ECE276A: Sensing & Estimation in Robotics
Lecture 15: Visual-Inertial SLAM

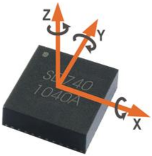
Instructor:
Nikolay Atanasov: natanasov@ucsd.edu

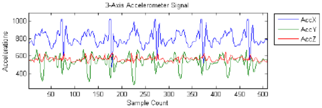
Teaching Assistants:
Qiaojun Feng: qif007@eng.ucsd.edu
Tianyu Wang: tiw161@eng.ucsd.edu
Ibrahim Akbar: iakbar@eng.ucsd.edu
You-Yi Jau: yjau@eng.ucsd.edu
Harshini Rajachander: hrajacha@eng.ucsd.edu

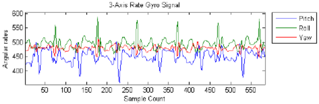
UC San Diego
JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

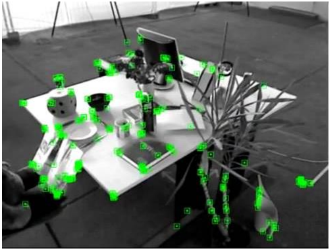
Visual-Inertial Localization and Mapping

- **Input**: IMU measurements of linear velocity $\mathbf{v}_t \in \mathbb{R}^3$ and rotational velocity $\omega_t \in \mathbb{R}^3$ and visual features $\mathbf{z}_t \in \mathbb{R}^{4 \times N_t}$ (left and right image pixels) extracted from stereo RGB images







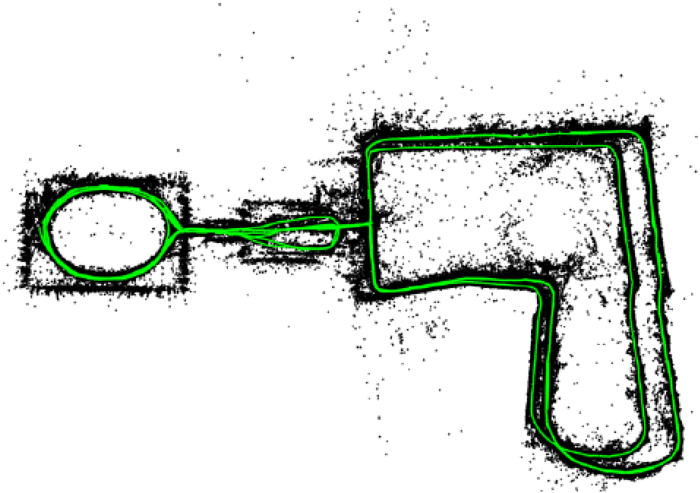


- **Assumption**: The transformation ${}_O T_I \in SE(3)$ from the IMU to the camera optical frame (extrinsic parameters) and the stereo camera calibration matrix M (intrinsic parameters) are known.

$$M := \begin{bmatrix} fs_u & 0 & c_u & 0 \\ 0 & fs_v & c_v & 0 \\ fs_u & 0 & c_u & -fs_u b \\ 0 & fs_v & c_v & 0 \end{bmatrix}$$

Visual-Inertial Localization and Mapping

- **Output**: the pose ${}_W T_I \in SE(3)$ of the IMU with respect to the world frame over time (green) and the world-frame coordiantes of the landmarks (black) that generated the visual features



Visual Mapping

- Consider the mapping-only problem first
- **Assumption**: the inverse IMU pose $T_t := {}_W T_{I,t}^{-1} \in SE(3)$ over time is known
- **Objective**: given the visual feature observations $\{\mathbf{z}_t\}_{t=0}^T$, estimate the homogeneous coordinates $\mathbf{m} \in \mathbb{R}^{4\times M}$ in the world frame of the landmarks that generated the visual observations
- **Assumption**: the data association $\pi_t : \{1,\ldots,M\} \to \{1,\ldots,N_t\}$ stipulating which landmarks were observed at each time t is known or provided by an external algorithm
- **Assumption**: the landmarks are static, i.e., it is not necessary to consider a motion model or a prediction step

Visual Mapping via the EKF

- **Prior**: $\mathbf{m} \mid \mathbf{z}_{0:t} \sim \mathcal{N}(\boldsymbol{\mu}_t, \Sigma_t)$ with $\boldsymbol{\mu}_t \in \mathbb{R}^{4\times M}$ and $\Sigma_t \in \mathbb{R}^{3M\times 3M}$
- The covariance is $3M \times 3M$ because only the 3-D part of the homogeneous coordiantes $\boldsymbol{\mu}_{t,j}$ is changing via a perturbation $\delta_{t,j} \in \mathbb{R}^3$:

$$\boldsymbol{\mu}_{t+1,j} = \boldsymbol{\mu}_{t,j} + D\delta_{t,j} \qquad D = \begin{bmatrix} I_3 \\ \mathbf{0}^T \end{bmatrix}$$

