ECE276A: Sensing & Estimation in Robotics
Lecture 3: Unconstrained Optimization

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Field

- A **field** is a set F with two binary operations, $+ : F \times F \mapsto F$ (addition) and $\cdot : F \times F \mapsto F$ (multiplication), which satisfy the following axioms:
  - **Associativity**: $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$, $\forall a, b, c \in F$
  - **Commutativity**: $a + b = b + a$ and $ab = ba$, $\forall a, b \in F$
  - **Identity**: $\exists 1, 0 \in F$ such that $a + 0 = a$ and $a1 = a$, $\forall a \in F$
  - **Inverse**: $\forall a \in F, \exists {-a} \in F$ such that $a + (-a) = 0$
    $\forall a \in F \setminus \{0\}, \exists a^{-1} \in F \setminus \{0\}$ such that $aa^{-1} = 1$
  - **Distributivity**: $a(b + c) = (ab) + (ac)$, $\forall a, b, c \in F$
- **Examples**: real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, rational numbers $\mathbb{Q}$

Vector Space

- A **vector space** over a field F is a set V with two binary operations, $+ : V \times V \mapsto V$ (addition) and $\cdot : F \times V \mapsto V$ (scalar multiplication), which satisfy the following axioms:
  - **Associativity**: $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$, $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$
  - **Compatibility**: $a(b\mathbf{x}) = (ab)\mathbf{x}$, $\forall a, b \in F$ and $\forall \mathbf{x} \in V$
  - **Commutativity**: $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{y}$, $\forall \mathbf{x}, \mathbf{y} \in V$
  - **Identity**: $\exists\ \mathbf{0} \in V$ and $1 \in F$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ and $1\mathbf{x} = \mathbf{x}$, $\forall \mathbf{x} \in V$
  - **Inverse**: $\forall \mathbf{x} \in V, \exists {-\mathbf{x}} \in V$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
  - **Distributivity**: $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + b\mathbf{y}$ and $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$, $\forall a, b \in F$ and $\forall \mathbf{x}, \mathbf{y} \in V$
- **Examples**: real vectors $\mathbb{R}^d$, complex vectors $\mathbb{C}^d$, rational vectors $\mathbb{Q}^d$, functions $\mathbb{R}^d \mapsto \mathbb{R}$

Basis and Dimension

- A **basis** of a vector space V over a field F is a set $B \subseteq V$ that satisfies:
  - **linear independence**: for all finite $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subseteq B$, if $a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m = 0$ for some $a_1, \ldots, a_m \in F$, then $a_1 = \cdots = a_m = 0$
  - B **spans** V: $\forall \mathbf{x} \in V$, $\exists\ \mathbf{x}_1, \ldots, \mathbf{x}_d \in B$ and unique $a_1, \ldots, a_d \in F$ such that $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_d\mathbf{x}_d$
- The **dimension** d of a vector space V is the cardinality of its bases

Inner Product and Norm

- An **inner product** on a vector space V over a field F is a function $\langle \cdot, \cdot \rangle : V \times V \mapsto F$ such that for all $a \in F$ and all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$:
  - $\langle a\mathbf{x}, \mathbf{y} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)
  - $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (additivity)
  - $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ (conjugate symmetry)
  - $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ (non-negativity)
  - $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = \mathbf{0}$ (definiteness)
- A **norm** on a vector space V over a field F is a function $\| \cdot \| : V \to \mathbb{R}$ such that for all $a \in F$ and all $\mathbf{x}, \mathbf{y} \in V$:
  - $\|a\mathbf{x}\| = |a|\|\mathbf{x}\|$ (absolute homogeneity)
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)
  - $\|\mathbf{x}\| \geq 0$ (non-negativity)
  - $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$ (definiteness)

Euclidean Vector Space

- A **Euclidean vector space** $\mathbb{R}^d$ is a vector space with finite dimension d over the real numbers $\mathbb{R}$

- A **Euclidean vector** $\mathbf{x} \in \mathbb{R}^d$ is a collection of scalars $x_i \in \mathbb{R}$ for $i = 1, \ldots, d$ organized as a column:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$$

- The **transpose** of $\mathbf{x} \in \mathbb{R}^d$ is organized as a row: $\mathbf{x}^\top = \begin{bmatrix} x_1 & \cdots & x_d \end{bmatrix}$

- The **Euclidean inner product** between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ is:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{d} x_i y_i$$

- The **Euclidean norm** of a vector $\mathbf{x} \in \mathbb{R}^d$ is $\|\mathbf{x}\|_2 := \sqrt{\mathbf{x}^\top \mathbf{x}}$ and satisfies:
    - $\max_{1 \le i \le d} |x_i| \le \|\mathbf{x}\|_2 \le \sqrt{d} \max_{1 \le i \le d} |x_i|$
    - $|\mathbf{x}^\top \mathbf{y}| \le \|\mathbf{x}\|_2 \|\mathbf{y}\|_2$ (Cauchy-Schwarz Inequality)

