ECE276A: Sensing & Estimation in Robotics
Lecture 7: Rotations

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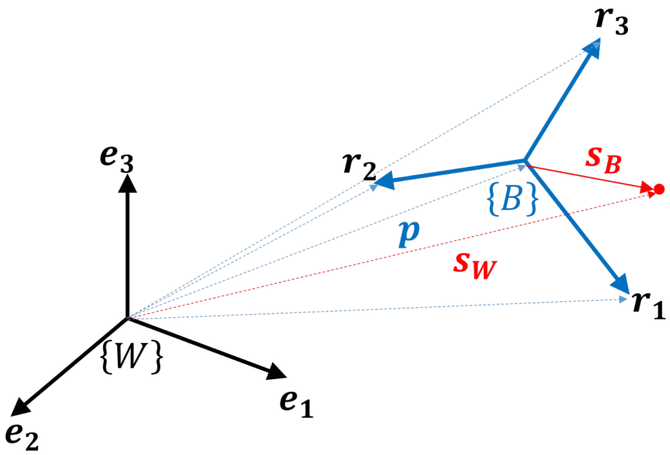
Rigid Body Motion

- Consider a moving object in a fixed **world reference frame** W
- **Rigid object**: it is sufficient to specify the motion of one point $p(t) \in \mathbb{R}^3$ and 3 coordinate axes $(r_1(t), r_2(t), r_3(t))$ attached to that point (**body reference frame** B)



Rigid Body Motion

- A rigid body is free to translate (3 degrees of freedom) and rotate (3 degrees of freedom)
- The **pose** $g(t)$ of a moving rigid object at time t is determined by
  1. The position $p(t) \in \mathbb{R}^3$ of the body frame B relative to the world frame W
  2. The orientation $R(t) \in SO(3)$ of B relative to W
- The body of a robot may be composed of multiple connected rigid bodies, each having their own pose. We will assume that the robot is a single rigid body.
- **Rigid body motion** is a family of transformations $g(t) : \mathbb{R}^3 \to \mathbb{R}^3$ that describe how the coordinates of points on the object change in time

Special Euclidean Group

- Rigid body motion preserves both distances (vector norms) and orientation (vector cross products)
- **Euclidean Group** $E(3)$: a set of maps $g : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve the norm of any two vectors
- **Special Euclidean Group** $SE(3)$: a set of maps $g : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve the norm and cross product of any two vectors
- The set of rigid body motions forms a group because:
  - We can combine several motions to generate a new one (**closure**)
  - We can execute a motion that leaves the object at the same state (**identity element**)
  - We can move rigid objects from one place to another and then reverse the action (**inverse element**)
- The space $\mathbb{R}^3$ of translations/positions is familiar
- How do we describe orientation?

Special Euclidean Group

- A **group** is a set G with an associated operator $\odot$ (group law of G) that satisfies:
  - **Closure**: $a \odot b \in G$, $\forall a, b \in G$
  - **Identity element**: $\exists! e \in G$ (unique) such that $e \odot a = a \odot e = a$
  - **Inverse element**: for $a \in G$, $\exists b \in G$ such that $a \odot b = b \odot a = e$
  - **Associativity**: $(a \odot b) \odot c = a \odot (b \odot c)$, $\forall a, b, c, \in G$

- $SE(3)$ is a group of maps $g : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve:
  1. Norm: $\|g(u) - g(v)\| = \|v - u\|, \forall u, v \in \mathbb{R}^3$
  2. Cross product: $g(u) \times g(v) = g(u \times v), \forall u, v \in \mathbb{R}^3$

- **Corollary**: $SE(3)$ elements also preserve:
  1. Angle: $u^T v = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2\right) \Rightarrow u^T v = g(u)^T g(v)$, $\forall u, v \in \mathbb{R}^3$
  2. Volume: $\forall u, v, w \in \mathbb{R}^3$, $g(u)^T(g(v) \times g(w)) = u^T(v \times w)$
     (volume of parallelepiped spanned by u, v, w)

Cross product

- The **cross product** of two vectors $\omega, \beta \in \mathbb{R}^3$ is also a vector in $\mathbb{R}^3$:

$$\omega \times \beta := \begin{bmatrix} \omega_2\beta_3 - \omega_3\beta_2 \\ \omega_3\beta_1 - \omega_1\beta_3 \\ \omega_1\beta_2 - \omega_2\beta_1 \end{bmatrix}$$

- For fixed ω, the cross product can be represented by a *linear* map $\omega \times \beta = \hat{\omega}\beta$ for $\hat{\omega} \in \mathbb{R}^{3\times 3}$

- The **hat map** $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms an $\mathbb{R}^3$ vector to a skew-symmetric matrix:

$$\hat{\omega} := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- The vector space $\mathbb{R}^3$ and the space of skew-symmetric 3×3 matrices $\mathfrak{so}(3)$ are isomorphic, i.e., there exists a one-to-one map (the hat map) that preserves their structure.