- **Observation Model**: with measurement noise $v_t \sim \mathcal{N}(0, V)$

$$\mathbf{z}_{t,i} = h(T_t, \mathbf{m}_j) + v_t := M\pi\left({}_O T_I T_t \mathbf{m}_j\right) + v_t$$

- Projection function and its derivative:

$$\pi(\mathbf{q}) := \frac{1}{q_3}\mathbf{q} \in \mathbb{R}^4 \qquad \frac{d\pi}{d\mathbf{q}}(\mathbf{q}) = \frac{1}{q_3}\begin{bmatrix} 1 & 0 & -\frac{q_1}{q_3} & 0 \\ 0 & 1 & -\frac{q_2}{q_3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{q_4}{q_3} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

Visual Mapping via the EKF

- **EKF Update**: with slight abuse of notation

$$K_t = \Sigma_t H_t^T \left(H_t \Sigma_t H_t^T + I \otimes V\right)^{-1}$$
$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t + DK_t \left(\mathbf{z}_t - M\pi\left({}_OT_I T_t \boldsymbol{\mu}_t\right)\right)$$
$$\Sigma_{t+1} = (I - K_t H_t)\Sigma_t$$

$$I \otimes V := \begin{bmatrix} V & & \\ & \ddots & \\ & & V \end{bmatrix}$$

- We need the observation model Jacobian $H_t \in \mathbb{R}^{4N_t \times 3M}$ evaluated at $\boldsymbol{\mu}_t$
- Let the elements of $H_t \in \mathbb{R}^{4N_t \times 3M}$ corresponding to different observations i and different landmarks j be $H_{i,j,t} \in \mathbb{R}^{4 \times 3}$
- The first-order Taylor series approximation to observation i at time t using the perturbation $\delta_{t,j}$ of the position of landmark j is:

$$\mathbf{z}_{t,i} = M\pi\left({}_OT_I T_t \left(\boldsymbol{\mu}_{t,j} + D\delta_{t,j}\right)\right)$$
$$\approx \underbrace{M\pi\left({}_OT_I T_t \boldsymbol{\mu}_{t,j}\right)}_{\hat{\mathbf{z}}_{t,i}} + \underbrace{M\frac{d\pi}{d\mathbf{q}}\left({}_OT_I T_t \boldsymbol{\mu}_{t,j}\right) {}_OT_I T_t D}_{H_{i,j,t}} \delta_{t,j}$$

Visual Mapping via the EKF (Summary)

- Prior: $\boldsymbol{\mu}_t \in \mathbb{R}^{4\times M}$ and $\Sigma_t \in \mathbb{R}^{3M\times 3M}$
- Known: calibration matrix M, extrinsics ${}_OT_I \in SE(3)$, (inverse) IMU pose $T_t \in SE(3)$, dilation matrix D, new observation $z_t \in \mathbb{R}^{4\times N_t}$
- Compute the predicted observation based on $\boldsymbol{\mu}_t$ and known correspondences:

$$\hat{\mathbf{z}}_{t,i} := M\pi\left({}_OT_I T_t \boldsymbol{\mu}_{t,j}\right) \in \mathbb{R}^4 \qquad \text{for } i = 1, \ldots, N_t$$

- Compute the Jacobian of $\hat{\mathbf{z}}_{t,i}$ with respect to $\mathbf{m}_j$ evaluated at $\boldsymbol{\mu}_{t,j}$:

$$H_{i,j,t} = \begin{cases} M\frac{d\pi}{d\mathbf{q}}\left({}_OT_I T_t \boldsymbol{\mu}_{t,j}\right) {}_OT_I T_t D & \text{if observation } i \text{ corresponds to landmark } j \text{ at time } t \\ \mathbf{0} \in \mathbb{R}^{4\times 3} & \text{otherwise} \end{cases}$$

- Perform the EKF update:

$$\begin{aligned} K_t &= \Sigma_t H_t^T \left(H_t \Sigma_t H_t^T + I \otimes V\right)^{-1} \\ \boldsymbol{\mu}_{t+1} &= \boldsymbol{\mu}_t + DK_t\left(\mathbf{z}_t - \hat{\mathbf{z}}_t\right) \\ \Sigma_{t+1} &= (I - K_t H_t)\Sigma_t \end{aligned} \qquad I \otimes V := \begin{bmatrix} V & & \\ & \ddots & \\ & & V \end{bmatrix}$$

