t) p_{t|t}(x_t) dx_t}^{\textbf{Predict: } p_{t+1|t}(x_{t+1})}}_{\textbf{Update}}$$

- **Joint distribution**:

$$p(x_{0:T}, z_{0:T}, u_{0:T-1}) = \underbrace{p_{0|-1}(x_0)}_{\text{prior}} \prod_{t=0}^{T} \underbrace{p_h(z_t \mid x_t)}_{\text{observation model}} \prod_{t=1}^{T} \underbrace{p_f(x_t \mid x_{t-1}, u_{t-1})}_{\text{motion model}}$$

Nonlinear Kalman Filter

- A **nonlinear Kalman filter** is a Bayes filter for which:
  - The prior pdf $p_{0|-1}$ is Gaussian
  - The motion model is ~~linear in the state and~~ affected by Gaussian noise
  - The observation model is ~~linear in the state and~~ affected by Gaussian noise
  - The process noise w_t and measurement noise v_t are independent of each other, of the state x_t and across time
  - The posterior pdf is **forced to be Gaussian via approximation**

- **Prior**: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$

- **Motion Model**: $x_{t+1} = f(x_t, u_t, w_t), \quad w_t \sim \mathcal{N}(0, W)$

- **Observation Model**: $z_t = h(x_t, v_t), \quad v_t \sim \mathcal{N}(0, V)$

- **Challenge**: the predicted and updated pdfs are not Gaussian and can no longer be evaluated in closed form

- **Moment matching**: we can force the predicted and updated pdfs to be Gaussian by evaluating their first and second moments and approximating them with Gaussians with the same moments

Nonlinear Kalman Filter Prediction

- Prior state distribution: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$
- Motion model: $x_{t+1} = f(x_t, u_t, w_t), \qquad w_t \sim \mathcal{N}(0, W)$
- Force a Gaussian predicted pdf: $x_{t+1} \mid z_{0:t}, u_{0:t} \sim \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$

$$
\begin{aligned}
\mu_{t+1|t} &= \mathbb{E}\left[x_{t+1} \mid z_{0:t}, u_{0:t}\right] \\
&= \int \int f(x, u_t, w)\phi(x; \mu_{t|t}, \Sigma_{t|t})\phi(w; 0, W)dxdw \\
\Sigma_{t+1|t} &= \mathbb{E}\left[\left(x_{t+1} - \mu_{t+1|t}\right)\left(x_{t+1} - \mu_{t+1|t}\right)^T \mid z_{0:t}, u_{0:t}\right] \\
&= \mathbb{E}\left[x_{t+1}x_{t+1}^T \mid z_{0:t}, u_{0:t}\right] - \mu_{t+1|t}\mu_{t+1|t}^T \\
&= \int \int f(x, u_t, w)f(x, u_t, w)^T \phi(x; \mu_{t|t}, \Sigma_{t|t})\phi(w; 0, W)dxdw - \mu_{t+1|t}\mu_{t+1|t}^T
\end{aligned}
$$

Nonlinear Kalman Filter Update

- Prior state distribution: $x_{t+1} \mid z_{0:t}, u_{0:t} \sim \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$
- Observation model: $z_{t+1} = h(x_{t+1}, v_{t+1}), \qquad v_{t+1} \sim \mathcal{N}(0, V)$
- The Gaussian distribution which approximates the joint distribution of $x_{t+1} \mid z_{0:t}, u_{0:t}$ and z_{t+1} via moment matching is:

$$\begin{pmatrix} x_{t+1} \mid z_{0:t}, u_{0:t} \\ z_{t+1} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{t+1|t} \\ m_{t+1|t} \end{pmatrix}, \begin{bmatrix} \Sigma_{t+1|t} & C_{t+1|t} \\ C_{t+1|t}^T & S_{t+1|t} \end{bmatrix} \right)$$

$$m_{t+1|t} := \int \int h(x, v)\phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V)dxdv$$

$$S_{t+1|t} := \int \int (h(x, v) - m_{t+1|t})(h(x, v) - m_{t+1|t})^T \phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V)dxdv$$

$$C_{t+1|t} := \int \int (x - \mu_{t+1|t})(h(x, v) - m_{t+1|t})^T \phi(x; \mu_{t+1|t}, \Sigma_{t+1|t})\phi(v; 0, V)dxdv$$

