ECE276A: Sensing & Estimation in Robotics
Lecture 11: Kalman Filtering

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Bayes Filter

- **Markov Assumptions**:
  - **Motion model**: given $\mathbf{x}_t$, $\mathbf{u}_t$, the state $\mathbf{x}_{t+1}$ is independent of the history $\mathbf{x}_{0:t-1}$, $\mathbf{z}_{0:t-1}$, $\mathbf{u}_{0:t-1}$:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t, \mathbf{w}_t) \sim p_f(\cdot \mid \mathbf{x}_t, \mathbf{u}_t)$$

  - **Observation model**: given $\mathbf{x}_t$, the observation $\mathbf{z}_t$ is independent of the history $\mathbf{x}_{0:t-1}$, $\mathbf{z}_{0:t-1}$, $\mathbf{u}_{0:t-1}$:

$$\mathbf{z}_t = h(\mathbf{x}_t, \mathbf{v}_t) \sim p_h(\cdot \mid \mathbf{x}_t)$$

- **Prior**: $p_{t|t}(\mathbf{x}_t) := p(\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1})$
- **Prediction**: $p_{t+1|t}(\mathbf{x}) = \int p_f(\mathbf{x} \mid \mathbf{s}, \mathbf{u}_t) p_{t|t}(\mathbf{s}) d\mathbf{s}$
- **Update**: $p_{t+1|t+1}(\mathbf{x}) = \frac{p_h(\mathbf{z}_{t+1}|\mathbf{x}) p_{t+1|t}(\mathbf{x})}{\int p_h(\mathbf{z}_{t+1}|\mathbf{s}) p_{t+1|t}(\mathbf{s}) d\mathbf{s}}$

Kalman Filter

- A Bayes filter with the following **assumptions**:
  - The prior pdf $p_{t|t}$ is Gaussian
  - The motion model is linear in the state $\mathbf{x}_t$ with Gaussian noise $\mathbf{w}_t$
  - The observation model is linear in the state $\mathbf{x}_t$ with Gaussian noise $\mathbf{v}_t$
  - The motion noise $\mathbf{w}_t$ and observation noise $\mathbf{v}_t$ are independent of each other, of the state $\mathbf{x}_t$, and across time
- **Prior**: $\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1} \sim \mathcal{N}(\boldsymbol{\mu}_{t|t}, \Sigma_{t|t})$
- **Motion Model**:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t, \mathbf{w}_t) := F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim \mathcal{N}(0, W)$$

$$\mathbf{x}_{t+1} \mid \mathbf{x}_t, \mathbf{u}_t \sim \mathcal{N}(F\mathbf{x}_t + G\mathbf{u}_t, W), \quad F \in \mathbb{R}^{d_x \times d_x}, G \in \mathbb{R}^{d_x \times d_u}, \; W \in \mathbb{R}^{d_x \times d_x}$$

- **Observation Model**:

$$\mathbf{z}_t = h(\mathbf{x}_t, \mathbf{v}_t) := H\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim \mathcal{N}(0, V)$$

$$\mathbf{z}_t \mid \mathbf{x}_t \sim \mathcal{N}(H\mathbf{x}_t, V), \quad H \in \mathbb{R}^{d_z \times d_x}, \; V \in \mathbb{R}^{d_z \times d_z}$$

Matrix Manipulation

- The following results are necessary for deriving the Kalman filter:
  - **Matrix inversion lemma**:

$$(A + BDC)^{-1} = A^{-1} - A^{-1}B\left(D^{-1} + CA^{-1}B\right)^{-1}CA^{-1}$$

  - **Matrix block inversion**:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & \underbrace{D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1}}_{(D-CA^{-1}B)^{-1}} \end{bmatrix}$$

  - **Schur complement**: $\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(S_A) = \det(D)\det(S_D)$
    - $S_A = D - CA^{-1}B$
    - $S_D = A - BD^{-1}C$
  - **Square completion**:

$$\frac{1}{2}\mathbf{x}^\top A\mathbf{x} + \mathbf{b}^\top\mathbf{x} + \mathbf{c} = \frac{1}{2}\left(\mathbf{x} + A^{-1}\mathbf{b}\right)^\top A\left(\mathbf{x} + A^{-1}\mathbf{b}\right) - \frac{1}{2}\mathbf{b}^\top A^{-1}\mathbf{b} + \mathbf{c}$$

