ECE276A: Sensing & Estimation in Robotics
Lecture 16: Localization and Odometry from Point Features

Instructor:

Nikolay Atanasov: natanasov@ucsd.edu

Teaching Assistants:

Qiaojun Feng: qif007@eng.ucsd.edu
Tianyu Wang: tiw161@eng.ucsd.edu
Ibrahim Akbar: iakbar@eng.ucsd.edu
You-Yi Jau: yjau@eng.ucsd.edu
Harshini Rajachander: hrajacha@eng.ucsd.edu

UC San Diego
JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

Localization and Odometry from Point Features

- Observation model: relates a sensor observation z_i obtained from robot position p and orientation θ or R with the position m_i of the landmark that generated z_i:
  - **Position Sensor**: $z_i = R^T(m_i - p)$
  - **Range Sensor**: $z_i = \|m_i - p\|_2$
  - **Bearing Sensor**: $z_i = \arctan\left(\frac{m_{i,y} - p_y}{m_{i,x} - p_x}\right) - \theta$
  - **Camera Sensor**: $z_i = K\pi\left(R^T(m_i - p)\right)$

- **Localization Problem**: Given landmark positions m_i and measurements z_i at one time instance, determine the global robot position p and orientation θ or R

- **Odometry Problem**: Given measurements $z_{i,t}$, $z_{i,t+1}$ at two time instances, determine the relative position ${}_tp_{t+1}$ and orientation ${}_t\theta_{t+1}$ or ${}_tR_{t+1}$ between the two robot frames at time t and $t+1$

2-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $p \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $m_1, m_2 \in \mathbb{R}^2$ (world frame) and **relative position** measurements (body frame):

$$z_i = R^T(\theta)(m_i - p) \in \mathbb{R}^2, \quad i = 1, 2$$

- Let $\delta z := z_1 - z_2$ and $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so that:

$$m_1 - m_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (z_1 - z_2) = \begin{bmatrix} \delta z & J\delta z \end{bmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

- As long as $\det \begin{bmatrix} \delta z & J\delta z \end{bmatrix} = \|\delta z\|_2^2 = \|m_1 - m_2\|_2^2 \neq 0$, we can compute:

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{\|\delta z\|_2^2} \begin{bmatrix} \delta z_x & \delta z_y \\ -\delta z_y & \delta z_x \end{bmatrix} (m_1 - m_2) \qquad \boxed{\theta = \mathbf{atan2}(\sin\theta, \cos\theta)}$$

- Given the orientation θ, we can then obtain the robot position:

$$\boxed{p = \frac{1}{2}\left((m_1 + m_2) - R(\theta)(z_1 + z_2)\right)}$$

3-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $m_1, m_2, m_3 \in \mathbb{R}^3$ (world frame) and **relative position** measurements (body frame):

$$z_i = R^T(m_i - p) \in \mathbb{R}^3, \quad i = 1, 2, 3$$

- Let $m_{ij} := m_i - m_j$ and $z_{ij} = z_i - z_j$ and compute:

$$m_{12} \times m_{13} = (Rz_{12}) \times (Rz_{13}) = R(z_{12} \times z_{13})$$

- The vector $m_{12} \times m_{13}$ provides orthogonal information to m_1 and m_2 and can be used to estimate the orientation R **as long as the three features are not all on the same line**:

$$\begin{bmatrix} m_1 & m_2 & m_{12} \times m_{13} \end{bmatrix} = R \begin{bmatrix} z_1 & z_2 & z_{12} \times z_{13} \end{bmatrix}$$

$$\boxed{R = \begin{bmatrix} m_1 & m_2 & m_{12} \times m_{13} \end{bmatrix} \begin{bmatrix} z_1 & z_2 & z_{12} \times z_{13} \end{bmatrix}^{-1}}$$

- Given the orientation R, we can then obtain the robot position:

$$\boxed{p = \frac{1}{3} \sum_{i=1}^{3} (m_i - Rz_i)}$$

3-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: n landmark positions $m_i \in \mathbb{R}^3$ (world frame) and **relative position** measurements (body frame):

$$z_i = R^T(m_i - p) \in \mathbb{R}^3, \quad i = 1, \ldots, n$$

- Define the landmark centroids in the world and body frames:

$$\bar{m} := \frac{1}{n}\sum_{i=1}^{n} m_i \qquad \bar{z} := \frac{1}{n}\sum_{i=1}^{n} z_i \qquad \boxed{\bar{m} = p + R\bar{z}}$$

- Let $\delta m_i := m_i - \bar{m}$ and $\delta z_i := z_i - \bar{z}$ so that $\delta m_i = R\delta z_i$ for $i = 1, \ldots, n$
- Estimate the orientation via least-squares:

$$\min_R \sum_{i=1}^{n} \|\delta m_i - R\delta z_i\|_2^2 = \min_R \sum_{i=1}^{n} \delta m_i^T \delta m_i - 2\delta m_i^T R\delta z_i - \delta z_i^T \underbrace{R^T R}_{I_{3\times 3}} \delta z_i$$

Kabsch Algorithm

- Find transformation p, R to match two sets $\{m_i\}$ and $\{z_i\}$ of 3-D points
- Given the rotation R, the optimal translation is: $\boxed{p = \bar{m} - R\bar{z}}$
- Need to solve a least-squares problem in $SO(3)$ to determine R:

$$\max_R \sum_{i=1}^{n} \delta m_i^T R \delta z_i = \operatorname{tr}\left(Q^T R\right) \qquad \text{where } Q^T := \sum_{i=1}^{n} \delta z_i \delta m_i^T$$
$$\text{s.t. } R^T R = I_{3\times 3}, \ \det(R) = 1$$