- The conditional Gaussian distribution of $x_{t+1} \mid z_{t+1}, z_{0:t}, u_{0:t}$ is then:

$$\begin{aligned} \mu_{t+1|t+1} &:= \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t}) \\ \Sigma_{t+1|t+1} &:= \Sigma_{t+1|t} - K_{t+1|t}S_{t+1|t}K_{t+1|t}^T \\ K_{t+1|t} &:= C_{t+1|t}S_{t+1|t}^{-1} \end{aligned}$$

Extended and Unscented Kalman Filters

- The **EKF** and **UKF** use different methods to approximate the five integrals required to implement the nonlinear Kalman filter
- The **EKF** uses a first-order Taylor series approximation to the motion and observation models:

$$f(x_t, u_t, w_t) \approx f(\mu_{t|t}, u_t, 0) + \left[\frac{df}{dx}(\mu_{t|t}, u_t, 0)\right](x_t - \mu_{t|t}) + \left[\frac{df}{dw}(\mu_{t|t}, u_t, 0)\right](w_t - 0)$$

$$h(x_{t+1}, v_{t+1}) \approx h(\mu_{t+1|t}, 0) + \left[\frac{dh}{dx}(\mu_{t+1|t}, 0)\right](x_{t+1} - \mu_{t+1|t}) + \left[\frac{dh}{dv}(\mu_{t+1|t}, 0)\right](v_{t+1} - 0)$$

- The **UKF** uses a set of **sigma points** that capture the mean and covariance of the prior Gaussian pdfs to approximate the integrals via a sum. This resembles Monte Carlo approximation but the sigma points are selected **deterministically**.

Extended Kalman Filter Prediction

- Let $F_t := \frac{df}{dx}(\mu_{t|t}, u_t, 0)$ and $Q_t := \frac{df}{dw}(\mu_{t|t}, u_t, 0)$ so that:

$$f(x_t, u_t, w_t) \approx f(\mu_{t|t}, u_t, 0) + F_t(x_t - \mu_{t|t}) + Q_t w_t$$

- Then, the predicted mean and cov can be computed in closed form:

$$\begin{aligned}
\mu_{t+1|t} &\approx \iint \left(f(\mu_{t|t}, u_t, 0) + F_t(x - \mu_{t|t}) + Q_t w\right) \phi(x; \mu_{t|t}, \Sigma_{t|t})\phi(w; 0, W)dxdw \\
&= f(\mu_{t|t}, u_t, 0) + F_t \left(\int x\phi(x; \mu_{t|t}, \Sigma_{t|t})dx - \mu_{t|t}\right) + Q_t \int w\phi(w; 0, W)dw \\
&= \boxed{f(\mu_{t|t}, u_t, 0)}
\end{aligned}$$

$$\begin{aligned}
\Sigma_{t+1|t} \approx& \iint \left(f(\mu_{t|t}, u_t, 0) + F_t(x - \mu_{t|t}) + Q_t w\right) \left(f(\mu_{t|t}, u_t, 0) + F_t(x - \mu_{t|t}) + Q_t w\right)^T \phi(x; \mu_{t|t}, \Sigma_{t|t})\phi(w; 0, W)dxdw \\
&- \mu_{t+1|t}\mu_{t+1|t}^T \\
=& f(\mu_{t|t}, u_t, 0) \left(\int (x - \mu_{t|t})^T \phi(x; \mu_{t|t}, \Sigma_{t|t})dx\right) F_t^T + F_t \left(\int (x - \mu_{t|t})\phi(x; \mu_{t|t}, \Sigma_{t|t})dx\right) f(\mu_{t|t}, u_t, 0)^T \\
&+ F_t \left(\int (x - \mu_{t|t})(x - \mu_{t|t})^T \phi(x; \mu_{t|t}, \Sigma_{t|t})dx\right) F_t^T + Q_t \left(\int ww^T \phi(w; 0, W)dw\right) Q_t^T \\
=& \boxed{F_t \Sigma_{t|t} F_t^T + Q_t W Q_t^T}
\end{aligned}$$