Matrices

- A **matrix** $A \in \mathbb{R}^{m \times n}$ is a rectangular array of scalars $A_{ij} \in \mathbb{R}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$
- The entries of the **transpose** $A^\top \in \mathbb{R}^{n \times m}$ of a matrix $A \in \mathbb{R}^{m \times n}$ are $A^\top_{ij} = A_{ji}$. The transpose satisfies: $(AB)^\top = B^\top A^\top$
- The **trace** of a matrix $A \in \mathbb{R}^{n \times n}$ is the sum of its diagonal entries:

$$\operatorname{tr}(A) := \sum_{i=1}^{n} A_{ii} \qquad \operatorname{tr}(ABC) = \operatorname{tr}(BCA) = \operatorname{tr}(CAB)$$

- The **Frobenius inner product** between two matrices $X, Y \in \mathbb{R}^{m \times n}$ is:

$$\langle X, Y \rangle = \operatorname{tr}(X^\top Y)$$

- The **Frobenius norm** of a matrix $X \in \mathbb{R}^{m \times n}$ is: $\|X\|_F := \sqrt{\operatorname{tr}(X^\top X)}$

Matrix Determinant and Inverse

- The **determinant** of a matrix $A \in \mathbb{R}^{n \times n}$ is:

$$\det(A) := \sum_{j=1}^{n} A_{ij}\mathbf{cof}_{ij}(A) \qquad \det(AB) = \det(A)\det(B) = \det(BA)$$

  where $\mathbf{cof}_{ij}(A)$ is the **cofactor** of the entry A_{ij} and is equal to $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ submatrix that results when the i^{th}-row and j^{th}-col of A are removed. This recursive definition uses the fact that the determinant of a scalar is the scalar itself.

- The **adjugate** is the transpose of the cofactor matrix:

$$\mathbf{adj}(A) := \mathbf{cof}(A)^\top$$

- The **inverse** A^{-1} of A exists iff $\det(A) \neq 0$ and satisfies:

$$A^{-1} = \frac{\mathbf{adj}(A)}{\det(A)} \qquad (AB)^{-1} = B^{-1}A^{-1}$$

Matrix Inversion Lemma

- **Square completion**:

$$\frac{1}{2}x^\top Ax + b^\top x + c = \frac{1}{2}\left(x + A^{-1}b\right)^\top A\left(x + A^{-1}b\right) + c - \frac{1}{2}b^\top A^{-1}b$$

- **Woodbury matrix identity**:

$$(A + BDC)^{-1} = A^{-1} - A^{-1}B\left(CA^{-1}B + D^{-1}\right)^{-1}CA^{-1}$$

- **Block matrix inversion**:

$$\begin{aligned}
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}^{-1} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix}^{-1} \\
&= \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix} \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & -\left(A - BD^{-1}C\right)^{-1}BD^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & D^{-1} + D^{-1}C\left(A - BD^{-1}C\right)^{-1}BD^{-1} \end{bmatrix}
\end{aligned}$$

Eigenvalue Decomposition

- For any $A \in \mathbb{R}^{n\times n}$, if there exists $\mathbf{q} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and $\lambda \in \mathbb{C}$ such that:

$$A\mathbf{q} = \lambda\mathbf{q}$$

  then $\mathbf{q}$ is an **eigenvector** corresponding to the **eigenvalue** λ.

- A real matrix can have complex eigenvalues and eigenvectors, which appear in conjugate pairs.

- Eigenvectors are not unique since for any $c \in \mathbb{C} \setminus \{0\}$, $c\mathbf{q}$ is an eigenvector corresponding to the same eigenvalue.

- The n eigenvalues of $A \in \mathbb{R}^{n\times n}$ are precisely the n roots of the **characteristic polynomial** of A:

$$p(\lambda) := \det(\lambda I - A)$$

- We can put all n equations $A\mathbf{q}_i = \lambda_i\mathbf{q}_i$ to obtain the **eigen decomposition** of A:

$$A = Q\Lambda Q^{-1}$$

Eigenvalue Decomposition

- The roots of a polynomial are continuous functions of its coefficients and hence the eigenvalues of a matrix are continuous functions of its entries.

$$\mathrm{tr}(A) := \sum_{i=1}^{n} \lambda_i \qquad \det(A) := \prod_{i=1}^{n} \lambda_i$$

- $A^\top$ has the same eigenvalues and eigenvectors as A
- $A^\top A$ has the same eigenvectors as A but its eigenvalues are λ^2
- A^k for $k = 1, 2, \ldots$ has the same eigenvectors as A but its eigenvalues are λ^k
- A^{-1} has the same eigenvectors as A but its eigenvalues are λ^{-1}
- The eigenvalues of A are invariant under any unitary transform $U^* A U$ for $U^* U = U U^* = I$
- If A is symmetric ($A^\top = A$), then all its eigenvalues are real and all its eigenvectors are orthogonal ($Q^{-1} = Q^\top$)

Singular Value Decomposition

- An eigen-decomposition does not exist for $A \in \mathbb{R}^{m \times n}$
- $A \in \mathbb{R}^{m \times n}$ with rank $r \leq \min\{m, n\}$ can be diagonalized by two orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ via **singular value decomposition**:

$$A = U \Sigma V^\top \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & \end{bmatrix} \in \mathbb{R}^{m \times n}$$

- U contains the m orthogonal eigenvectors of the symmetric matrix $AA^\top \in \mathbb{R}^{m \times m}$ and satisfies $U^\top U = UU^\top = I$
- V contains the n orthogonal eigenvectors of the symmetric matrix $A^\top A \in \mathbb{R}^{n \times n}$ and satisfies $V^\top V = VV^\top = I$
- Σ contains the singular values $\sigma_i = \sqrt{\lambda_i}$, equal to the square roots of the r non-zero eigenvalues λ_i of $AA^\top$ or $A^\top A$, on its diagonal
- If A is normal ($A^\top A = AA^\top$), its singular values are related to its eigenvalues via $\sigma_i = |\lambda_i|$