Hat Map Properties

- **Lemma**: A matrix $M \in \mathbb{R}^{3\times 3}$ is skew-symmetric iff $M = \hat{\omega}$ for some $\omega \in \mathbb{R}^3$.
- The inverse of the hat map is the **vee operator**, $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$, that extracts the components of the vector $\omega = \hat{\omega}^\vee$ from the matrix $\hat{\omega}$.
- For any $x, y \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$, the hat map satisfies:
  - $\hat{x}y = x \times y = -y \times x = -\hat{y}x$
  - $\hat{x}^2 = xx^T - x^T x \, I_{3\times 3}$
  - $\hat{x}^{2k+1} = (-x^T x)^k \hat{x}$
  - $-\frac{1}{2}\operatorname{tr}(\hat{x}\hat{y}) = x^T y$
  - $\hat{x}A + A^T\hat{x} = ((\operatorname{tr}(A)I_{3\times 3} - A)x)^\wedge$
  - $\operatorname{tr}(\hat{x}A) = \frac{1}{2}\operatorname{tr}(\hat{x}(A - A^T)) = -x^T(A - A^T)^\vee$
  - $\widehat{Ax} = \det(A)A^{-T}\hat{x}A^{-1}$

3-D Orientation

- The orientation of a body frame B is determined by the coordinates of the three orthogonal vectors $r_1 = g(e_1)$, $r_2 = g(e_2)$, $r_3 = g(e_3)$ relative to the world frame W, i.e., by the 3×3 matrix:

$$R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

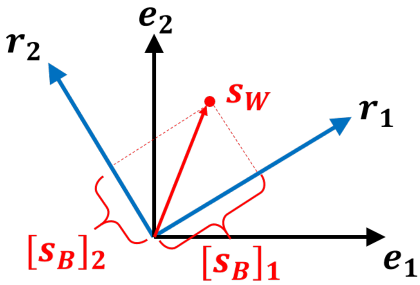
- Consider a point $s \in \mathbb{R}^3$ with coordinates s_B in $\{B\}$ and s_W in $\{W\}$

- Pure 2D rotation:

$$s_W = [s_B]_1 r_1 + [s_B]_2 r_2$$

- 3D translation p and rotation R:

$$\begin{aligned} s_W &= [s_B]_1 r_1 + [s_B]_2 r_2 + [s_B]_3 r_3 + p \\ &= R s_B + p \end{aligned}$$



Special Orthogonal Lie Group $SO(3)$

- Since r_1, r_2, r_3 form an orthonormal basis: $r_i^T r_j = \delta_{ij}$
- R is an **orthogonal matrix** $R^T R = RR^T = I$
- R's inverse is its transpose: $R^{-1} = R^T$
- $\det(R) = r_1^T(r_2 \times r_3) = 1$
- R belongs to the **special orthogonal group**:

$$SO(3) := \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det(R) = 1\}$$

Special Orthogonal Lie Group $SO(n)$

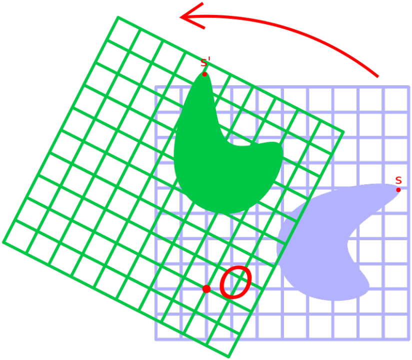
- $SO(n) := \{R \in \mathbb{R}^{n\times n} \mid R^T R = I, \det(R) = 1\}$
- Closed under multiplication: $R_1 R_2 \in SO(n)$
- Identity: $I \in SO(n)$
- Inverse: $R^{-1} = R^T \in SO(n)$
- Associative property: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
- **Manifold structure**: n^2 parameters with $n(n+1)/2$ constraints (due to $R^T R = I$) and hence $n(n-1)/2$ degrees of freedom
- Distances are preserved:
  $\|x - y\|_2^2 = \|R(x-y)\|_2^2 = (x-y)^T R^T R (x-y) \Rightarrow R^T R = I$
- No reflections allowed, i.e., a right-handed coordinate system is kept:
  $R(x \times y) = (Rx) \times (Ry) = \widehat{Rx} Ry = \det(R) R \hat{x} R^T Ry \Rightarrow \det(R) = 1$