Extended Kalman Filter Update

- Let $H_{t+1} := \frac{dh}{dx}(\mu_{t+1|t}, 0)$ and $R_{t+1} := \frac{dh}{dv}(\mu_{t+1|t}, 0)$ so that:

$$h(x_{t+1}, v_{t+1}) \approx h(\mu_{t+1|t}, 0) + H_{t+1}(x_{t+1} - \mu_{t+1|t}) + R_{t+1} v_{t+1}$$

- The joint distribution of x_{t+1} and z_{t+1} can be computed in closed form:

$$\begin{aligned}
m_{t+1|t} &:= \int\int h(x,v)\phi(x;\mu_{t+1|t},\Sigma_{t+1|t})\phi(v;0,V)dxdv \approx \boxed{h(\mu_{t+1|t},0)} \\
S_{t+1|t} &:= \int\int (h(x,v) - m_{t+1|t})(h(x,v) - m_{t+1|t})^T\phi(x;\mu_{t+1|t},\Sigma_{t+1|t})\phi(v;0,V)dxdv \\
&\approx \boxed{H_{t+1}\Sigma_{t+1|t}H_{t+1}^T + R_{t+1}VR_{t+1}^T} \\
C_{t+1|t} &:= \int\int (x - \mu_{t+1|t})(h(x,v) - m_{t+1|t})^T\phi(x;\mu_{t+1|t},\Sigma_{t+1|t})\phi(v;0,V)dxdv \\
&\approx \boxed{\Sigma_{t+1|t}H_{t+1}^T}
\end{aligned}$$

- The conditional Gaussian distribution of $x_{t+1} \mid z_{t+1}$ is then:

$$\begin{aligned}
\mu_{t+1|t+1} &:= \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t}) \\
\Sigma_{t+1|t+1} &:= \Sigma_{t+1|t} - K_{t+1|t}S_{t+1|t}K_{t+1|t}^T \\
K_{t+1|t} &:= C_{t+1|t}S_{t+1|t}^{-1}
\end{aligned}$$

Extended Kalman Filter

Prior: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$

Motion model:
$$x_{t+1} = f(x_t, u_t, w_t), \quad w_t \sim \mathcal{N}(0, W)$$
$$F_t := \frac{df}{dx}(\mu_{t|t}, u_t, 0), \quad Q_t := \frac{df}{dw}(\mu_{t|t}, u_t, 0)$$

Obs. model:
$$z_t = h(x_t, v_t), \quad v_t \sim \mathcal{N}(0, V)$$
$$H_t := \frac{dh}{dx}(\mu_{t|t-1}, 0), \quad R_t := \frac{dh}{dv}(\mu_{t|t-1}, 0)$$

Prediction:
$$\mu_{t+1|t} = f(\mu_{t|t}, u_t, 0)$$
$$\Sigma_{t+1|t} = F_t \Sigma_{t|t} F_t^T + Q_t W Q_t^T$$

Update:
$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - h(\mu_{t+1|t}, 0))$$
$$\Sigma_{t+1|t+1} = (I - K_{t+1|t} H_{t+1}) \Sigma_{t+1|t}$$

Kalman Gain: $K_{t+1|t} := \Sigma_{t+1|t} H_{t+1}^T \left(H_{t+1} \Sigma_{t+1|t} H_{t+1}^T + R_{t+1} V R_{t+1}^T \right)^{-1}$

Unscented Transform

- The **unscented transform** (Julier et al., 1995, 2000) is a numerical method for approximating the joint distribution of a Gaussian random variable $y \in \mathbb{R}^d$ and a nonlinear transformation g of it:

$$s = g(y), \qquad \begin{pmatrix} y \\ s \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ m_U \end{pmatrix}, \begin{bmatrix} \Sigma & C_U \\ C_U^T & S_U \end{bmatrix}\right)$$