Matrix Pseudo Inverse

- The **pseudo-inverse** $A^\dagger \in \mathbb{R}^{n\times m}$ of $A \in \mathbb{R}^{m\times n}$ can be obtained from its SVD $A = U\Sigma V^\top$:

$$A^\dagger = V\Sigma^\dagger U^T \qquad \Sigma^\dagger = \begin{bmatrix} 1/\sigma_1 & & & \\ & \ddots & & \\ & & 1/\sigma_r & \\ & & & \end{bmatrix} \in \mathbb{R}^{n\times m}$$

- The pseudo-inverse $A^\dagger \in \mathbb{R}^{n\times m}$ satisfies the Moore-Penrose conditions:
  - $AA^\dagger A = A$
  - $A^\dagger AA^\dagger = A^\dagger$
  - $\left(AA^\dagger\right)^\top = AA^\dagger$
  - $\left(A^\dagger A\right)^\top = A^\dagger A$

Linear System of Equations

- Consider the linear system of equations $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$ with SVD $A = U\Sigma V^\top$ and rank r

- The **column space** or **image** of A is $im(A) \subseteq \mathbb{R}^m$ and is spanned by the r columns of U corresponding to non-zero singular values

- The **null space** or **kernel** of A is $ker(A) \subseteq \mathbb{R}^n$ and is spanned by the $n - r$ columns of V corresponding to zero singular values

- The **row space** or **co-image** of A is $im(A^\top) \subseteq \mathbb{R}^n$ and is spanned by the r columns of V corresponding to non-zero singular values

- The **left null space** or **co-kernel** of A is $ker(A^\top) \subseteq \mathbb{R}^m$ and is spanned by the $m - r$ columns of U corresponding to zero singular values

- The **domain** of A is $\mathbb{R}^n = ker(A) \oplus im(A^\top)$

- The **co-domain** of A is $\mathbb{R}^m = ker(A^\top) \oplus im(A)$

Solution of Linear System of Equations

- Consider the linear system of equations $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$ with SVD $A = U\Sigma V^\top$ and rank r
- If $\mathbf{b} \in im(A)$, i.e., $\mathbf{b}^\top \mathbf{v} = 0$ for all $\mathbf{v} \in ker(A^\top)$, then $A\mathbf{x} = \mathbf{b}$ has **one or infinitely many solutions** $\mathbf{x} = A^\dagger \mathbf{b} + (I - A^\dagger A)\mathbf{y}$ for any $\mathbf{y} \in \mathbb{R}^n$
- If $\mathbf{b} \notin im(A)$, then **no solution exists** and $\mathbf{x} = A^\dagger \mathbf{b}$ is an approximate solution with minimum $\|\mathbf{x}\|$ and $\|A\mathbf{x} - \mathbf{b}\|$ norms
- If $m = n = r$, then $A\mathbf{x} = \mathbf{b}$ has a **unique solution** $\mathbf{x} = A^\dagger \mathbf{b} = A^{-1}\mathbf{b}$

Positive Semidefinite Matrices

- The product $\mathbf{x}^\top A\mathbf{x}$ for $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$ is called a **quadratic form** and A can be assumed **symmetric**, $A = A^\top$, because:

$$\frac{1}{2}\mathbf{x}^\top(A + A^\top)\mathbf{x} = \mathbf{x}^\top A\mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^n$$

- A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is **positive semidefinite** if $\mathbf{x}^\top A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

- A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is **positive definite** if it is positive semidefinite and if $\mathbf{x}^\top A\mathbf{x} = 0$ implies $\mathbf{x} = 0$.

- All eigenvalues of a symmetric positive semidefinite matrix are non-negative.

- All eigenvalues of a symmetric positive definite matrix are positive.

Schur Complement

- The Schur complement of block D of $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is $S_D = A - BD^{-1}C$
- The Schur complement of block A of $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is $S_A = D - CA^{-1}B$
- Let $M = \begin{bmatrix} A & B \\ B^\top & D \end{bmatrix}$ be symmetric. Then:
  - $M \succ 0 \Leftrightarrow A \succ 0, S_A = D - B^\top A^{-1} B \succ 0$
  - $M \succ 0 \Leftrightarrow D \succ 0, S_D = A - BD^{-1} B^\top \succ 0$
  - $M \succeq 0 \Leftrightarrow A \succeq 0, S_A \succeq 0, (I - AA^\dagger)B = 0$
  - $M \succeq 0 \Leftrightarrow D \succeq 0, S_D \succeq 0, (I - DD^\dagger)B^\top = 0$

Derivatives (numerator layout)

- Derivatives of $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^{m \times n}$ by scalar $x \in \mathbb{R}$:

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} \frac{dy_1}{dx} \\ \vdots \\ \frac{dy_m}{dx} \end{bmatrix} \in \mathbb{R}^{m \times 1} \qquad \frac{dY}{dx} = \begin{bmatrix} \frac{dY_{11}}{dx} & \cdots & \frac{dY_{1n}}{dx} \\ \vdots & \ddots & \vdots \\ \frac{d\mathbf{Y}_{m1}}{dx} & \cdots & \frac{d\mathbf{Y}_{mn}}{dx} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