Visual-Inertial Odometry

- Now, consider the localization-only problem
- **Assumption**: the homogeneous coordiantes $\mathbf{m} \in \mathbb{R}^{4\times M}$ in the world frame of the landmarks are known
- **Objective**: given the IMU measurements $\{u_t\}_{t=0}^{T}$ with $u_t := [\mathbf{v}_t^T, \ \omega_t^T]^T$ and the visual feature observations $\{\mathbf{z}_t\}_{t=0}^{T}$, estimate the inverse IMU pose $T_t := {}_W T_{I,t}^{-1} \in SE(3)$ over time
- **Assumption**: the data association $\pi_t : \{1, \ldots, M\} \rightarrow \{1, \ldots, N_t\}$ stipulating which landmarks were observed at each time t is known or provided by an external algorithm

Visual-Inertial Odometry via the EKF

- **Prior**: $T_t | z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\boldsymbol{\mu}_{t|t}, \Sigma_{t|t})$ with $\boldsymbol{\mu}_{t|t} \in SE(3)$ and $\Sigma_{t|t} \in \mathbb{R}^{6\times 6}$

- The covariance is 6×6 because only the six degrees of freedom of $\boldsymbol{\mu}_{t|t}$ are changing via a perturbation $\xi_t \in \mathbb{R}^6$:

$$\boldsymbol{\mu}_{t+1|t} = \exp\left(\hat{\xi}_t\right)\boldsymbol{\mu}_{t|t} \approx (I + \hat{\xi}_t)\boldsymbol{\mu}_{t|t}$$

- **Motion Model**: with time discretization τ and noise $w_t \sim \mathcal{N}(0, W)$

$$T_{t+1} = \exp\left((\tau(-u_t + w_t))^\wedge\right) T_t \qquad u_t := \begin{bmatrix} v_t \\ \omega_t \end{bmatrix}$$

- Note that u_t is negative above since T_t is the inverse IMU pose:

$$\begin{aligned} {}_W\dot{T}_B &= {}_W T_B \hat{\zeta}_B \\ -{}_B\dot{T}_W &= ({}_B T_W)\left({}_W\dot{T}_B\right)({}_B T_W) \quad = \hat{\zeta}_B \left({}_B T_W\right) \end{aligned}$$

Perturbed Pose Kinematics

- Consider what happens with the pose kinematics
$$\dot{T} = \hat{\zeta} T$$
if the pose is expressed as a nominal pose $\bar{T} \in SE(3)$ and small perturbation $\hat{\xi} \in \mathfrak{se}(3)$:
$$T = \exp(\hat{\xi})\,\bar{T}$$
and the twist is expressed as a nominal twist $\bar{\zeta} \in \mathbb{R}^6$ and a small perturbation $w \in \mathbb{R}^6$:
$$\zeta = \bar{\zeta} + w$$
- The perturbed kinematics $\dot{T} = \hat{\zeta} T$ can be broken into nominal and perturbation kinematics:
$$\begin{aligned} \text{nominal}: \quad & \dot{\bar{T}} = \hat{\bar{\zeta}}\bar{T} \\ \text{perturbation}: \quad & \dot{\xi} = \curlywedge{\bar{\zeta}}\xi + w \end{aligned} \qquad \overset{\curlywedge}{\zeta} := \begin{bmatrix} \hat{\omega} & \hat{v} \\ 0 & \hat{\omega} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$
- This is useful to separate the effect of the noise w_t from the motion of the deterministic part of T_t. See Barfoot Ch. 7.2 for details.

EKF Prediction Step

- Using the perturbation idea from the previous slide, converted to discrete time, we can re-write the motion model in terms of nominal kinematics of the mean of T_t and zero-mean perturbation kinematics:

$$\boldsymbol{\mu}_{t+1|t} = \exp\left(-\tau \hat{u}_t\right) \boldsymbol{\mu}_{t|t} \qquad \overset{\curlywedge}{u}_t := \begin{bmatrix} \hat{\omega}_t & \hat{v}_t \\ 0 & \hat{\omega}_t \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$\xi_{t+1|t} = \exp\left(-\tau \overset{\curlywedge}{u}_t\right) \xi_{t|t} + \tau w_t$$

- **EKF Prediction Step**:

$$\boldsymbol{\mu}_{t+1|t} = \exp\left(-\tau \hat{u}_t\right) \boldsymbol{\mu}_{t|t} \qquad u_t := \begin{bmatrix} v_t \\ \omega_t \end{bmatrix}$$

$$\Sigma_{t+1|t} = \mathbb{E}[\xi_{t+1|t}\xi_{t+1|t}^T] = \exp\left(-\tau \overset{\curlywedge}{u}_t\right) \Sigma_{t|t} \exp\left(-\tau \overset{\curlywedge}{u}_t\right)^T + \tau^2 W$$