- Let $Q = Z\Sigma M^T$ be a singular value decomposition with $\Sigma_{ii} \geq 0$, $\det M = \pm 1$, and $\det Z = \pm 1$
- Define a <u>unitary</u> matrix $U := Z^T R M \in \mathbb{R}^{n\times n}$.
- $\operatorname{tr}(Q^T R) = \operatorname{tr}(\Sigma Z^T R M) = \operatorname{tr}(\Sigma U) = \sum_{i=1}^{n} \Sigma_{ii} U_{ii}$ and since $\Sigma_{ii} \geq 0$ and $\det(U) = \pm 1$, the objective is maximized for:

$$U = Z^T R M = I_{n\times n} \quad \underset{\text{reflection}}{\overset{\text{avoids}}{\Rightarrow}} \quad R = Z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(ZM^T) \end{bmatrix} M^T$$

Iterative Closest Point (ICP)

- Kabsch assumes **known point correspondences** (data association)
- The ICP Algorithm finds a rigid body transformation to match two sets $\{m_i\}$ and $\{z_i\}$ of 3-D points with **unknown correspondences**
- Start with (p_0, R_0) (**sensitive to initial guess**) and iterate
  1. Given (p, R), find correspondences $m_i \leftrightarrow z_j$ based on **closest points**:

$$j^* = \arg\min_j \|m_i - (Rz_j + p)\|_2^2$$

  2. Given correspondences, find (p, R) using the **Kabsch** algorithm

Probabilistic ICP

- Place a small probabilistic ball around each m_i to define a mixture distribution for the data:

$$p(x) = \sum_i \alpha_i \pi(x; m_i, \sigma_i^2 I_{3\times 3})$$

- Find parameters (p, R) to max the likelihood of $\{Rz_j + p\}$ under $p(x)$:

$$\max_{p,R} \sum_j \log \sum_i \alpha_i \pi(Rz_j + p; m_i, \sigma_i^2 I_{3\times 3})$$

- Use **EM**!
- ICP is a special case with $\sigma_i^2 \to 0$
- **Robustness**: use $\exp\left(-\frac{|x-m_i|^\beta}{2\sigma_i^2}\right)$ with $\beta \in (0,2)$ instead of $\exp\left(-\frac{|x-m_i|^2}{2\sigma_i^2}\right)$

2-D Odometry from Relative Position Measurements

- **Goal**: determine the relative transformation ${}_tp_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: relative position measurements $z_{t,1}, z_{t,2} \in \mathbb{R}^2$ and $z_{t+1,1}, z_{t+1,2} \in \mathbb{R}^2$ at consecutive time steps to two **unknown** landmarks
- If we consider the robot frame at time t to be the "world frame", **this problem is the same as 2-D localization from relative position measurements** with $m_i := z_{t,i}$, $z_i := z_{t+1,i}$, $p := {}_tp_{t+1}$, $\theta := {}_t\theta_{t+1}$

3-D Odometry from Relative Position Measurements

- **Goal**: determine the relative transformation ${}_t p_{t+1} \in \mathbb{R}^3$ and ${}_t R_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t
- **Given**: relative position measurements $z_{t,i} \in \mathbb{R}^3$ and $z_{t+1,i} \in \mathbb{R}^3$ at consecutive time steps to n **unknown** landmarks
- If we consider the robot frame at time t to be the "world frame", **this problem is the same as 3-D localization from relative position measurements** with $m_i := z_{t,i}$, $z_i := z_{t+1,i}$, $p := {}_t p_{t+1}$, $R := {}_t R_{t+1}$

Summary: Rel. Position Measurements $z_i = R^T(m_i - p)$

- **Localization**

$m_1, m_2, z_1, z_2 \in \mathbb{R}^2$	$(m_1 - m_2) = R(\theta)(z_1 - z_2)$ $p = \frac{1}{2}\sum_{i=1}^{2}(m_i - Rz_i)$
$m_1, z_i \in \mathbb{R}^3, \; i = 1, 2, 3$ $m_{ij} := m_i - m_j, \;\; z_{ij} := z_i - z_j$	$\begin{bmatrix} m_1 & m_2 & m_{12} \times m_{13} \end{bmatrix} = R \begin{bmatrix} z_1 & z_2 & z_{12} \times z_{13} \end{bmatrix}$ $p = \frac{1}{3}\sum_{i=1}^{3}(m_i - Rz_i)$
$m_i, z_i \in \mathbb{R}^3, \; i = 1, \ldots, n$ $\delta m_i := m_i - \frac{1}{n}\sum_{j=1}^{n} m_j,$ $\delta z_i := z_i - \frac{1}{n}\sum_{j=1}^{n} z_j$	$R = \underset{R \in SO(3)}{\arg\max} \sum_{i=1}^{n} \delta m_i^T R \delta z_i$ $\xlongequal[SVD\left(\sum_{i=1}^{n} \delta m_i \delta z_i^T\right) = Z\Sigma M^T]{\text{Kabsch algorithm}} Z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(ZM^T) \end{bmatrix} M^T$ $p = \frac{1}{n}\sum_{i=1}^{n}(m_i - Rz_i)$