- Derivatives of $y \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^m$ by vector $\mathbf{x} \in \mathbb{R}^p$:

$$\frac{dy}{d\mathbf{x}} = \underbrace{\begin{bmatrix} \frac{dy}{dx_1} & \cdots & \frac{dy}{dx_p} \end{bmatrix}}_{[\nabla_{\mathbf{x}} y]^\top \text{ (gradient transpose)}} \in \mathbb{R}^{1 \times p} \qquad \frac{d\mathbf{y}}{d\mathbf{x}} = \underbrace{\begin{bmatrix} \frac{dy_1}{dx_1} & \cdots & \frac{dy_1}{dx_p} \\ \vdots & \ddots & \vdots \\ \frac{dy_m}{dx_1} & \cdots & \frac{dy_m}{dx_p} \end{bmatrix}}_{\text{Jacobian}} \in \mathbb{R}^{m \times p}$$

- Derivative of $y \in \mathbb{R}$ by matrix $\mathbf{X} \in \mathbb{R}^{p \times q}$:

$$\frac{dy}{dX} = \begin{bmatrix} \frac{dy}{dX_{11}} & \cdots & \frac{dy}{dX_{p1}} \\ \vdots & \ddots & \vdots \\ \frac{dy}{dX_{1q}} & \cdots & \frac{dy}{dX_{pq}} \end{bmatrix} \in \mathbb{R}^{q \times p}$$

Matrix Derivatives Example

- $\frac{d}{dX_{ij}} X = \mathbf{e}_i \mathbf{e}_j^\top$
- $\frac{d}{d\mathbf{x}} A\mathbf{x} = A$
- $\frac{d}{d\mathbf{x}} \mathbf{x}^\top A \mathbf{x} = \mathbf{x}^\top (A + A^\top)$
- $\frac{d}{dx} M^{-1}(x) = -M^{-1}(x) \frac{dM(x)}{dx} M^{-1}(x)$
- $\frac{d}{dX} \operatorname{tr}(AX^{-1}B) = -X^{-1}BAX^{-1}$
- $\frac{d}{dX} \log\det X = X^{-1}$

Matrix Derivatives Example

- $\frac{d}{d\mathbf{x}} A\mathbf{x} = \begin{bmatrix} \frac{d}{dx_1}\sum_{j=1}^n A_{1j}x_j & \cdots & \frac{d}{dx_n}\sum_{j=1}^n A_{1j}x_j \\ \vdots & \ddots & \vdots \\ \frac{d}{dx_1}\sum_{j=1}^n A_{mj}x_j & \cdots & \frac{d}{dx_n}\sum_{j=1}^n A_{mj}x_j \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$
- $\frac{d}{d\mathbf{x}} \mathbf{x}^\top A\mathbf{x} = \mathbf{x}^\top A^\top \frac{d\mathbf{x}}{d\mathbf{x}} + \mathbf{x}^\top \frac{dA\mathbf{x}}{d\mathbf{x}} = \mathbf{x}^\top (A^\top + A)$
- $M(x)M^{-1}(x) = I \quad \Rightarrow \quad 0 = \left[\frac{d}{dx} M(x)\right] M^{-1}(x) + M(x)\left[\frac{d}{dx} M^{-1}(x)\right]$
- $$\begin{aligned} \frac{d}{dX_{ij}} \operatorname{tr}(AX^{-1}B) &= \operatorname{tr}(A\frac{d}{dX_{ij}}X^{-1}B) = -\operatorname{tr}(AX^{-1}\mathbf{e}_i\mathbf{e}_j^\top X^{-1}B) \\ &= -\mathbf{e}_j^\top X^{-1}BAX^{-1}\mathbf{e}_i = -\mathbf{e}_i^\top \left(X^{-1}BAX^{-1}\right)^\top \mathbf{e}_j \end{aligned}$$
- $$\begin{aligned} \frac{d}{dX_{ij}} \log\det X &= \frac{1}{\det(X)} \frac{d}{dX_{ij}} \sum_{k=1}^n X_{ik}\mathbf{cof}_{ik}(X) \\ &= \frac{1}{\det(X)}\mathbf{cof}_{ij}(X) = \frac{1}{\det(X)}\mathbf{adj}_{ji}(X) = \mathbf{e}_i^\top X^{-T}\mathbf{e}_j \end{aligned}$$

Unconstrained Optimization

- Many problems we encounter in this course, lead to an **unconstrained optimization problem** over the Euclidean vector space $\mathbb{R}^d$:

$$\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x})$$

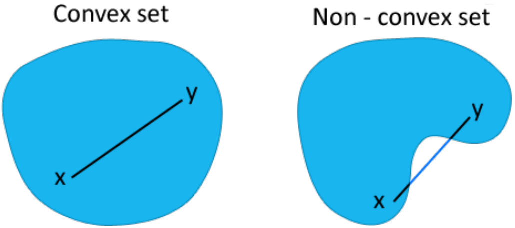
- A **global minimizer** $\mathbf{x}^* \in \mathbb{R}^d$ satisfies $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$. The value $f(\mathbf{x}^*)$ is called **global minimum**.
- A **local minimizer** $\mathbf{x}^* \in \mathbb{R}^d$ satisfies $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{N}(\mathbf{x}^*)$, where $\mathcal{N}(\mathbf{x}^*) \subset \mathbb{R}^d$ is a neighborhood around $\mathbf{x}^*$ (e.g., an open ball with small radius centered at $\mathbf{x}^*$). The value $f(\mathbf{x}^*)$ is called **local minimum**.
- The objective function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is **differentiable** if the gradient:

$$\nabla f(\mathbf{x}) := \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f(\mathbf{x})}{\partial x_d} \end{bmatrix}^\top \in \mathbb{R}^d$$

  exists at each $\mathbf{x} \in \mathbb{R}^d$
- A **critical point** $\bar{\mathbf{x}} \in \mathbb{R}^d$ satisfies $\nabla f(\bar{\mathbf{x}}) = 0$ or $\nabla f(\bar{\mathbf{x}}) = \text{undefined}$
- All minimizers are critical points but not all critical points are minimizers. A critical point is either a local maximizer, a local minimizer, or neither (saddle point).

Convexity

- A set $\mathcal{D} \subseteq \mathbb{R}^d$ is **convex** if $\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in \mathcal{D}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, $\lambda \in [0,1]$
- A convex set contains the line segment between any two points in it



- A function $f : \mathcal{D} \mapsto \mathbb{R}$ with $\mathcal{D} \subseteq \mathbb{R}^d$ is **convex** if:
  - $\mathcal{D}$ is a convex set
  - $f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, $\lambda \in [0,1]$
- **First-order convexity condition**: a differentiable $f : \mathcal{D} \mapsto \mathbb{R}$ with convex $\mathcal{D}$ is convex iff $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}$
- **Second-order convexity condition**: a twice-differentiable $f : \mathcal{D} \mapsto \mathbb{R}$ with convex $\mathcal{D}$ is convex iff $\nabla^2 f(\mathbf{x}) \succeq 0$ for all $\mathbf{x} \in \mathcal{D}$

Descent Direction

- Consider the **unconstrained optimization problem**:

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x})$$

Descent Direction Theorem

Suppose f is differentiable at $\bar{\mathbf{x}}$. If $\exists\ \delta\mathbf{x}$ such that $\nabla f(\bar{\mathbf{x}})^\top \delta\mathbf{x} < 0$, then $\exists\ \epsilon > 0$ such that $f(\bar{\mathbf{x}} + \alpha\delta\mathbf{x}) < f(\bar{\mathbf{x}})$ for all $\alpha \in (0, \epsilon)$.

- The vector $\delta\mathbf{x}$ is called a **descent direction**
- The theorem states that if a descent direction exists at $\bar{\mathbf{x}}$, then it is possible to move to a new point that has a lower f value
- **Steepest descent direction**: $\delta\mathbf{x} := -\frac{\nabla f(\bar{\mathbf{x}})}{\|\nabla f(\bar{\mathbf{x}})\|}$
- Based on this theorem, we can derive conditions for determining the optimality of $\bar{\mathbf{x}}$

Optimality Conditions

First-order Necessary Condition

Suppose f is differentiable at $\bar{\mathbf{x}}$. If $\bar{\mathbf{x}}$ is a local minimizer, then $\nabla f(\bar{\mathbf{x}}) = 0$.

Second-order Necessary Condition

Suppose f is twice-differentiable at $\bar{\mathbf{x}}$. If $\bar{\mathbf{x}}$ is a local minimizer, then $\nabla f(\bar{\mathbf{x}}) = 0$ and $\nabla^2 f(\bar{\mathbf{x}}) \succeq 0$.

Second-order Sufficient Condition

Suppose f is twice-differentiable at $\bar{\mathbf{x}}$. If $\nabla f(\bar{\mathbf{x}}) = 0$ and $\nabla^2 f(\bar{\mathbf{x}}) \succ 0$, then $\bar{\mathbf{x}}$ is a local minimizer.

Necessary and Sufficient Condition

Suppose f is differentiable at $\bar{\mathbf{x}}$. If f is **convex**, then $\bar{\mathbf{x}}$ is a global minimizer **if and only if** $\nabla f(\bar{\mathbf{x}}) = 0$.

Descent Optimization Methods

- A critical point of f can be obtained by solving $\nabla f(\mathbf{x}) = 0$ but an explicit solution may be difficult to derive
- **Descent methods**: iterative methods to obtain a solution of $\nabla f(\mathbf{x}) = 0$
- Given an initial guess $\mathbf{x}^{(k)}$, take a step of size $\alpha^{(k)} > 0$ along a descent direction $\delta\mathbf{x}^{(k)}$:
$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha^{(k)} \delta\mathbf{x}^{(k)}$$
- Different methods differ in the way $\delta\mathbf{x}^{(k)}$ and $\alpha^{(k)}$ are chosen
- $\delta\mathbf{x}^{(k)}$ needs to be a descent direction: $\nabla f(\mathbf{x}^{(k)})^\top \delta\mathbf{x}^{(k)} < 0$, $\forall \mathbf{x}^{(k)} \neq \mathbf{x}^*$
- $\alpha^{(k)}$ needs to ensure sufficient decrease in f to guarantee convergence:
    - The best step size choice is $\alpha^{(k)} \in \arg\min_{\alpha > 0} f(\mathbf{x}^{(k)} + \alpha\delta\mathbf{x}^{(k)})$
    - In practice, $\alpha^{(k)}$ is obtained via approximate **line search** methods