Adjoints

- The **adjoint** of $T = \begin{bmatrix} R & p \\ \mathbf{0}^T & 1 \end{bmatrix} \in SE(3)$ is:

$$\mathcal{T} = Ad(T) = \begin{bmatrix} R & \hat{p}R \\ \mathbf{0} & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- $Ad(SE(3)) := \{\mathcal{T} = Ad(T) \mid T \in SE(3)\}$ is a matrix Lie group
- The adjoint of $\hat{\xi} = \begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathfrak{se}(3)$ is:

$$ad(\hat{\xi}) = \overset{\curlywedge}{\xi} = \begin{bmatrix} \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\rho}} \\ \mathbf{0} & \hat{\boldsymbol{\theta}} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- $ad(\mathfrak{se}(3)) := \{\Phi = ad(\hat{\xi}) \in \mathbb{R}^{6\times 6} \mid \hat{\xi} \in \mathfrak{se}(3)\}$ is the Lie algebra associated with $Ad(SE(3))$

- The relationship between $\overset{\curlywedge}{\xi}$ and $\mathcal{T}$ is specified by the exponential map:

$$\mathcal{T} = \exp\left(\overset{\curlywedge}{\xi}\right) = I + \overset{\curlywedge}{\xi}\mathcal{J}_L(\xi) \qquad \mathcal{J}_L(\xi) = \mathcal{T}\mathcal{J}_R(\xi) = \mathcal{J}_R(-\xi)$$

Pose Lie Groups and Lie Algebras

$$
\begin{array}{cccc}
 & \text{Lie algebra} & & \text{Lie group} \\
4 \times 4 & \boldsymbol{\xi}^{\wedge} \in \mathfrak{se}(3) & \xrightarrow{\exp} & \mathbf{T} \in SE(3) \\
 & \Big\downarrow \text{ad} & & \Big\downarrow \text{Ad} \\
6 \times 6 & \boldsymbol{\xi}^{\curlywedge} \in \text{ad}(\mathfrak{se}(3)) & \xrightarrow{\exp} & \boldsymbol{\mathcal{T}} \in \text{Ad}(SE(3))
\end{array}
$$

$$
\mathcal{T} = Ad\underbrace{\left(\exp(\hat{\xi})\right)}_{T} = \exp\underbrace{\left(ad(\hat{\xi})\right)}_{\overset{\curlywedge}{\xi}} \qquad \xi = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6
$$

$$
= Ad\left(\exp\left(\begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}\right)\right) = \exp\left(ad\left(\begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}\right)\right)
$$

$$
= Ad\left(\begin{bmatrix} \exp(\hat{\boldsymbol{\theta}}) & J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix}\right) = \exp\left(\begin{bmatrix} \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\rho}} \\ \mathbf{0} & \hat{\boldsymbol{\theta}} \end{bmatrix}\right)
$$

$$
= \begin{bmatrix} \exp(\hat{\boldsymbol{\theta}}) & (J_L(\boldsymbol{\theta})\boldsymbol{\rho})^{\wedge}\exp(\hat{\boldsymbol{\theta}}) \\ \mathbf{0} & \exp(\hat{\boldsymbol{\theta}}) \end{bmatrix}
$$

Rodrigues Formula for the Adjoint of $SE(3)$

- The exponential map is **surjective** but **not injective**, i.e., every element of $Ad(SE(3))$ can be generated from multiple elements of $ad(\mathfrak{se}(3))$
- **Rodrigues Formula**: using $(\overset{\curlywedge}{\xi})^5 + 2\|\boldsymbol{\theta}\|^2(\overset{\curlywedge}{\xi})^3 + \|\boldsymbol{\theta}\|^4\overset{\curlywedge}{\xi} = 0$ we can obtain a direct expression of $\mathcal{T} \in Ad(SE(3))$ in terms of $\xi = \begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\theta}\end{bmatrix} \in \mathbb{R}^6$:

$$\mathcal{T} = \exp\left(\overset{\curlywedge}{\xi}\right) = \begin{bmatrix}\exp(\hat{\boldsymbol{\theta}}) & (J_L(\boldsymbol{\theta})\boldsymbol{\rho})^\wedge \exp(\hat{\boldsymbol{\theta}}) \\ \mathbf{0} & \exp(\hat{\boldsymbol{\theta}})\end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!}(\overset{\curlywedge}{\xi})^n$$

$$= I + \left(\frac{3\sin\|\boldsymbol{\theta}\| - \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|}\right)\overset{\curlywedge}{\xi} + \left(\frac{4 - \|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\| - 4\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^2}\right)(\overset{\curlywedge}{\xi})^2$$

$$+ \left(\frac{\sin\|\boldsymbol{\theta}\| - \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^3}\right)(\overset{\curlywedge}{\xi})^3 + \left(\frac{2 - \|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\| - 2\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^4}\right)(\overset{\curlywedge}{\xi})^4$$