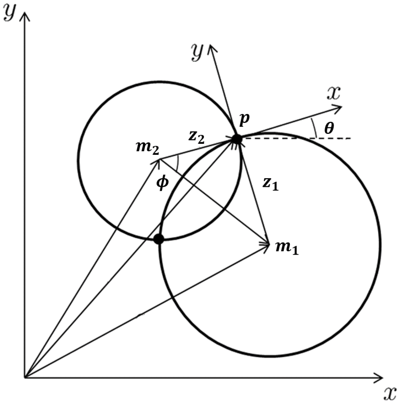
- **Odometry**: same with $m_i = z_{t,i}, \; z_i := z_{t+1,i}, \; p := {}_t p_{t+1}, \; R := {}_t R_{t+1}$

2-D Localization from Range Measurements

- **Goal**: determine the robot position $p \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $m_1, m_2 \in \mathbb{R}^2$ (world frame) and **range** measurements (body frame):

$$z_i = \|m_i - p\|_2 \in \mathbb{R}, \quad i = 1, 2$$

- Because all possible positions whose distance to m_1 is z_1 is a circle, the possible robot positions are given by the intersection of two circles



2-D Localization from Range Measurements

- Applying the law of cosines to the triangle gives:

$$z_2^2 = z_1^2 + \|m_2 - m_1\|_2^2 - 2z_1\|m_2 - m_1\|_2 \cos\phi$$

- Solving for ϕ and then the circle intersection points provides the possible robot positions:

$$p = m_2 + z_2 R(\pm\phi)\frac{m_1 - m_2}{\|m_1 - m_2\|_2}$$

- The orientation of the robot θ is **not identifiable**

2-D Localization from Range Measurements

- **Pose disambiguation**: the robot can make a move with known translation p_Δ (measured in the frame at time t) and take two new range measurements
- There are 2 possible robot positions at each time frame for a total of 4 combinations but comparing $\|p_{t+1} - p_t\|_2$ to the known $\|p_\Delta\|_2$ leaves only <u>two valid options</u> (and we cannot distinguish between them)
- To obtain the orientation, we use geometric constraints:

$$p_{t+1} - p_t = R(\theta_t)p_\Delta = \begin{bmatrix} p_{\Delta,x} & -p_{\Delta,y} \\ p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} \begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix}$$

- As long as $\det \begin{bmatrix} p_{\Delta,x} & -p_{\Delta,y} \\ p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} = \|p_\Delta\|_2^2 \neq 0$, we can compute:

$$\begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix} = \frac{1}{\|p_\Delta\|_2^2} \begin{bmatrix} p_{\Delta,x} & p_{\Delta,y} \\ -p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} (p_{t+1} - p_t)$$

$$\theta_t = \mathbf{atan2}(\sin\theta_t, \cos\theta_t)$$

3-D Localization from Range Measurements

- **Goal**: determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $m_1, m_2, m_3 \in \mathbb{R}^3$ (world frame) and **range** measurements (body frame):

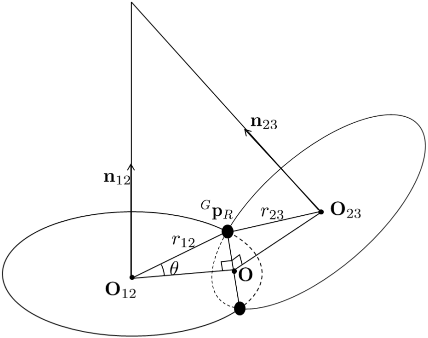
$$z_i = \|m_i - p\|_2 \in \mathbb{R}, \quad i = 1, 2, 3$$

- All possible positions whose distance to m_1 is z_1 is a sphere
- The possible robot positions are the intersections of three spheres
- To find the intersection of 3 spheres, we first find the intersection of sphere one and two (a circle) and of sphere two and three (a circle). The intersection of these two circles gives the possible robot positions.
- **Degenerate case**: all landmarks are on the same line – the intersection of the spheres is a circle with infinitely many possible robot positions

3-D Localization from Range Measurements

- **Intersecting circle of spheres with radii z_1 and z_2**: center O_{12}, radius r_{12}, normal vector n_{12} (perpendicular to the circle plane)
- Law of Cosines: $z_2^2 = z_1^2 + \|m_2 - m_1\|_2^2 - 2z_1\|m_2 - m_1\|_2 \cos\theta_{12}$
- Geometric relationships:

$$O_{12} = m_1 + z_1 \cos\theta_{12} n_{12}$$
$$r_{12} = z_1|\sin(\theta_{12})|$$
$$n_{12} = \frac{m_2 - m_1}{\|m_2 - m_1\|_2}$$



- **Intersecting circle of spheres with radii z_2 and z_3**: center O_{23}, radius r_{23}, normal vector n_{23} (perpendicular to the circle plane):

$$O_{23} = m_2 + z_2 \cos\theta_{23} n_{23} \qquad r_{23} = z_2|\sin(\theta_{23})| \qquad n_{23} = \frac{m_3 - m_2}{\|m_3 - m_2\|_2}$$

3-D Localization from Range Measurements

- The intersecting points of the two circles can be obtained from the geometric relationships:

$$\begin{aligned} n_{12}^T(O_{12} - O) &= 0 \\ n_{23}^T(O_{23} - O) &= 0 \\ (n_{12} \times n_{23})^T(O_{12} - O) &= 0 \end{aligned} \qquad \begin{bmatrix} n_{12}^T \\ n_{23}^T \\ (n_{12} \times n_{23})^T \end{bmatrix} O = \begin{bmatrix} n_{12}^T O_{12} \\ n_{23}^T O_{23} \\ (n_{12} \times n_{23})^T O_{12} \end{bmatrix}$$