Gradient Descent (First-Order Method)

- **Idea**: $-\nabla f(\mathbf{x}^{(k)})$ points in the direction of steepest local descent
- **Gradient descent**: let $\delta\mathbf{x}^{(k)} := -\nabla f(\mathbf{x}^{(k)})$ and iterate:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \alpha^{(k)}\nabla f(\mathbf{x}^{(k)})$$

- **Step size**: a good choice for $\alpha^{(k)}$ is $\frac{1}{L}$, where $L > 0$ is the Lipschitz constant of $\nabla f(\mathbf{x})$:

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{x}')\| \leq L\|\mathbf{x} - \mathbf{x}'\| \qquad \forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$$

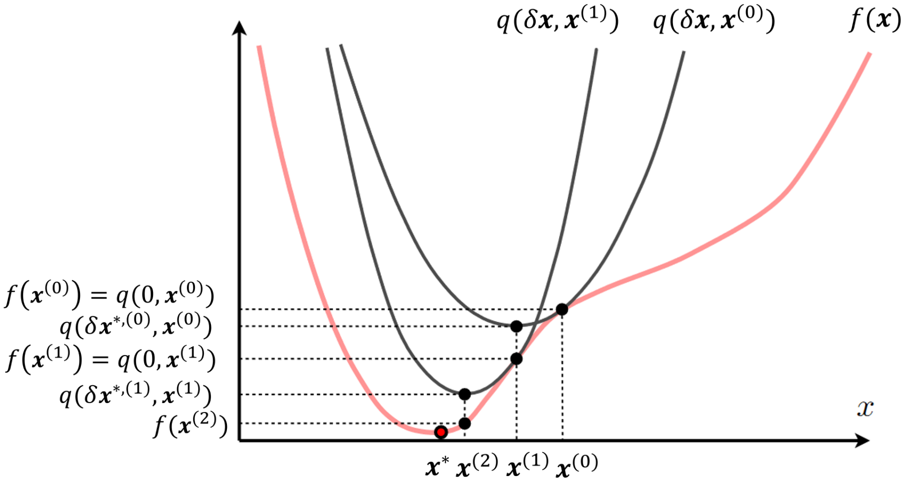
Newton's Method (Second-Order Method)

- **Newton's method**: iteratively approximates f by a quadratic function
- Since $\delta\mathbf{x}$ is a 'small' change to the initial guess $\mathbf{x}^{(k)}$, we can approximate f using a Taylor-series expansion:

$$
\begin{aligned}
f(\mathbf{x}^{(k)} + \delta\mathbf{x}) &\approx f(\mathbf{x}^{(k)}) + \underbrace{\left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \bigg|_{\mathbf{x}=\mathbf{x}^{(k)}} \right)}_{\text{Gradient Transpose}} \delta\mathbf{x} + \frac{1}{2}\delta\mathbf{x}^\top \underbrace{\left(\frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}^\top} \bigg|_{\mathbf{x}=\mathbf{x}^{(k)}} \right)}_{\text{Hessian}} \delta\mathbf{x} \\
&=: \quad \underbrace{q(\delta\mathbf{x}, \mathbf{x}^{(k)})}_{\text{quadratic function in } \delta\mathbf{x}}
\end{aligned}
$$

- The symmetric Hessian matrix $\nabla^2 f(\mathbf{x}^{(k)})$ needs to be positive-definite for this method to work.

Newton's Method (Second-Order Method)



Newton's Method (Second-Order Method)

- Find $\delta\mathbf{x}$ that minimizes the quadratic approximation to $f(\mathbf{x}^{(k)} + \delta\mathbf{x})$
- Since this is an unconstrained optimization problem, $\delta\mathbf{x}$ can be determined by setting the derivative with respect to $\delta\mathbf{x}$ to zero:

$$0 = \frac{\partial q(\delta\mathbf{x}, \mathbf{x}^{(k)})}{\partial \delta\mathbf{x}} = \left(\left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right) + \delta\mathbf{x}^\top \left(\left.\frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}^\top}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right)$$

$$\Rightarrow \quad \left(\left.\frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}^\top}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right) \delta\mathbf{x} = -\left(\left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right)^\top$$

- The above is a linear system of equations and can be solved when the Hessian is invertible, i.e., $\nabla^2 f(\mathbf{x}^{(k)}) \succ 0$:

$$\delta\mathbf{x} = -\left[\nabla^2 f(\mathbf{x}^{(k)}) \right]^{-1} \nabla f(\mathbf{x}^{(k)})$$

- **Newton's method**:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \alpha^{(k)} \left[\nabla^2 f(\mathbf{x}^{(k)}) \right]^{-1} \nabla f(\mathbf{x}^{(k)})$$

Newton's Method (Comments)

- Newton's method, like any other descent method, converges only to a **local** minimum
- **Damped Newton phase**: when the iterates are "far away" from the optimal point, the function value is decreased sublinearly, i.e., the step sizes $\alpha^{(k)}$ are small
- **Quadratic convergence phase**: when the iterates are "sufficiently close" to the optimum, full Newton steps are taken, i.e., $\alpha^{(k)} = 1$, and the function value converges quadratically to the optimum
- A **disadvantage** of Newton's method is the need to form the Hessian, which can be numerically ill-conditioned or very computationally expensive in high-dimensional problems