2-D Rotation

- A 2-D rotation of point $s \in \mathbb{R}^2$ through an angle θ can be described by a rotation matrix $R(\theta) \in SO(2)$:

$$s_W = R(\theta) s_B := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} s_B$$

- $\theta > 0$: counterclockwise rotation



- There is a one-to-one correspondence between 2-D rotation matrices and unit-norm complex numbers:

$$e^{i\theta}([s_B]_1 + i[s_B]_2) = ([s_B]_1 \cos\theta - [s_B]_2 \sin\theta) + i([s_B]_1 \sin\theta + [s_B]_2 \cos\theta)$$

Principal 3D Rotations

- A rotation by an angle ϕ around the x-axis is represented by:

$$R_x(\phi) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

- A rotation by an angle θ around the y-axis is represented by:

$$R_y(\theta) := \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

- A rotation by an angle ψ around the z-axis is represented by:

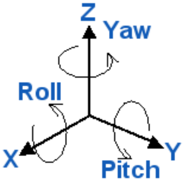
$$R_z(\psi) := \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angle Parameterization

- One way to parameterize rotation is to use three angles that specify the rotations around the principal axes
- There are 24 different ways to apply these rotations
  - **Extrinsic axes**: the rotation axes remain fixed/global/static
  - **Intrinsic axes**: the rotation axes move with the rotations
  - Each of the two groups (intrinsic and extrinsic) can be divided into:
    - **Euler Angles**: rotation about one axis, then a second and then the first
    - **Tait-Bryan Angles**: rotation about all three axes
  - The Euler and Tait-Bryan Angles each have 6 possible choices for each of the extrinsic/intrinsic groups leading to $2 * 2 * 6 = 24$ possible conventions to specify a rotation sequence with three given angles
- For simplicity we will refer to all these 24 conventions as **Euler Angles** and will explicitly specify:
  - r (rotating = intrinsic) or s (static = extrinic)
  - xyz or zyx or zxz, etc. (axes about which to perform the rotation in the specified order)

Euler Angle Convention

- Spin (θ), nutation (γ), precession (ψ) sequence:
  - A rotation ψ about the original *z*-axis
  - A rotation γ about the intermediate *x*-axis
  - A rotation θ about the transformed *z*-axis
- Roll (ϕ), pitch (θ), yaw (ψ):
  - A rotation ϕ about the original *x*-axis
  - A rotation θ about the intermediate *y*-axis
  - A rotation ψ about the transformed *z*-axis



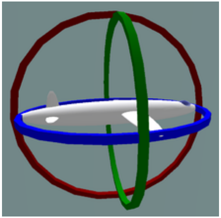
- We will call **Euler Angles** the **roll** (ϕ), **pitch** (θ), **yaw** (ψ) angles specifying an **XYZ extrinsic** rotation or equivalently a **ZYX intrinsic** rotation:

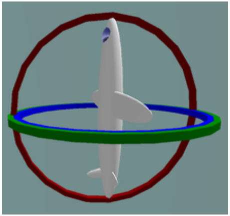
$$R = R_z(\psi)R_y(\theta)R_x(\phi)$$

$$= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Gimbal Lock

- Angle parameterizations have **singularities** (not one-to-one), which can result in **gimbal lock**, e.g., if $\theta = 90^\circ$, the roll and yaw become associated with the same degree of freedom and cannot be uniquely determined.
- The gimbal lock is a problem only if we want to recover the rotation angles from a rotation matrix