- As long as the three landmarks are not on the same line, we can uniquely solve for O:

$$\det \begin{bmatrix} n_{12}^T \\ n_{23}^T \\ (n_{12} \times n_{23})^T \end{bmatrix} \neq 0 \qquad \Leftrightarrow \qquad n_{12} \text{ and } n_{23} \text{ not colinear}$$

- The two possible robot positions are:

$$p = O_{12} + r_{12} R(n_{12}, \pm\theta) \frac{O - O_{12}}{\|O - O_{12}\|_2} \qquad \cos\theta = \frac{\|O - O_{12}\|_2}{r_{12}}$$

- As in the 2-D case, the robot orientation R is **not identifiable**

3-D Localization from Range Measurements

- **Pose disambiguation**: the robot can make a move with known translation $p_\Delta \in \mathbb{R}^3$ and rotation $R_\Delta \in SO(3)$ and take three new range measurements

- As in the 2-D case, after eliminating the impossible robot positions, we should be left with only two options for p_t and p_{t+1}

- Given p_t, p_{t+1}, p_Δ, and R_Δ, we can now obtain R_t

$$p_{t+1} = p_t + R_t p_\Delta$$

- This is not sufficient because the rotation about p_Δ is not identifiable

- The robot needs to **move a second time** to a third pose p_{t+2}, R_{t+2} with known translation $p_{\Delta,2} \in \mathbb{R}^3$ and take three more range measurements to the three landmarks:

$$p_{t+2} = p_{t+1} + R_{t+1} p_{\Delta,2} = p_{t+1} + R_t R_\Delta p_{\Delta,2}$$

3-D Localization from Range Measurements

- Putting the previous two equations together:

$$\begin{aligned} p_{t+1} - p_t &= R_t p_\Delta \\ p_{t+2} - p_{t+1} &= R_t R_\Delta p_{\Delta,2} \end{aligned}$$

- Taking a cross product between the two:

$$(p_{t+1} - p_t) \times (p_{t+2} - p_{t+1}) = R_t(p_\Delta \times R_\Delta p_{\Delta,2})$$

- As long as $U := [p_\Delta,\ R_\Delta p_{\Delta,2},\ p_\Delta \times R_\Delta p_{\Delta,2})]$ is nonsingular, i.e., p_Δ and $R_\Delta p_{\Delta,2}$ are not co-linear or equivalently **the three robot positions are not on the same line**, we can determine the robot orientation:

$$\boxed{R_t = [(p_{t+1} - p_t),\ (p_{t+2} - p_{t+1}),\ (p_{t+1} - p_t) \times (p_{t+2} - p_{t+1})]U^{-1}}$$

2-D Odometry from Range Measurements

- **Goal**: determine the relative transformation ${}_t p_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: range measurements $z_{t,i} \in \mathbb{R}$ and $z_{t+1,i} \in \mathbb{R}$ at consecutive time steps to n **unknown** landmarks
- Let $m_{t+1,i}$ be the relative position to the i-th landmark at $t+1$ so that:

$$
\begin{aligned}
z_{t+1,i} &= \|m_{t+1,i}\|_2 \\
z_{t,i} &= \|{}_t p_{t+1} + R({}_t\theta_{t+1}) m_{t+1,i}\|_2
\end{aligned}
$$

- Squaring and combining these equations, we get:

$$
[{}_t p_{t+1}]^T \, {}_t p_{t+1} + 2 m_{t+1,i}^T R^T({}_t\theta_{t+1}) {}_t p_{t+1} = z_{t,i}^2 - z_{t+1,i}^2, \qquad i = 1, \ldots, n
$$

- We have n equations with $n+3$ unknowns (3 for the relative pose and n for the unknown directions to the landmarks at $t+1$), which is **not solvable**.

3-D Odometry from Range Measurements

- **Goal**: determine the relative transformation ${}_t p_{t+1} \in \mathbb{R}^3$ and ${}_t R_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t
- **Given**: range measurements $z_{t,i} \in \mathbb{R}$ and $z_{t+1,i} \in \mathbb{R}$ at consecutive time steps to n **unknown** landmarks
- Following the same derivation as in the 2-D case, we obtain:

$$[{}_t p_{t+1}]^T \, {}_t p_{t+1} + 2 m_{t+1,i}^T \, [{}_t R_{t+1}]^T \, {}_t p_{t+1} = z_{t,i}^2 - z_{t+1,i}^2, \qquad i = 1, \ldots, n$$

- We have n equations with $2n+6$ unknowns (6 for the relative pose and $2n$ for the unknown directions to the landmarks at $t+1$), which is **not solvable**.