- Choose a set of $2d+1$ **sigma points** using the i-th columns of the square root $\sqrt{\Sigma}$ of the covariance $\Sigma = \sqrt{\Sigma}\sqrt{\Sigma}^T$ (**note**: $\sqrt{\Sigma}$ is lower-triangular and can be obtained, e.g., via **Cholesky factorization**):

$$\mathcal{Y}^{(0)} = \mu, \qquad \mathcal{Y}^{(i)} = \mu \pm \sqrt{d+\lambda}\left[\sqrt{\Sigma}\right]_i, \quad i = 1, \dots, d$$

$$\lambda := \alpha^2(d+k) - d, \qquad (\alpha, k) : \text{determine sigma points spread, e.g.,}$$

$$\alpha \in [0.0001, 1], k = 0$$

Unscented Transform

- The sigma points capture the shape of the original distribution of y well and can be propagated through the nonlinear function g to estimate the mean and covariance of the transformed variable

- The mean and covariance of $s = g(y)$ are estimated using the sigma points $\mathcal{Y}^{(0)} = \mu$ and $\mathcal{Y}^{(i)} = \mu \pm \sqrt{d+\lambda}\left[\sqrt{\Sigma}\right]_i$, $i = 1, \ldots, d$:

$$m_U = \sum_{i=0}^{2d} W_i^{(m)} g\left(\mathcal{Y}^{(i)}\right), \qquad W_0^{(m)} = \frac{\lambda}{d+\lambda}, \qquad W_i^{(m)} = \frac{1}{2(d+\lambda)}, \quad i = 1, \ldots, 2d$$

$$S_U = \sum_{i=0}^{2d} W_i^{(c)} \left(g\left(\mathcal{Y}^{(i)}\right) - m_U\right)\left(g\left(\mathcal{Y}^{(i)}\right) - m_U\right)^T, \qquad W_0^{(c)} = \frac{\lambda}{d+\lambda} + (1 - \alpha^2 + \beta)$$

$$\text{e.g., } \alpha \in [0.0001, 1], \ \beta \in \{0, 2\}$$

$$C_U = \sum_{i=0}^{2d} W_i^{(c)} \left(\mathcal{Y}^{(i)} - \mu\right)\left(g\left(\mathcal{Y}^{(i)}\right) - m_U\right)^T, \qquad W_i^{(c)} = \frac{1}{2(d+\lambda)}, \quad i = 1, \ldots, 2d$$

Unscented Kalman Filter Prediction

- **Prior**: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$
- **Motion Model**: $x_{t+1} = f(x_t, u_t, w_t), \quad w_t \sim \mathcal{N}(0, W)$

$$\begin{pmatrix} \mathcal{X}^{(0)}_{t|t} \\ \mathcal{W}^{(0)} \end{pmatrix} = \begin{pmatrix} \mu_{t|t} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{X}^{(i)}_{t|t} \\ \mathcal{W}^{(i)} \end{pmatrix} = \begin{pmatrix} \mu_{t|t} \\ 0 \end{pmatrix} \pm \sqrt{(d_x + d_w)} \begin{bmatrix} \sqrt{\Sigma_{t|t}} & 0 \\ 0 & \sqrt{W} \end{bmatrix}_i$$

- **Prediction**: for $\alpha = 1$, $k = 0$, $\lambda = 0$, and $\beta = 2$

$$\mu_{t+1|t} = \sum_{i=0}^{2(d_x+d_w)} W_i^{(m)} f\left(\mathcal{X}^{(i)}_{t|t}, u_t, \mathcal{W}^{(i)}\right)$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2(d_x+d_w)} W_i^{(c)} \left(f\left(\mathcal{X}^{(i)}_{t|t}, u_t, \mathcal{W}^{(i)}\right) - \mu_{t+1|t}\right) \left(f\left(\mathcal{X}^{(i)}_{t|t}, u_t, \mathcal{W}^{(i)}\right) - \mu_{t+1|t}\right)^T$$

- **Weights**:

$$W_0^{(m)} = 0, \qquad W_i^{(m)} = \frac{1}{2(d_x + d_w)}, \quad i = 1, \ldots, 2(d_x + d_w)$$

$$W_0^{(c)} = 2, \qquad W_i^{(c)} = \frac{1}{2(d_x + d_w)}, \quad i = 1, \ldots, 2(d_x + d_w)$$