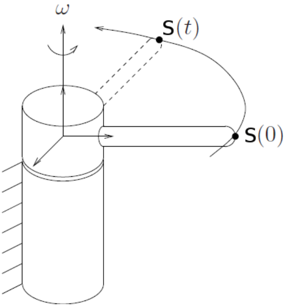




Axis-Angle Parameterization

- Every rotation can be represented by a **rotation vector** $\omega \in \mathbb{R}^3$ as a rotation about an axis $\xi := \frac{\omega}{\|\omega\|_2}$ through angle $\theta := \|\omega\|_2$
- Consider a point s rotating about an axis ξ at constant unit velocity ($\|\omega\|_2 = 1$):

$$\dot{s}(t) = \xi \times s(t) = \hat{\xi} s(t), \quad s(0) = s_0$$
$$\Rightarrow \; s(t) = e^{\hat{\xi} t} s_0 = R(t) s_0$$



- **Rotation kinematics**: if $\omega \in \mathbb{R}^3$ is constant (world frame) angular velocity of a body $\{B\}$, then the body orientation changes as follows:

$$\dot{R}(t) = \hat{\omega} R(t) \quad \Rightarrow \quad R(t) = \exp(\hat{\omega} t) R(t_0)$$

- **Axis-angle representation**: a rotation around the axis $\xi := \frac{\omega}{\|\omega\|_2}$ through an angle $\theta := \|\omega\|_2$ can thus be represented as:

$$R = \exp(\hat{\omega})$$

- **The matrix exponential defines a map from $\mathfrak{so}(3)$ to $SO(3)$.**

Quaternions (Hamilton Convention)

- **Quaternions**: $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ generalize complex numbers $\mathbb{C} = \mathbb{R} + \mathbb{R}i$

  $$q = q_s + q_1 i + q_2 j + q_3 k = [q_s,\ \mathbf{q}_v] \qquad ij = -ji = k,\ i^2 = j^2 = k^2 = -1$$

- Just as in 2-D, 3-D rotations can be represented using "complex numbers", i.e., **unit-norm** quaternions $\{q \in \mathbb{H} \mid q_s^2 + \mathbf{q}_v^T \mathbf{q}_v = 1\}$

- To represent rotations, the quaternion space embeds a 3-D space into a 4-D space (**no singularities**) and introduces a unit norm constraint. The space of quaternions is a **double covering** of $SO(3)$ because two unit quaternions correspond to the same rotation: $R(q) = R(-q)$.

- A rotation matrix $R \in SO(3)$ can be obtained from a unit quaternion q:

  $$R(q) = E(q)G(q)^T \quad E(q) = [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \quad G(q) = [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v]$$

Quaternion Conversions

- A rotation around a unit axis $\xi := \frac{\omega}{\|\omega\|} \in \mathbb{R}^3$ by angle $\theta := \|\omega\|$ can be represented by a unit quaternion:

$$q = \left[\cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)\xi\right]$$

- A rotation around a unit axis $\xi \in \mathbb{R}^3$ by angle θ can be recovered from a unit quaternion q:

$$\theta = 2\arccos(q_s) \qquad \xi = \begin{cases} \frac{1}{\sin(\theta/2)}\mathbf{q}_v, & \text{if } \theta \neq 0 \\ 0, & \text{if } \theta = 0 \end{cases}$$

- The inverse transformation above has a singularity at $\theta = 0$ because there are infinitely many rotation axes that can be used or equivalently the transformation from an axis-angle representation to a quaternion representation is many-to-one

Quaternion Properties

Addition $q + p = [q_s + p_s,\ \mathbf{q}_v + \mathbf{p}_v]$

Multiplication $q \circ p = \left[q_s p_s - \mathbf{q}_v^T \mathbf{p}_v,\ q_s \mathbf{p}_v + p_s \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v\right]$

Conjugate $\bar{q} = [q_s,\ -\mathbf{q}_v]$

Norm $|q| := \sqrt{q_s^2 + \mathbf{q}_v^T \mathbf{q}_v} \quad |q \circ p| = |q||p|$

Inverse $q^{-1} = \frac{\bar{q}}{|q|^2}$

Rotation $[0,\ \mathbf{x}'] = q \circ [0,\ \mathbf{x}] \circ q^{-1} = [0,\ R(q)\mathbf{x}]$

Rot. Velocity $\dot{q} = \frac{1}{2}[0,\ \omega] \circ q = \frac{1}{2} E(q)^T \omega = \frac{1}{2} q \circ [0,\ \omega_B] = \frac{1}{2} G(q)^T \omega_B$