Summary: Range Measurements $z_i = \|m_i - p\|_2$

- **2-D Localization**: given $m_1, m_2 \in \mathbb{R}^2$ and $z_1, z_2 \in \mathbb{R}$
  1. Law of Cosines: $z_2^2 = z_1^2 + \|m_2 - m_1\|_2^2 - 2z_1\|m_2 - m_1\|_2 \cos\theta$
  2. Position: $p = m_2 + z_2 R(\pm\theta)\frac{m_1 - m_2}{\|m_1 - m_2\|_2}$
  3. Move with known p_Δ, θ_Δ (in frame t)
  4. Orientation: $(p_{t+1} - p_t) = R(\theta_t)p_\Delta$
- **3-D Localization**: given $m_1, m_2, m_3 \in \mathbb{R}^3$ and $z_1, z_2, z_3 \in \mathbb{R}$
  1. Intersection of 2 circles with centers O_{12}, O_{23}, radii r_{12}, r_{23}, normals n_{12}, n_{23} obtained via Law of Cosines and point O on intersecting line:
  $$\begin{bmatrix} n_{12}^T \\ n_{23}^T \\ (n_{12} \times n_{23})^T \end{bmatrix} O = \begin{bmatrix} n_{12}^T O_{12} \\ n_{23}^T O_{23} \\ (n_{12} \times n_{23})^T O_{12} \end{bmatrix}$$
  2. Position: $p = O_{12} + r_{12}R(n_{12}, \pm\theta)\frac{O - O_{12}}{\|O - O_{12}\|_2}$, where $\cos\theta = \frac{\|O - O_{12}\|_2}{r_{12}}$
  3. Move twice with known $p_\Delta, R_\Delta, p_{\Delta,2}, R_{\Delta,2}$
  4. Orientation: as long as $U := [p_\Delta,\ R_\Delta p_{\Delta,2},\ p_\Delta \times R_\Delta p_{\Delta,2})]$ is nonsingular:
  $$R_t = [(p_{t+1} - p_t),\ (p_{t+2} - p_{t+1}),\ (p_{t+1} - p_t) \times (p_{t+2} - p_{t+1})]U^{-1}$$
- **Odometry**: not solvable

2-D Localization from Bearing Measurements

- **Goal**: determine the robot position $p \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $m_1, m_2 \in \mathbb{R}^2$ (world frame) and **bearing** measurements (body frame):

$$z_i = \arctan\left(\frac{m_{i,y} - p_y}{m_{i,x} - p_x}\right) - \theta \in \mathbb{R}, \quad i = 1, 2$$

- The bearing constraints are equivalent to:

$$\frac{m_i - p}{\|m_i - p\|_2} = \begin{bmatrix} \cos(z_i + \theta) \\ \sin(z_i + \theta) \end{bmatrix} = R(z_i + \theta) e_1 \quad \Rightarrow \quad R^T(z_i)(m_i - p) = \|m_i - p\|_2 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

- To eliminate θ, the two constraints can be combined via:

$$\begin{aligned} 0 &= \|m_1 - p\|_2 \begin{bmatrix} \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \|m_2 - p\|_2 \\ &= \|m_1 - p\|_2 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}^T R\left(\frac{\pi}{2}\right) \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \|m_2 - p\|_2 \end{aligned}$$

2-D Localization from Bearing Measurements

- The previous equation is quadratic in p:

$$(m_1 - p)^T R(z_1) R\left(\frac{\pi}{2}\right) R^T(z_2)(m_2 - p) = 0$$

- Let $\eta := z_1 - z_2 + \pi/2$, so that:

$$p^T R(\eta) p - \left(m_1^T R(\eta) + m_2^T R^T(\eta)\right) p + m_1^T R(\eta) m_2 = 0$$

- Use the following to solve the quadratic equation:
  - $p^T R(\eta) p = \cos(\eta) p^T p$
  - $p^T p + 2b^T p + c = (p + b)^T (p + b) + c - b^T b$

- As long as $\cos(\eta) \neq 0$, i.e., **the robot and the two landmarks are not on the same line**:

$$(p - p_0)^T (p - p_0) = \left(p_0^T p_0 - \frac{1}{\cos(\eta)} m_1^T R(\eta) m_2\right) \qquad p_0 := \frac{1}{2\cos(\eta)}\left(R^T(\eta) m_1 + R(\eta) m_2\right)$$

- The position p lies on one of the two possible circles containing m_1 and m_2

2-D Localization from Bearing Measurements

- **Pose disambiguation**: obtain a third bearing measurement:

$$R^T(z_i)(m_i - p) = \|m_i - p\|_2 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad i = 1, 2, 3$$

- Find β and γ such that $R^T(z_1) + \beta R^T(z_2) + \gamma R^T(z_3) = 0$. Then:

$$\underbrace{R^T(z_1)m_1 + \beta R^T(z_2)m_2 + \gamma R^T(z_3)m_3}_{:=u} - \underbrace{\left[R^T(z_1) + \beta R^T(z_2) + \gamma R^T(z_3)\right]}_{0} p$$

$$= (\|m_1 - p\|_2 + \beta\|m_2 - p\|_2 + \gamma\|m_3 - p\|_2) \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

- We can compute θ as $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \frac{u}{\|u\|_2}$ and recover p from:

$$R^T(z_i)(m_i - p) = \|m_i - p\|_2 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad i = 1, 2, 3$$

3-D Localization from Bearing Measurements (P3P)

- **Goal**: determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $m_i \in \mathbb{R}^3$ (world frame) and pixel measurements $z_i \in \mathbb{R}^3$ (homogeneous coordinates, body frame) obtained from a (calibrated pinhole) camera:

$$z_i = \frac{1}{\lambda_i} R^T (m_i - p) \qquad\qquad \lambda_i = \|R^T(m_i - p)\|_2 = \text{unknown scale}$$

- If we determine λ_i, we can transform the P3P problem to 3-D localization from relative position measurements

Find the depths λ_i via Grunert's method (1841)

- Cosines of the angles among the bearing vectors z_1, z_2, z_3:

$$\cos(\gamma_{ij}) = \frac{z_i^T z_j}{\|z_i\|_2 \|z_j\|_2} \quad \Rightarrow \quad \cos(\gamma_{ij}) = z_i^T z_j$$