Unscented Kalman Filter Update

- **Prior**: $x_{t+1} \mid z_{0:t}, u_{0:t} \sim \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$
- **Observation Model**: $z_{t+1} = h(x_{t+1}, v_{t+1}), \quad v_{t+1} \sim \mathcal{N}(0, V)$

$$\begin{pmatrix} \mathcal{X}^{(0)}_{t+1|t} \\ \mathcal{V}^{(0)} \end{pmatrix} = \begin{pmatrix} \mu_{t+1|t} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{X}^{(i)}_{t+1|t} \\ \mathcal{V}^{(i)} \end{pmatrix} = \begin{pmatrix} \mu_{t+1|t} \\ 0 \end{pmatrix} \pm \sqrt{(d_x + d_v)} \begin{bmatrix} \sqrt{\Sigma_{t+1|t}} & 0 \\ 0 & \sqrt{V} \end{bmatrix}_i$$

- **Update**: $\begin{aligned} \mu_{t+1|t+1} &= \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t})) \\ \Sigma_{t+1|t+1} &= \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K^T_{t+1|t} \end{aligned}$
- **Kalman Gain**: $K_{t+1|t} = C_{t+1|t} S^{-1}_{t+1|t}$

$$m_{t+1|t} = \sum_{i=0}^{2(d_x+d_v)} W_i^{(m)} h\left(\mathcal{X}^{(i)}_{t+1|t}, \mathcal{V}^{(i)}\right)$$

$$S_{t+1|t} = \sum_{i=0}^{2(d_x+d_v)} W_i^{(c)} \left(h\left(\mathcal{X}^{(i)}_{t+1|t}, \mathcal{V}^{(i)}\right) - m_{t+1|t}\right) \left(h\left(\mathcal{X}^{(i)}_{t+1|t}, \mathcal{V}^{(i)}\right) - m_{t+1|t}\right)^T$$

$$C_{t+1|t} = \sum_{i=0}^{2(d_x+d_v)} W_i^{(c)} \left(\mathcal{X}^{(i)}_{t+1|t} - \mu_{t+1|t}\right) \left(h\left(\mathcal{X}^{(i)}_{t+1|t}, \mathcal{V}^{(i)}\right) - m_{t+1|t}\right)^T$$

Unscented Kalman Filter (additive noise)

Prior $\quad x_{t+1} \mid z_{0:t}, u_{0:t} \sim \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$

Motion model $\quad x_{t+1} = f(x_t, u_t) + w_t, \quad w_t \sim \mathcal{N}(0, W)$

Obs. model $\quad z_{t+1} = h(x_{t+1}) + v_{t+1}, \quad v_{t+1} \sim \mathcal{N}(0, V)$

Predict

$$\mu_{t+1|t} = \sum_{i=0}^{2d_x} W_i^{(m)} f\left(\mathcal{X}_{t|t}^{(i)}, u_t\right), \quad \mathcal{X}_{t|t}^{(0)} = \mu_{t|t}, \quad \mathcal{X}_{t|t}^{(i)} = \mu_{t|t} \pm \sqrt{d_x + \lambda}\left[\sqrt{\Sigma_{t|t}}\right]_i$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2d_x} W_i^{(c)} \left(f\left(\mathcal{X}_{t|t}^{(i)}, u_t\right) - \mu_{t+1|t}\right)\left(f\left(\mathcal{X}_{t|t}^{(i)}, u_t\right) - \mu_{t+1|t}\right)^T + W$$

Update

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t})$$

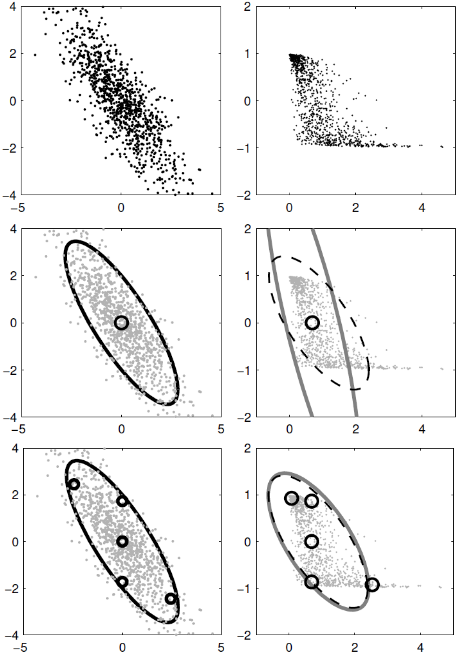
$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K_{t+1|t}^T$$