Exp $\exp(q) := e^{q_s} \left[\cos \|\mathbf{q}_v\|,\ \frac{\mathbf{q}_v}{\|\mathbf{q}_v\|} \sin \|\mathbf{q}_v\|\right]$

Log $\log(q) := \left[\log |q|,\ \frac{\mathbf{q}_v}{\|\mathbf{q}_v\|} \arccos \frac{q_s}{|q|}\right]$

- **Exp**: constructs q from rotation vector $\omega \in \mathbb{R}^3$: $q = \exp\left(\left[0,\ \frac{\omega}{2}\right]\right)$
- **Log**: recovers a rotation vector $\omega \in \mathbb{R}^3$ from q: $[0,\ \omega] = 2\log(q)$

Rigid Body Pose

- Let B be a body frame whose position and orientation with respect to the world frame W are $p \in \mathbb{R}^3$ and $R \in SO(3)$, respectively.
- The coordinates of a point $s_B \in \mathbb{R}^3$ in the body frame B can be converted to the world frame by first rotating the point and then translating it to the world frame: $s_W = Rs_B + p$.
- **Homogeneous coordinates**: the rigid-body transformation is not linear but **affine**. It can be converted to linear by appending 1 to the coordinates of a point s:

$$\begin{bmatrix} s_W \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_B \\ 1 \end{bmatrix}$$

- Each entry of a homogeneous point representations can be multiplied by a scale factor λ which allows representing points arbitrarily far away from the origin as $\lambda \to 0$: $\begin{bmatrix} \lambda s_B \\ \lambda \end{bmatrix}$
- To recover the original coordinates, divide the first three entries by λ

Special Euclidean Group $SE(3)$

- The pose of a rigid body can thus be described by a matrix:

$$SE(3) := \left\{ T := \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \middle| \; R \in SO(3), p \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times 4}$$

- Using homogeneous coordinates, it can be verified that $SE(3)$ satisfies all requirements of a group:
  - $T_1 T_2 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix} \in SE(3)$
  - $\begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} \in SE(3)$
  - $\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \in SE(3)$
  - $(T_1 T_2) T_3 = T_1 (T_2 T_3)$ for all $T_1, T_2, T_3 \in SE(3)$

Composing Transformations

- Let the pose of a rigid body be ${}_{\{W\}}T_{\{B\}} := \begin{bmatrix} {}_{\{W\}}R_{\{B\}} & {}_{\{W\}}p_{\{B\}} \\ 0 & 1 \end{bmatrix}$
- The subscripts indicate that **the pose a rigid body in the world frame specifies a transformation from the body to the world frame**
- Given a robot with pose T, a point s_B in the robot body frame has world frame coordinates:

$$s_W = Rs_B + p \quad \text{equivalent to} \quad \begin{bmatrix} s_W \\ 1 \end{bmatrix} = T \begin{bmatrix} s_B \\ 1 \end{bmatrix}$$

- Give a robot with pose ${}_{\{W\}}T_{\{1\}}$ at time t_1 and ${}_{\{W\}}T_{\{2\}}$ at time t_2, the relative transformation from the inertial frame $\{2\}$ at time t_2 to the inertial frame $\{1\}$ at time t_1 is:

$$\begin{aligned} {}_{\{1\}}T_{\{2\}} &= {}_{\{1\}}T_{\{W\}} \times {}_{\{W\}}T_{\{2\}} = \left({}_{\{W\}}T_{\{1\}}\right)^{-1} \times {}_{\{W\}}T_{\{2\}} \\ &= \begin{bmatrix} {}_{\{W\}}R_{\{1\}}^T & -{}_{\{W\}}R_{\{1\}}^T \times {}_{\{W\}}p_{\{1\}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}_{\{W\}}R_{\{2\}} & {}_{\{W\}}p_{\{2\}} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Summary

	Rotation $SO(3)$	**Pose** $SE(3)$
Representation	$R : \begin{cases} R^T R = I \\ \det(R) = 1 \end{cases}$	$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$
Transformation	$s_W = R s_B$	$s_W = R s_B + p$
Inverse	$R^{-1} = R^T$	$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$
Composition	${}_W R_{t_k} = {}_W R_{t_0} \prod_{i=0}^{k-1} {}_{t_i} R_{t_{i+1}}$	${}_W T_{t_k} = {}_W T_{t_0} \prod_{i=0}^{k-1} {}_{t_i} T_{t_{i+1}}$