EKF Update Step

- **Prior**: $T_{t+1} | z_{0:t}, u_{0:t} \sim \mathcal{N}(\boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t})$ with $\boldsymbol{\mu}_{t+1|t} \in SE(3)$ and $\Sigma_{t+1|t} \in \mathbb{R}^{6\times 6}$
- **Observation Model**: with measurement noise $v_t \sim \mathcal{N}(0, V)$

$$\mathbf{z}_{t+1,i} = h(T_{t+1}, \mathbf{m}_j) + v_{t+1} := M\pi \left({}_O T_I T_{t+1} \mathbf{m}_j \right) + v_{t+1}$$

- The observation model is the same as in the visual mapping problem but this time the variable of interest is the inverse IMU pose $T_{t+1} \in SE(3)$ instead of the landmark positions $\mathbf{m} \in \mathbb{R}^{4\times M}$
- We need the observation model Jacobian $H_{t+1|t} \in \mathbb{R}^{4N_t \times 6}$ with respect to the inverse IMU pose, evaluated at $\boldsymbol{\mu}_{t+1|t}$

EKF Update Step

- Let the elements of $H_{t+1|t} \in \mathbb{R}^{4N_t \times 6}$ corresponding to different observations i be $H_{i,t+1|t} \in \mathbb{R}^{4 \times 6}$
- The first-order Taylor series approximation of observation i at time $t+1$ using the IMU pose perturbation $\xi_{t+1|t+1}$ is:

$$\begin{aligned}
\mathbf{z}_{t+1,i} &= M\pi\left({}_O T_I \exp\left(\hat{\xi}_{t+1|t+1}\right) \boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right) \\
&\approx M\pi\left({}_O T_I \left(I + \hat{\xi}_{t+1|t+1}\right) \boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right) \\
&= M\pi\left({}_O T_I \boldsymbol{\mu}_{t+1|t}\mathbf{m}_j + {}_O T_I \left(\boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right)^{\odot} \xi_{t+1|t+1}\right) \\
&\approx \underbrace{M\pi\left({}_O T_I \boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right)}_{\hat{\mathbf{z}}_{t+1,i}} + \underbrace{M\frac{d\pi}{d\mathbf{q}}\left({}_O T_I \boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right) {}_O T_I \left(\boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right)^{\odot}}_{H_{i,t+1|t}} \xi_{t+1|t+1}
\end{aligned}$$

where for homogeneous coordiantes $r \in \mathbb{R}^4$ and $\hat{\xi} \in \mathfrak{se}(3)$:

$$\hat{\xi} r = r^{\odot} \xi \qquad \begin{bmatrix} s \\ \lambda \end{bmatrix}^{\odot} = \begin{bmatrix} \lambda I & -\hat{s} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 6}$$

EKF Update Step

- **Prior**: $\boldsymbol{\mu}_{t+1|t} \in SE(3)$ and $\Sigma_{t+1|t} \in \mathbb{R}^{6\times 6}$
- Known: calibration matrix M, extrinsics ${}_O T_I \in SE(3)$, homogeneous coordinate landmark positions $\mathbf{m} \in \mathbb{R}^{4\times M}$, new observation $z_{t+1} \in \mathbb{R}^{4\times N_t}$
- Compute the predicted observation based on $\boldsymbol{\mu}_{t+1|t}$ and known correspondences:

$$\hat{\mathbf{z}}_{t+1,i} := M\pi\left({}_O T_I \boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right) \qquad \text{for } i = 1, \ldots, N_t$$

- Compute the Jacobian of $\hat{\mathbf{z}}_{t+1,i}$ with respect to T_{t+1} evaluated at $\boldsymbol{\mu}_{t+1|t}$

$$H_{i,t+1|t} = M\frac{d\pi}{d\mathbf{q}}\left({}_O T_I \boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right) {}_O T_I \left(\boldsymbol{\mu}_{t+1|t}\mathbf{m}_j\right)^{\odot} \in \mathbb{R}^{4\times 6}$$