Gauss-Newton's Method

- **Gauss-Newton** is an approximation to Newton's method that avoids computing the Hessian. It is applicable when the objective function has the following quadratic form:

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^\top \mathbf{e}(\mathbf{x}) \qquad \mathbf{e}(\mathbf{x}) \in \mathbb{R}^m$$

- The Jacobian and Hessian matrices are:

$$\text{Jacobian:} \qquad \left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} = \mathbf{e}(\mathbf{x}^{(k)})^\top \left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right)$$

$$\text{Hessian:} \qquad \left.\frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}\partial \mathbf{x}^\top}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} = \left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right)^\top \left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right) + \sum_{i=1}^{m} e_i(\mathbf{x}^{(k)}) \left(\left.\frac{\partial^2 e_i(\mathbf{x})}{\partial \mathbf{x}\partial \mathbf{x}^\top}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right)$$

Gauss-Newton's Method

- Near the minimum of f, the second term in the Hessian is small relative to the first and the Hessian can be approximated according to:

$$\left.\frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}^\top}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} \approx \left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right)^\top \left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right)$$

- The above does not involve any second derivatives

- Setting the gradient of this new quadratic approximation of f with respect to $\delta\mathbf{x}$ to zero, leads to the system:

$$\left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right)^\top \left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right) \delta\mathbf{x} = -\left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}\right)^\top \mathbf{e}(\mathbf{x}^{(k)})$$

- **Gauss-Newton's method**:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha^{(k)} \delta\mathbf{x}$$

Gauss-Newton's Method (Alternative Derivation)

- Another way to think about the Gauss-Newton method is to start with a Taylor expansion of $\mathbf{e}(\mathbf{x})$ instead of $f(\mathbf{x})$:

$$\mathbf{e}(\mathbf{x}^{(k)} + \delta\mathbf{x}) \approx \mathbf{e}(\mathbf{x}^{(k)}) + \left(\left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right) \delta\mathbf{x}$$

- Substituting into f leads to:

$$f(\mathbf{x}^{(k)} + \delta\mathbf{x}) \approx \frac{1}{2} \left(\mathbf{e}(\mathbf{x}^{(k)}) + \left(\left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right) \delta\mathbf{x} \right)^\top \left(\mathbf{e}(\mathbf{x}^{(k)}) + \left(\left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right) \delta\mathbf{x} \right)$$

- Minimizing this with respect to $\delta\mathbf{x}$ leads to the same system as before:

$$\left(\left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right)^\top \left(\left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right) \delta\mathbf{x} = -\left(\left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right)^\top \mathbf{e}(\mathbf{x}^{(k)})$$

Levenberg-Marquardt's Method

- The **Levenberg-Marquardt** modification to the Gauss-Newton method uses a positive diagonal matrix D to condition the Hessian approximation:

$$\left(\left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right)^{\top} \left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right) + \lambda D \right) \delta \mathbf{x} = - \left(\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} \right)^{\top} \mathbf{e}(\mathbf{x}^{(k)})$$

- When $\lambda \geq 0$ is large, the descent vector $\delta \mathbf{x}$ corresponds to a very small step in the direction of steepest descent. This helps when the Hessian approximation is poor or poorly conditioned by providing a meaningful direction.

Levenberg-Marquardt's Method (Summary)

- An iterative optimization approach for the unconstrained problem:

$$\min_{\mathbf{x}} f(\mathbf{x}) := \frac{1}{2}\sum_j \mathbf{e}_j(\mathbf{x})^\top \mathbf{e}_j(\mathbf{x}) \qquad \mathbf{e}_j(\mathbf{x}) \in \mathbb{R}^{m_j},\ \mathbf{x} \in \mathbb{R}^n$$

- Given an initial guess $\mathbf{x}^{(k)}$, determine a descent direction $\delta\mathbf{x}$ by solving:

$$\left(\sum_j J_j(\mathbf{x}^{(k)})^\top J_j(\mathbf{x}^{(k)}) + \lambda D\right)\delta\mathbf{x} = -\left(\sum_j J_j(\mathbf{x}^{(k)})^\top \mathbf{e}_j(\mathbf{x}^{(k)})\right)$$

  where $J_j(\mathbf{x}) := \frac{\partial \mathbf{e}_j(\mathbf{x})}{\partial \mathbf{x}} \in \mathbb{R}^{m_j \times n}$, $\lambda \geq 0$, $D \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, e.g., $D = \mathbf{diag}\left(\sum_j J_j(\mathbf{x}^{(k)})^\top J_j(\mathbf{x}^{(k)})\right)$

- Obtain an updated estimate according to:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha^{(k)}\delta\mathbf{x}$$