- Let $\epsilon_{ij} := \|m_i - m_j\|_2$ be the lengths of the triangle formed in the world frame by m_1, m_2, m_3. Applying the law of cosines gives:

$$\lambda_i^2 + \lambda_j^2 - 2\lambda_i \lambda_j \cos(\gamma_{ij}) = \epsilon_{ij}^2 \qquad \text{for } \lambda_i := \|m_i - p\|_2$$

- Let $\lambda_2 = u\lambda_1$ and $\lambda_3 = v\lambda_1$ so that:

$$\begin{aligned} \lambda_1^2(u^2 + v^2 - 2uv\cos(\gamma_{23})) &= \epsilon_{23}^2 \\ \lambda_1^2(1 + v^2 - 2v\cos(\gamma_{13})) &= \epsilon_{13}^2 \\ \lambda_1^2(u^2 + 1 - 2u\cos(\gamma_{12})) &= \epsilon_{12}^2 \end{aligned}$$

- Equivalently

$$\lambda_1^2 = \frac{\epsilon_{23}^2}{u^2 + v^2 - 2uv\cos(\gamma_{23})} = \frac{\epsilon_{13}^2}{1 + v^2 - 2v\cos(\gamma_{13})} = \frac{\epsilon_{12}^2}{u^2 + 1 - 2u\cos(\gamma_{12})}$$

Find the depths λ_i via Grunert's method (1841)

- Cross-multiplying the second fraction, with the first and the third:

$$u^2 + \frac{\epsilon_{13}^2 - \epsilon_{23}^2}{\epsilon_{13}^2} v^2 - 2uv\cos(\gamma_{23}) + \frac{2\epsilon_{23}^2}{\epsilon_{13}^2} v\cos(\gamma_{13}) - \frac{\epsilon_{23}^2}{\epsilon_{13}^2} = 0 \tag{1}$$

$$u^2 - \frac{\epsilon_{12}^2}{\epsilon_{13}^2} v^2 + 2v\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos(\gamma_{13}) - 2u\cos(\gamma_{12}) + \frac{\epsilon_{13}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} = 0 \tag{2}$$

- Substituting (1) into (2):

$$u = \frac{\left(-1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right) v^2 - 2\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13})v + 1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}}{2(\cos(\gamma_{12}) - v\cos(\gamma_{23}))} \tag{3}$$

- Substituting (3) into (1), we get a fourth-order polynomial in v:

$$A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0$$

Polynomial Coefficients

$$A_4 = \left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} - 1\right)^2 - 4\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos^2(\gamma_{23})$$

$$A_3 = 4\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\left(1 - \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13}) - \left(1 - \frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{12}) + 2\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos^2(\gamma_{23})\cos(\gamma_{13})\right)$$

$$A_2 = 2\left(\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2 - 1 + 2\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2\cos^2(\gamma_{13}) + 2\left(\frac{\epsilon_{13}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos^2(\gamma_{23}) + 2\left(\frac{\epsilon_{13}^2 - \epsilon_{23}^2}{\epsilon_{13}^2}\right)\cos^2(\gamma_{12})\right.$$
$$\left. - 4\left(\frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{13})\cos(\gamma_{12})\right)$$

$$A_1 = 4\left(-\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\left(1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13}) - \left(1 - \frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{12}) + 2\frac{\epsilon_{23}^2}{\epsilon_{13}^2}\cos^2(\gamma_{12})\cos(\gamma_{13})\right)$$

$$A_0 = \left(1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2 - \frac{4\epsilon_{23}^2}{\epsilon_{13}^2}\cos^2(\gamma_{12})$$

- We can obtain up to 4 real solutions for v, which we can substitute in (3) to obtain u.
- We can recover λ_1 from u and v via the fractions relationship
- Having λ_1, $\lambda_2 := u\lambda_1$, and $\lambda_3 := v\lambda_1$ we have converted the P3P problem into 3-D localization from relative position measurements

3-D Localization from Bearing Measurements (PnP)

- **Goal**: determine the robot position $p \in \mathbb{R}^3$ and orientation $R \in SO(3)$

- **Given**: landmark positions $m_i \in \mathbb{R}^3$ (world frame) and pixel measurements $z_i \in \mathbb{R}^3$ (homogeneous coordinates) obtained from a (calibrated pinhole) camera for $i = 1, \ldots, n$:

$$z_i = \frac{1}{\lambda_i} R^T (m_i - p) \qquad \lambda_i = \|R^T (m_i - p)\|_2 = \text{unknown scale}$$

- The PnP can be formulated as a **constrained nonlinear least-squares** minimization:

$$\begin{aligned} \min_{\lambda_i, R, p} \; & \sum_{i=1}^{n} \|z_i - \frac{1}{\lambda_i} R^T (m_i - p)\|_2^2 \\ \text{s.t.} \;\; & R^T R = I, \quad \det R = 1, \quad \lambda_i = \|R^T (m_i - x)\|_2 \end{aligned}$$

Reformulation into a Polynomial System

- The constraints $\lambda_i z_i = R^T(m_i - p)$ can be re-written in matrix form as:

$$\underbrace{\begin{bmatrix} z_1 & & & -I \\ & \ddots & & \vdots \\ & & z_n & -I \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ -R^T p \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} R^T & & \\ & \ddots & \\ & & R^T \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}}_{d}$$

  where A and d are known or measured, x are the unknowns we wish to eliminate, and W is a block diagonal matrix of the unknown rotation R

- We can express p and λ_i in terms of the other quantities as follows:

$$x = (A^T A)^{-1} A^T W d = \begin{bmatrix} U \\ V \end{bmatrix} W d$$

  where $(A^T A)^{-1} A^T$ is partitioned so that the scale parameters are a function of U and the translation $-R^T p$ is a function of V.