Kalman gain $\quad K_{t+1|t} = C_{t+1|t} S_{t+1|t}^{-1}$

$$m_{t+1|t} = \sum_{i=0}^{2d_x} W_i^{(m)} h\left(\mathcal{X}_{t+1|t}^{(i)}\right), \quad \mathcal{X}_{t+1|t}^{(0)} = \mu_{t+1|t}, \quad \mathcal{X}_{t+1|t}^{(i)} = \mu_{t+1|t} \pm \sqrt{d_x + \lambda}\left[\sqrt{\Sigma_{t+1|t}}\right]_i$$

$$S_{t+1|t} = \sum_{i=0}^{2d_x} W_i^{(c)} \left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)\left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)^T + V$$

$$C_{t+1|t} = \sum_{i=0}^{2d_x} W_i^{(c)} \left(\mathcal{X}_{t+1|t}^{(i)} - \mu_{t+1|t}\right)\left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)^T$$



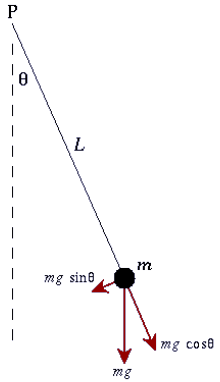
Applying a nonlinear transformation to the random variable on the left results in a new random variable (right)

EKF: Linearization-based approximation to the transformation formed by calculating the curvature at the mean (solid) and true covariance (dashed).

UKF: Unscented transform approximation to the transformation formed by propagating sigma points through the nonlinearity and estimating the covariance from them (solid) and true covariance (dashed).

Noisy Pendulum Tracking

- Consider a simple pendulum consisting of a mass m hanging from a string of length L and fixed at a pivot point P



- The differential equation for the pendulum motion can be obtained using Newton's second law for rotational systems which relates the net external torque τ (position $\times$ force) to the product of the moment of inertia $I = mL^2$ and the angular acceleration $\ddot{\theta}(t)$:

$$\tau = -mgL\sin\theta(t) = mL^2\ddot{\theta}(t) \quad \Rightarrow \quad \ddot{\theta}(t) = -\frac{g}{L}\sin\theta(t) + \underbrace{w(t)}_{\text{noise}\sim\mathcal{N}(0,q)}$$

- The model can be converted into a state-space model with state $x(t) := (\theta(t), \omega(t))^T$, where $\omega(t) := \dot{\theta}(t)$ as follows:

$$\frac{d}{dt}\begin{pmatrix}\theta(t)\\ \omega(t)\end{pmatrix} = \begin{bmatrix}\omega(t)\\ -\frac{g}{L}\sin(\theta(t))\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} w(t)$$

Discrete-time Model

- **Motion model**: a simple discretization of the pendulum state-space model with sampling period τ leads to:

$$x_{t+1} = \begin{pmatrix} \theta_{t+1} \\ \omega_{t+1} \end{pmatrix} = \underbrace{\begin{bmatrix} \theta_t + \tau\omega_t \\ \omega_t - \tau\frac{g}{L}\sin\theta_t \end{bmatrix}}_{f(x_t)} + w_t, \quad w_t \sim \mathcal{N}\left(0, \underbrace{q\begin{bmatrix} \frac{\tau^3}{3} & \frac{\tau^2}{2} \\ \frac{\tau^2}{2} & \tau \end{bmatrix}}_{W}\right)$$

- **Observation model**: consider estimating the angle θ_t and the velocity ω_t of the pendulum using measurements of its deviation from rest position, i.e.,:

$$z_t = \underbrace{L\sin(\theta_t)}_{h(x_t)} + v_t, \qquad v_t \sim \mathcal{N}(0, V)$$