- Perform the EKF update:

$$K_{t+1|t} = \Sigma_{t+1|t} H_{t+1|t}^T \left(H_{t+1|t}\Sigma_{t+1|t}H_{t+1|t}^T + I \otimes V\right)^{-1}$$

$$\boldsymbol{\mu}_{t+1|t+1} = \exp\left(\left(K_{t+1|t}(\mathbf{z}_{t+1} - \hat{\mathbf{z}}_{t+1})\right)^{\wedge}\right)\boldsymbol{\mu}_{t+1|t}$$

$$\Sigma_{t+1|t+1} = (I - K_{t+1|t}H_{t+1|t})\Sigma_{t+1|t}$$

$$H_{t+1|t} = \begin{bmatrix} H_{1,t+1|t} \\ \vdots \\ H_{N_{t+1},t+1|t} \end{bmatrix}$$

$SE(3)$ Geometry Review

$SO(3)$ Identities and Approximations

- Lie Algebra $\mathfrak{so}(3)$

$$
\begin{aligned}
\hat{\boldsymbol{\theta}} &= \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}^{\wedge} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \\
\hat{\boldsymbol{\theta}}^T &= -\hat{\boldsymbol{\theta}} \\
\hat{\boldsymbol{\theta}}\boldsymbol{\theta} &= 0 \\
(A\boldsymbol{\theta})^{\wedge} &= \hat{\boldsymbol{\theta}}\left(\operatorname{tr}(A)I - A\right) - A^T\hat{\boldsymbol{\theta}} \qquad A \in \mathbb{R}^{3\times 3} \\
\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} &= \boldsymbol{\phi}\boldsymbol{\theta}^T - \left(\boldsymbol{\theta}^T\boldsymbol{\phi}\right) I \qquad \boldsymbol{\phi} \in \mathbb{R}^3 \\
\hat{\boldsymbol{\theta}}^{2k+1} &= \left(-\boldsymbol{\theta}^T\boldsymbol{\theta}\right)^k \hat{\boldsymbol{\theta}}
\end{aligned}
$$

$SO(3)$ Identities and Approximations

- Lie Group $SO(3)$

$$
\begin{aligned}
R &= \exp\left(\hat{\boldsymbol{\theta}}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\hat{\boldsymbol{\theta}}^n = I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2 \approx I + \hat{\boldsymbol{\theta}} \\
R^{-1} &= R^T = \exp\left(-\hat{\boldsymbol{\theta}}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\hat{\boldsymbol{\theta}}\right)^n \approx I - \hat{\boldsymbol{\theta}} \\
\det(R) &= 1 \\
\operatorname{tr}(R) &= 2\cos\|\boldsymbol{\theta}\| + 1 \\
R\boldsymbol{\theta} &= \boldsymbol{\theta} \\
R\hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{\theta}}R \\
(R\mathbf{m})^\wedge &= R\hat{\mathbf{m}}R^T \qquad\qquad \mathbf{m} \in \mathbb{R}^3 \\
\exp\left((R\mathbf{m})^\wedge\right) &= R\exp\left(\hat{\mathbf{m}}\right)R^T
\end{aligned}
$$

$SO(3)$ Identities and Approximations

- Jacobian of $SO(3)$

$$J_L(\boldsymbol{\theta}) = I + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\|\boldsymbol{\theta}\| - \sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\right)\hat{\boldsymbol{\theta}}^2 \approx I + \frac{1}{2}\hat{\boldsymbol{\theta}}$$

$$J_L(\boldsymbol{\theta})^{-1} = I - \frac{1}{2}\hat{\boldsymbol{\theta}} + \left(\frac{1+\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2} - \frac{1}{2\|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}}^2 \approx I - \frac{1}{2}\hat{\boldsymbol{\theta}}$$

$$\exp\left((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^{\wedge}\right) \approx \exp\left((J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta})^{\wedge}\right)\exp\left(\hat{\boldsymbol{\theta}}\right)$$

$$R = I + \hat{\boldsymbol{\theta}} J_L(\boldsymbol{\theta})$$

$$J_L(\boldsymbol{\theta}) = R J_L(-\boldsymbol{\theta})$$

$SE(3)$ Identities and Approximations

- Lie Algebra $\mathfrak{se}(3)$

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \qquad \overset{\curlywedge}{\boldsymbol{\xi}} = ad(\hat{\boldsymbol{\xi}}) = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}^{\curlywedge} = \begin{bmatrix} \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\rho}} \\ \mathbf{0} & \hat{\boldsymbol{\theta}} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$\overset{\curlywedge}{\boldsymbol{\zeta}}\boldsymbol{\xi} = -\overset{\curlywedge}{\boldsymbol{\xi}}\boldsymbol{\zeta} \qquad \boldsymbol{\zeta} \in \mathbb{R}^6$$