Reformulation into a Polynomial System

$$x = (A^T A)^{-1} A^T W d = \begin{bmatrix} U \\ V \end{bmatrix} W d$$

- Exploiting the sparse structure of A, the matrices U and V can be computed in closed form
- Both λ_i and $-R^T p$ are linear functions of the unknown R^T:

$$\lambda_i = u_i^T W d \qquad -R^T p = V W d, \qquad i = 1, \ldots, n$$

  where u_i^T is the i-th row of U.
- We can rewrite the constraints $\lambda_i z_i = R^T(m_i - p)$ as:

$$\underbrace{u_i^T W d}_{\lambda_i} z_i = R^T m_i + \underbrace{V W d}_{-R^T p}$$

- We have reduced the number of unknowns from $6 + n$ to 3

Reformulation into a Polynomial System

- **Cayley-Gibbs-Rodrigues Rotation Parameterization**

$$R^T = \frac{\bar{C}}{1 + s^T s} \qquad \bar{C} = ((1 - s^T s) I_3 + 2\hat{s} + 2ss^T)$$

- The CGR parameters automatically satisfy the rotation matrix constraints, i.e., $R^T R = I$ and $\det(R) = 1$ and allow us to formulate an unconstrained least-squares minimization in s.

- Since R^T appears linearly in the equations, we can cancel the denominator $1 + s^T s$. This leads to the following formulation of the PnP problem:

$$\min_s J(s) = \sum_{i=1}^{n} \left\| u_i^T \begin{bmatrix} \bar{C} & & \\ & \ddots & \\ & & \bar{C} \end{bmatrix} dz_i - \bar{C} m_i - V \begin{bmatrix} \bar{C} & & \\ & \ddots & \\ & & \bar{C} \end{bmatrix} d \right\|^2$$

which contains all monomials up to degree four, i.e., $\{1, s_1, s_2, s_3, s_1 s_2, s_1 s_3, s_2 s_3, \ldots, s_1^4, s_2^4, s_3^4\}$.

Macaulay Matrix

- Since $J(s)$ is a fourth-order polynomial, the optimality conditions form a system of three third-order polynomials (derivatives with respect to s_1, s_2 and s_3).
- We use a **Macaulay resultant matrix** (matrix of polynomial coefficients) to find the roots of the third-order polynomials and hence compute all critical points of $J(s)$
- Since the polynomial system is of constant degree (independent of n), it is only necessary to compute the Macaulay matrix symbolically once.
- Online, the elements of the Macaulay matrix are formed from the data (linear operation in n) and the roots are determined via an eigen-decomposition of the Schur complement (dense 27×27 matrix) of the top block of the Macaulay matrix (sparse 120×120 matrix)

2-D Odometry from Bearing Measurements

- **Goal**: determine the relative transformation ${}_t p_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: bearing measurements $z_{t,i} \in \mathbb{R}^2$ and $z_{t+1,i} \in \mathbb{R}^2$ (unit vectors) at consecutive time steps to n **unknown** landmarks
- The measurements are related as follows:

$$d_{t,i} b_{t,i} = {}_t p_{t+1} + R({}_t\theta_{t+1}) d_{t+1,i} b_{t+1,i}, \qquad i = 1, \ldots, n$$

  where $d_{t,i}, d_{t+1,i}$ are the unknown distances to m_i.
- There are $2n$ equations and $2n+3$ unknowns, which means that this problem is **not solvable**.

3-D Odometry from Bearing Measurements

- **Goal**: determine the relative transformation ${}_t p_{t+1} \in \mathbb{R}^3$ and ${}_t R_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t
- **Given**: bearing measurements $z_{t,i} \in \mathbb{R}^3$ and $z_{t+1,i} \in \mathbb{R}^3$ (unit vectors) at consecutive time steps to n **unknown** landmarks ($n \geq 5$)
- **Essential matrix**: $E := \left[{}_t \hat{p}_{t+1}\right] \left[{}_t R_{t+1}\right]$
- **Epipolar constraint**: $0 = z_{t,i}^T E z_{t+1,i}$, for $i = 1, \ldots, n$
- **Idea**: recover the essential matrix between the two views first

3-D Odometry from Bearing Measurements (8-Pt Alg)

- The epipolar constraint $0 = z_{t,i}^T E z_{t+1,i}$ is linear in the elements of E:

$$0 = \bar{z}_i^T \mathbf{e}$$

  where $\mathbf{e} := \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{21} & E_{22} & E_{23} & E_{31} & E_{32} & E_{33} \end{bmatrix}^T$ and $\bar{z}_i := \mathbf{vec}\left(z_{t+1,i} z_{t,i}^T\right) \in \mathbb{R}^9$ where $\mathbf{vec}(\cdot)$ is a row-wise vectorization.

- Stacking $\bar{z}_i$'s from 8 point observations together, we obtain an 8×9 matrix $\bar{Z} := \begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_8 \end{bmatrix}^T$ leading to the following equation for $\mathbf{e}$:

$$\bar{Z}\mathbf{e} = 0$$

- Thus, $\mathbf{e}$ is a **singular vector** of $\bar{Z}$ associated to a singular value that equals zero.

- If at least 8 linearly independent vectors $\bar{z}_i$ are used to construct $\bar{Z}$, then the singular vector is unique (up to scalar multiplication) and $\mathbf{e}$ and E can be determined.