Extended Kalman Filter

- **Jacobians** of the motion and observation model for $x = (\theta, \omega)^T$:

$$F(x) := \begin{bmatrix} 1 & \tau \\ -\tau \frac{g}{L} \cos\theta & 1 \end{bmatrix} \qquad H(x) = \begin{bmatrix} L\cos\theta & 0 \end{bmatrix}$$

- **Prior**: $x_t \mid z_{0:t} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$

- **Prediction**:

$$\mu_{t+1|t} = f(\mu_{t|t}) = \begin{bmatrix} \mu^{\theta}_{t|t} + \tau \mu^{\omega}_{t|t} \\ \mu^{\omega}_{t|t} - \tau \frac{g}{L} \sin \mu^{\theta}_{t|t} \end{bmatrix}$$

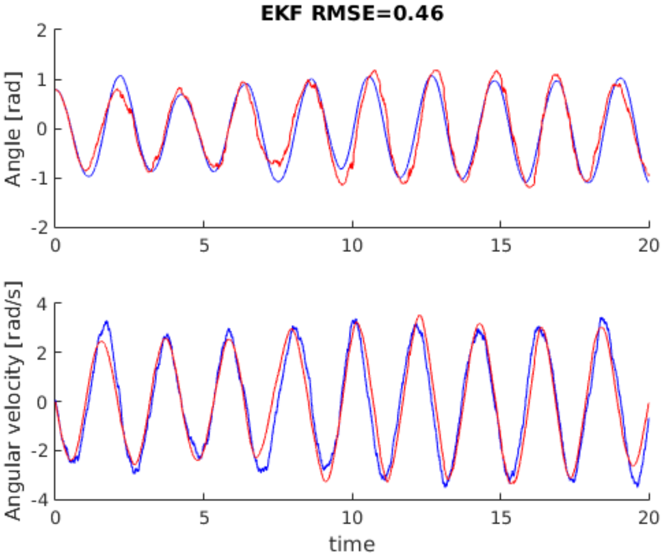
$$\Sigma_{t+1|t} = F(\mu_{t|t}) \Sigma_{t|t} F(\mu_{t|t})^T + W$$

Extended Kalman Filter

- **Innovation**: $\nu_{t+1|t} := z_{t+1} - L\sin(\mu^{\theta}_{t+1|t})$
- **Measurement/innovation covariance**: $S_{t+1|t} := H(\mu_{t+1|t})\Sigma_{t+1|t}H(\mu_{t+1|t})^T + V$
- **State-measurement cross-covariance**: $\Sigma_{t+1|t}H(\mu_{t+1|t})^T$
- **Kalman gain**: $K_{t+1|t} = \Sigma_{t+1|t}H(\mu_{t+1|t})^T S^{-1}_{t+1|t}$
- **Update**: $$\begin{aligned}\mu_{t+1|t+1} &= \mu_{t+1|t} + K_{t+1|t}\nu_{t+1|t}\\ \Sigma_{t+1|t+1} &= \Sigma_{t+1|t} - K_{t+1|t}H(\mu_{t+1|t})\Sigma_{t+1|t}\end{aligned}$$

EKF Performance

- $\tau = 0.001$, $q = 0.3$, $g = 9.81$, $L = 1$, $V = 0.64$
- Prediction at 1000 Hz, update at 20 Hz



Uscented Kalman Filter Prediction

- **Prior**: $x_t \mid z_{0:t} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$
- **Sigma points** with parameter $\lambda = \alpha^2(d_x + k) - d_x$ determining the sigma point spread (usual choice: $\alpha \in [0.0001, 1]$, $k = 0$, $\beta \in \{0, 2\}$)

$$\mathcal{X}_{t|t}^{(0)} = \mu_{t|t}, \qquad \mathcal{X}_{t|t}^{(i)} = \mu_{t|t} \pm \sqrt{(d_x + \lambda)} \left[\sqrt{\Sigma_{t|t}}\right]_i, \quad i = 1, \ldots, 2d_x$$

$$W_m^{(0)} = \frac{\lambda}{d_x + \lambda}, \qquad W_m^{(i)} = \frac{1}{2(d_x + \lambda)}, \quad i = 1, \ldots, 2d_x$$

$$W_c^{(0)} = \frac{\lambda}{d_x + \lambda} + (1 - \alpha^2 + \beta), \qquad W_c^{(i)} = \frac{1}{2(d_x + \lambda)}, \quad i = 1, \ldots, 2d_x$$