Unconstrained Optimization Example

- Let $f(\mathbf{x}) := \frac{1}{2}\sum_{j=1}^{n} \|A_j\mathbf{x} + b_j\|_2^2$ for $\mathbf{x} \in \mathbb{R}^d$ and assume $\sum_{j=1}^{n} A_j^\top A_j \succ 0$
- Solve the unconstrained optimization problem $\min_{\mathbf{x}} f(\mathbf{x})$ using:
  - The necessary and sufficient optimality condition for convex function f
  - Gradient descent
  - Newton's method
  - Gauss-Newton's method
- We will need $\nabla f(\mathbf{x})$ and $\nabla^2 f(\mathbf{x})$:

$$\frac{df(\mathbf{x})}{d\mathbf{x}} = \frac{1}{2}\sum_{j=1}^{n} \frac{d}{d\mathbf{x}}\|A_j\mathbf{x} + b_j\|_2^2 = \sum_{j=1}^{n} (A_j\mathbf{x} + b_j)^\top A_j$$

$$\nabla f(\mathbf{x}) = \frac{df(\mathbf{x})}{d\mathbf{x}}^\top = \left(\sum_{j=1}^{n} A_j^\top A_j\right)\mathbf{x} + \left(\sum_{j=1}^{n} A_j^\top b_j\right)$$

$$\nabla^2 f(\mathbf{x}) = \frac{d}{d\mathbf{x}}\nabla f(\mathbf{x}) = \sum_{j=1}^{n} A_j^\top A_j \succ 0$$

Necessary and Sufficient Optimality Condition

- Solve $\nabla f(\mathbf{x}) = 0$ for $\mathbf{x}$:

$$0 = \nabla f(\mathbf{x}) = \left(\sum_{j=1}^{n} A_j^\top A_j \right) \mathbf{x} + \left(\sum_{j=1}^{n} A_j^\top b_j \right)$$

$$\mathbf{x} = - \left(\sum_{j=1}^{n} A_j^\top A_j \right)^{-1} \left(\sum_{j=1}^{n} A_j^\top b_j \right)$$

- The solution above is unique since we assumed that $\sum_{j=1}^{n} A_j^\top A_j \succ 0$

Gradient Descent

- Start with an initial guess $\mathbf{x}^{(0)} = \mathbf{0}$
- At iteration k, gradient descent uses the descent direction $\delta\mathbf{x}^{(k)} = -\nabla f(\mathbf{x}^{(k)})$
- Given arbitary $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$, determine the Lipschitz constant of $\nabla f(\mathbf{x})$:

$$\|\nabla f(\mathbf{x}_1) - \nabla f(\mathbf{x}_2)\| = \left\| \left(\sum_{j=1}^{n} A_j^\top A_j \right) (\mathbf{x}_1 - \mathbf{x}_2) \right\| \leq \underbrace{\left\| \sum_{j=1}^{n} A_j^\top A_j \right\|}_{L} \|\mathbf{x}_1 - \mathbf{x}_2\|$$

- Choose step size $\alpha^{(k)} = \frac{1}{L}$ and iterate:

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} + \alpha^{(k)} \delta\mathbf{x}^{(k)} \\ &= \mathbf{x}^{(k)} - \frac{1}{L} \left(\sum_{j=1}^{n} A_j^\top A_j \right) \mathbf{x}^{(k)} - \frac{1}{L} \left(\sum_{j=1}^{n} A_j^\top b_j \right) \end{aligned}$$

Newton's Method

- Start with an initial guess $\mathbf{x}^{(0)} = \mathbf{0}$
- At iteration k, Newton's method uses the descent direction:

$$\begin{aligned}
\delta \mathbf{x}^{(k)} &= -\left[\nabla^2 f(\mathbf{x}^{(k)})\right]^{-1} \nabla f(\mathbf{x}^{(k)}) \\
&= -\mathbf{x}^{(k)} - \left(\sum_{j=1}^{n} A_j^\top A_j\right)^{-1} \left(\sum_{j=1}^{n} A_j^\top b_j\right)
\end{aligned}$$

  and updates the solution estimate via:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \delta \mathbf{x}^{(k)} = -\left(\sum_{j=1}^{n} A_j^\top A_j\right)^{-1} \left(\sum_{j=1}^{n} A_j^\top b_j\right)$$

- Note that for this problem, Newton's method converges in one iteration!

Gauss-Newton's Method

- $f(\mathbf{x})$ is of the form $\frac{1}{2}\sum_{j=1}^{n} \mathbf{e}_j(\mathbf{x})^\top \mathbf{e}_j(\mathbf{x})$ for $\mathbf{e}_j(\mathbf{x}) := A_j\mathbf{x} + b_j$
- The Jacobian of $\mathbf{e}_j(\mathbf{x})$ is $J_j(\mathbf{x}) = A_j$
- Start with an initial guess $\mathbf{x}^{(0)} = \mathbf{0}$
- At iteration k, Gauss-Newton's method uses the descent direction:

$$
\begin{aligned}
\delta\mathbf{x}^{(k)} &= -\left(\sum_{j=1}^{n} J_j(\mathbf{x}^{(k)})^\top J_j(\mathbf{x}^{(k)})\right)^{-1} \left(\sum_{j=1}^{n} J_j(\mathbf{x}^{(k)})^\top \mathbf{e}_j(\mathbf{x}^{(k)})\right) \\
&= -\left(\sum_{j=1}^{n} A_j^\top A_j\right)^{-1} \left(\sum_{j=1}^{n} A_j^\top (A_j\mathbf{x}^{(k)} + b_j)\right) \\
&= -\mathbf{x}^{(k)} - \left(\sum_{j=1}^{n} A_j^\top A_j\right)^{-1} \left(\sum_{j=1}^{n} A_j^\top b_j\right)
\end{aligned}
$$

- If $\alpha^{(k)} = 1$, in this problem, Gauss-Newton's method behaves exactly like Newton's method and coverges in one iteration!