$$\overset{\curlywedge}{\boldsymbol{\xi}}\boldsymbol{\xi} = 0$$

$$\hat{\boldsymbol{\xi}}^4 + \left(\mathbf{m}^T\mathbf{m}\right)\hat{\boldsymbol{\xi}}^2 = 0 \qquad \mathbf{m} \in \mathbb{R}^3$$

$$\left(\overset{\curlywedge}{\boldsymbol{\xi}}\right)^5 + 2\left(\mathbf{m}^T\mathbf{m}\right)\left(\overset{\curlywedge}{\boldsymbol{\xi}}\right)^3 + \left(\mathbf{m}^T\mathbf{m}\right)^2\overset{\curlywedge}{\boldsymbol{\xi}} = 0$$

$$\mathbf{m}^{\odot} = \begin{bmatrix} s \\ \lambda \end{bmatrix}^{\odot} = \begin{bmatrix} \lambda I & -\hat{s} \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \in \mathbb{R}^{4\times 6} \qquad \mathbf{m}^{\circledcirc} = \begin{bmatrix} s \\ \lambda \end{bmatrix}^{\circledcirc} = \begin{bmatrix} 0 & s \\ -\hat{s} & 0 \end{bmatrix} \in \mathbb{R}^{6\times 4}$$

$$\hat{\boldsymbol{\xi}}\mathbf{m} = \mathbf{m}^{\odot}\boldsymbol{\xi} \qquad \mathbf{m}^T\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}^T\mathbf{m}^{\circledcirc}$$

$SE(3)$ Identities and Approximations

- Lie Group $SE(3)$

$$
\begin{aligned}
T &= \exp\left(\hat{\boldsymbol{\xi}}\right) = \begin{bmatrix} \exp\left(\hat{\boldsymbol{\theta}}\right) & J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\hat{\boldsymbol{\xi}}^n = I + \hat{\xi} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\xi}^2 + \left(\frac{\|\boldsymbol{\theta}\| - \sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\right)\hat{\xi}^3 \approx I + \hat{\boldsymbol{\xi}} \\
T^{-1} &= \exp\left(-\hat{\boldsymbol{\xi}}\right) = \begin{bmatrix} \exp\left(-\hat{\boldsymbol{\theta}}\right) & -\exp\left(-\hat{\boldsymbol{\theta}}\right) J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\hat{\boldsymbol{\xi}}\right)^n \approx I - \hat{\boldsymbol{\xi}} \\
\det(T) &= 1 \\
\operatorname{tr}(T) &= 2\cos\|\boldsymbol{\theta}\| + 2 \\
T\hat{\boldsymbol{\xi}} &= \hat{\boldsymbol{\xi}}T
\end{aligned}
$$

$SE(3)$ Identities and Approximations

- Lie Group $Ad(SE(3))$

$$\mathcal{T} = Ad(T) = \exp\left(\overset{\curlywedge}{\boldsymbol{\xi}}\right) = \begin{bmatrix} \exp\left(\hat{\boldsymbol{\theta}}\right) & (J_L(\boldsymbol{\theta})\boldsymbol{\rho})^\wedge \exp\left(\hat{\boldsymbol{\theta}}\right) \\ \mathbf{0} & \exp\left(\hat{\boldsymbol{\theta}}\right) \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \overset{\curlywedge}{\boldsymbol{\xi}}^n = I + \left(\frac{3\sin\|\boldsymbol{\theta}\| - \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|}\right)\overset{\curlywedge}{\xi} + \left(\frac{4 - \|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\| - 4\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^2}\right)(\overset{\curlywedge}{\xi})^2$$

$$+ \left(\frac{\sin\|\boldsymbol{\theta}\| - \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^3}\right)(\overset{\curlywedge}{\xi})^3 + \left(\frac{2 - \|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\| - 2\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^4}\right)(\overset{\curlywedge}{\xi})^4 \approx I + \overset{\curlywedge}{\boldsymbol{\xi}}$$

$$\mathcal{T}^{-1} = \exp\left(-\overset{\curlywedge}{\boldsymbol{\xi}}\right) = \begin{bmatrix} \exp\left(-\hat{\boldsymbol{\theta}}\right) & -\exp\left(-\hat{\boldsymbol{\theta}}\right)(J_L(\boldsymbol{\theta})\boldsymbol{\rho})^\wedge \\ \mathbf{0} & \exp\left(-\hat{\boldsymbol{\theta}}\right) \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\overset{\curlywedge}{\boldsymbol{\xi}}\right)^n \approx I - \overset{\curlywedge}{\boldsymbol{\xi}}$$

$$\mathcal{T}\boldsymbol{\xi} = \boldsymbol{\xi} \qquad \mathcal{T}\overset{\curlywedge}{\boldsymbol{\xi}} = \overset{\curlywedge}{\boldsymbol{\xi}}\mathcal{T}$$