- **Prediction**:

$$\mu_{t+1|t} = \sum_{i=0}^{2d_x} W_m^{(i)} f\left(\mathcal{X}_{t|t}^{(i)}\right)$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2d_x} W_c^{(i)} \left(f\left(\mathcal{X}_{t|t}^{(i)}\right) - \mu_{t+1|t}\right)\left(f\left(\mathcal{X}_{t|t}^{(i)}\right) - \mu_{t+1|t}\right)^T + W$$

Uscented Kalman Filter Update

- **Sigma points**:

$$\mathcal{X}^{(0)}_{t+1|t} = \mu_{t+1|t}, \qquad \mathcal{X}^{(i)}_{t+1|t} = \mu_{t+1|t} \pm \sqrt{(d_x + \lambda)} \left[\sqrt{\Sigma_{t+1|t}}\right]_i, \quad i = 1, \ldots, 2d_x$$

- **Expected measurement**: $m_{t+1|t} = \sum_{i=0}^{2d_x} W_m^{(i)} h\left(\mathcal{X}^{(i)}_{t+1|t}\right)$
- **Innovation**: $\nu_{t+1|t} := z_{t+1} - m_{t+1|t}$
- **Measurement/innovation covariance**:

$$S_{t+1|t} = \sum_{i=0}^{2d_x} W_c^{(i)} \left(h\left(\mathcal{X}^{(i)}_{t+1|t}\right) - m_{t+1|t}\right)\left(h\left(\mathcal{X}^{(i)}_{t+1|t}\right) - m_{t+1|t}\right)^T + V$$

- **State-measurement cross-covariance**:

$$C_{t+1|t} = \sum_{i=0}^{2d_x} W_c^{(i)} \left(\mathcal{X}^{(i)}_{t+1|t} - \mu_{t+1|t}\right)\left(h\left(\mathcal{X}^{(i)}_{t+1|t}\right) - m_{t+1|t}\right)^T$$

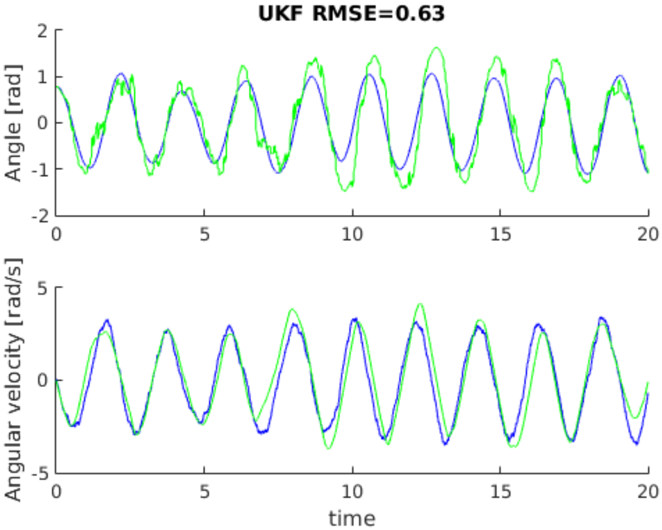
- **Kalman gain**: $K_{t+1|t} = C_{t+1|t} S^{-1}_{t+1|t}$
- **Update**:

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}\nu_{t+1|t}$$

$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K^T_{t+1|t}$$

UKF Performance

- $\tau = 0.001$, $q = 0.3$, $g = 9.81$, $L = 1$, $V = 0.64$
- Prediction at 1000 Hz, update at 20 Hz



UKF vs EKF Predicted Covariance

- Prior: $\mathcal{N}\left(\begin{pmatrix} \frac{\pi}{4} \\ -1 \end{pmatrix}, \begin{bmatrix} 2 & -0.3 \\ -0.3 & 0.5 \end{bmatrix}\right)$
- One prediction step with parameters $\tau = 1$, $g = 9.81$, $L = 1$