3-D Odometry from Bearing Measurements (5-Pt Alg)

- The essential matrix E can be recovered from $\bar{Z}\mathbf{e} = 0$, even if only 5 linearly independent vectors $\bar{z}_i$ are available using the fact that:

$$0 = EE^T E - \frac{1}{2}\operatorname{tr}(EE^T)E$$

- Stacking $\bar{z}_i$'s together, we obtain a 5×9 matrix $\bar{Z} := \begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_5 \end{bmatrix}^T$
- The right nullspace of $\bar{Z}$ has dimension 4 and the vectors that span the nullspace (obtained from SVD or QR decomposition) correspond to 3×3 matrices N_i, $i = 1, \ldots, 4$ so that

$$E = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + \alpha_4 N_4, \qquad \alpha_i \in \mathbb{R}$$

- Since the measurements are scale-invariant, we can arbitrarily fix $\alpha_4 = 1$
- Substituting $E = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + N_4$, we obtain 9 cubic-in-$\alpha_i$ equations and can recover up to 10 solutions for E

3-D Odometry from Bearing Measurements (5-Pt Alg)

- Once E is recovered, ${}_t p_{t+1}$ and ${}_t R_{t+1}$ can be computed from the singular value decomposition of E

- **Pose recovery from the essential matrix**: There are exactly two relative poses corresponding to a non-zero essential matrix $E = U\mathbf{diag}(\sigma, \sigma, 0)V^T$:

$$({}_t\hat{p}_{t+1}, {}_tR_{t+1}) = \left(UR_z\left(\frac{\pi}{2}\right)\mathbf{diag}(\sigma,\sigma,0)U^T, UR_z^T\left(\frac{\pi}{2}\right)V^T\right)$$
$$({}_t\hat{p}_{t+1}, {}_tR_{t+1}) = \left(UR_z\left(-\frac{\pi}{2}\right)\mathbf{diag}(\sigma,\sigma,0)U^T, UR_z^T\left(-\frac{\pi}{2}\right)V^T\right)$$

- Only one of these will place the points in front of both cameras

- The ambiguity can be resolved by intersecting the measurements of a single point and verifying which solution places it on the positive optical z-axis of both cameras

Summary: Bearing Measurements $z_i = \frac{1}{\lambda_i} R^T(m_i - p)$

- **2-D Localization**: given $m_1, m_2 \in \mathbb{R}^2$ and $z_1, z_2 \in [-\pi, \pi]$
  1. 2-D bearing: $\frac{1}{\lambda_i} R^T(\theta)(m_i - p) = R(z_i)e_1$
  2. Eliminate θ:
     $$0 = \lambda_1 e_1^T R(\theta) R\left(\frac{\pi}{2}\right) R(\theta) e_1 \lambda_2 = (m_1 - p)^T R(z_1) R\left(\frac{\pi}{2}\right) R^T(z_2)(m_2 - p)$$
  3. The position p in on one of two circles containing m_1 and m_2 and we need a third bearing measurement z_3 to disambiguate it
  4. Find β, γ such that $R^T(z_1) + \beta R^T(z_2) + \gamma R^T(z_3) = 0$ and combine $R^T(z_i)(m_i - p) = \lambda_i \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ to solve for θ
  5. Orientation: $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \frac{u}{\|u\|_2}$ for $u = R^T(z_1)m_1 + \beta R^T(z_2)m_2 + \gamma R^T(z_3)m_3$

- **3-D Localization (P3P)**: $m_i \in \mathbb{R}^3$, $z_i \in \mathbb{R}^3$ (homogeneous), $i = 1, 2, 3$
  1. Convert P3P to relative position localization by determining the depths $\lambda_1, \lambda_2, \lambda_3$ via Grunert's method
  2. Define the angles γ_{ij} among z_1, z_2, z_3 and apply the law of cosines: $\lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j \cos(\gamma_{ij}) = \|m_1 - m_j\|_2^2$
  3. Let $\lambda_2 = u\lambda_1$ and $\lambda_3 = v\lambda_1$ and combine the 3 equations to get a fourth order polynomial: $A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0$

Summary: Bearing Measurements $z_i = \frac{1}{\lambda_i} R^T (m_i - p)$

- **3-D Localization (PnP)**
  1. Rewrite $\lambda_i z_i = R^T(m_i - p)$ in matrix form and solve for $x := (\lambda_1, \ldots, \lambda_n, -R^T p)^T$ in terms of R
  2. The equations for λ_i and $-R^T p$ turn out to be linear in R so we are left with one equation with 3 unknowns (the 3 degrees of freedom of R)
  3. Obtain a fourth order polynomial $J(s)$ in terms of the Cayley-Gibbs-Rodrigues rotation parameterization s
  4. Compute a Macaulay matrix of the coefficients of $J(s)$ symbolically once. Online, determine the roots of $J(s)$ via an eigen-decomposition of the Schur complement of the Macaulay matrix.

- **2-D Odometry**: not solvable

- **3-D Odometry**: 5-point or 8-point algorithm:
  1. Obtain E from the epipolar constraint: $0 = \mathbf{vec}\left(z_{t+1,i} z_{t,i}^T\right)^T \mathbf{vec}(E)$, $i = 1, \ldots, 5$ and the property $0 = EE^T E - \frac{1}{2}\operatorname{tr}(EE^T)E$
  2. Recover two possible camera poses based on $SVD(E) = U\mathbf{diag}(\sigma, \sigma, 0)V^T$ and choose the one that places the measurements in front of both cameras