$$(\mathcal{T}\boldsymbol{\zeta})^\wedge = T\hat{\boldsymbol{\zeta}}T^{-1} \qquad (\overset{\curlywedge}{\mathcal{T}\boldsymbol{\zeta}}) = \mathcal{T}\overset{\curlywedge}{\boldsymbol{\zeta}}\mathcal{T}^{-1} \qquad \boldsymbol{\zeta} \in \mathbb{R}^6$$

$$\exp\left((\mathcal{T}\boldsymbol{\zeta})^\wedge\right) = T\exp\left(\hat{\boldsymbol{\zeta}}\right)T^{-1} \qquad \exp\left((\overset{\curlywedge}{\mathcal{T}\boldsymbol{\zeta}})\right) = \mathcal{T}\exp\left(\overset{\curlywedge}{\boldsymbol{\zeta}}\right)\mathcal{T}^{-1}$$

$$(T\mathbf{m})^\odot = T\mathbf{m}^\odot\mathcal{T}^{-1}$$

$$\left((T\mathbf{m})^\odot\right)^T (T\mathbf{m})^\odot = \mathcal{T}^{-T}\left(\mathbf{m}^\odot\right)^T \mathbf{m}^\odot\mathcal{T}^{-1}$$

$SE(3)$ Identities and Approximations

- Jacobian of $SE(3)$

$$\mathcal{J}_L(\xi) = \begin{bmatrix} J_L(\boldsymbol{\theta}) & Q_L(\xi) \\ 0 & J_L(\boldsymbol{\theta}) \end{bmatrix}$$

$$= I + \left(\frac{4 - \|\boldsymbol{\theta}\| \sin \|\boldsymbol{\theta}\| - 4 \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^2} \right) \overset{\curlywedge}{\boldsymbol{\xi}} + \left(\frac{4\|\boldsymbol{\theta}\| - 5 \sin \|\boldsymbol{\theta}\| + \|\boldsymbol{\theta}\| \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^3} \right) \overset{\curlywedge}{\boldsymbol{\xi}}^2$$

$$+ \left(\frac{2 - \|\boldsymbol{\theta}\| \sin \|\boldsymbol{\theta}\| - 2 \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^4} \right) \overset{\curlywedge}{\boldsymbol{\xi}}^3 + \left(\frac{2\|\boldsymbol{\theta}\| - 3 \sin \|\boldsymbol{\theta}\| + \|\boldsymbol{\theta}\| \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^5} \right) \overset{\curlywedge}{\boldsymbol{\xi}}^4$$

$$\approx I + \frac{1}{2} \overset{\curlywedge}{\boldsymbol{\xi}}$$

$$\mathcal{J}_L(\xi)^{-1} = \begin{bmatrix} J_L(\boldsymbol{\theta})^{-1} & -J_L(\boldsymbol{\theta})^{-1} Q_L(\xi) J_L(\boldsymbol{\theta})^{-1} \\ \mathbf{0} & J_L(\boldsymbol{\theta})^{-1} \end{bmatrix} \approx I - \frac{1}{2} \overset{\curlywedge}{\boldsymbol{\xi}}$$

$$Q_L(\xi) = \frac{1}{2} \hat{\boldsymbol{\rho}} + \left(\frac{\|\boldsymbol{\theta}\| - \sin \|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3} \right) \left(\hat{\boldsymbol{\theta}} \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}} \right)$$

$$+ \left(\frac{\|\boldsymbol{\theta}\|^2 + 2 \cos \|\boldsymbol{\theta}\| - 2}{2\|\boldsymbol{\theta}\|^4} \right) \left(\hat{\boldsymbol{\theta}}^2 \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}}^2 - 3 \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}} \right)$$

$$+ \left(\frac{2\|\boldsymbol{\theta}\| - 3 \sin \|\boldsymbol{\theta}\| + \|\boldsymbol{\theta}\| \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^5} \right) \left(\hat{\boldsymbol{\theta}} \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}}^2 + \hat{\boldsymbol{\theta}}^2 \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}} \right)$$

$$\mathcal{T} = I + \overset{\curlywedge}{\boldsymbol{\xi}} \mathcal{J}_L(\boldsymbol{\xi}) \qquad \mathcal{J}_L(\boldsymbol{\xi}) \overset{\curlywedge}{\boldsymbol{\xi}} = \overset{\curlywedge}{\boldsymbol{\xi}} \mathcal{J}_L(\boldsymbol{\xi}) \qquad \mathcal{J}_L(\boldsymbol{\xi}) = \mathcal{T} \mathcal{J}_L(-\boldsymbol{\xi})$$