Gaussian Distribution

- **Gaussian random vector** $X \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$
  - parameters:
    - **mean** $\boldsymbol{\mu} \in \mathbb{R}^d$
    - **covariance** $\Sigma \in \mathbb{S}^d_{\succ 0}$ (symmetric positive definite matrix)
  - pdf: $\phi(\mathbf{x}; \boldsymbol{\mu}, \Sigma) := \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$
  - expectation: $\mathbb{E}[X] = \int \mathbf{x} \phi(\mathbf{x}; \boldsymbol{\mu}, \Sigma) d\mathbf{x} = \boldsymbol{\mu}$
  - (co)variance: $Var[X] = \mathbb{E}\left[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^\top\right] = \Sigma$

Information Form of the Gaussian Distribution

- An alternative Gaussian parametrization: $X \sim \mathcal{G}(\boldsymbol{\nu}, \Omega)$
  - parameters:
    - **information mean** $\boldsymbol{\nu} := \Sigma^{-1}\boldsymbol{\mu} \in \mathbb{R}^d$
    - **information matrix**: $\Omega := \Sigma^{-1} \in \mathbb{S}^d_{\succ 0}$
  - pdf: obtained using square completion:

$$\phi(\mathbf{x}; \boldsymbol{\nu}, \Omega) = \sqrt{\frac{\det(\Omega)}{(2\pi)^d}} \exp\left(-\frac{1}{2}\left(\mathbf{x}^\top \Omega \mathbf{x} - 2\boldsymbol{\nu}^\top \mathbf{x} + \boldsymbol{\nu}^\top \Omega^{-1} \boldsymbol{\nu}\right)\right)$$
$$\propto \exp\left(-\frac{1}{2}\mathbf{x}^\top \Omega \mathbf{x} + \boldsymbol{\nu}^\top \mathbf{x}\right)$$

  - expectation: $\mathbb{E}[X] = \Omega^{-1}\boldsymbol{\nu}$
  - (co)variance: $Var[X] = \Omega^{-1}$

Covariance and Information Matrix Relationship

- The matrix block inversion relates the elements of Ω and Σ
- Let $\Omega = \begin{bmatrix} \Omega_{AA} & \Omega_{AB} \\ \Omega_{AB}^\top & \Omega_{BB} \end{bmatrix} = \Sigma^{-1} = \begin{bmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{AB}^\top & \Sigma_{BB} \end{bmatrix}^{-1}$
- The blocks are related via:

$$
\begin{aligned}
\Omega_{AA} &= (\Sigma_{AA} - \Sigma_{AB}\Sigma_{BB}^{-1}\Sigma_{AB}^\top)^{-1} \\
\Omega_{AB} &= -\Omega_{AA}\Sigma_{AB}\Sigma_{BB}^{-1} \\
\Omega_{BB} &= \Sigma_{BB}^{-1} + \Sigma_{BB}^{-1}\Sigma_{AB}^\top\Omega_{AA}\Sigma_{AB}\Sigma_{BB}^{-1} \\
&= (\Sigma_{BB} - \Sigma_{AB}^\top\Sigma_{AA}^{-1}\Sigma_{AB})^{-1}
\end{aligned}
$$

- The determinants are related via:

$$
\begin{aligned}
\det(\Sigma) = \det(\Sigma_{AA})\det(\Omega_{BB}^{-1}) = \det(\Sigma_{BB})\det(\Omega_{AA}^{-1}) \\
\det(\Omega) = \det(\Omega_{AA})\det(\Sigma_{BB}^{-1}) = \det(\Omega_{BB})\det(\Sigma_{AA}^{-1})
\end{aligned}
$$

Gaussian Marginals and Conditionals

- Consider a **joint Gaussian distribution**:

$$\mathbf{x} := \begin{pmatrix} \mathbf{x}_A \\ \mathbf{x}_B \end{pmatrix} \sim \mathcal{N}\left(\underbrace{\begin{pmatrix} \boldsymbol{\mu}_A \\ \boldsymbol{\mu}_B \end{pmatrix}}_{\mu}, \underbrace{\begin{bmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{AB}^\top & \Sigma_{BB} \end{bmatrix}}_{\Sigma} \right)$$

- The **marginal distributions** are also Gaussian:

$$\mathbf{x}_A \sim \mathcal{N}(\boldsymbol{\mu}_A, \Sigma_{AA}) \qquad \qquad \mathbf{x}_B \sim \mathcal{N}(\boldsymbol{\mu}_B, \Sigma_{BB})$$

- The **conditional distributions** are also Gaussian:

$$\mathbf{x}_B \mid \mathbf{x}_A \sim \mathcal{N}\left(\boldsymbol{\mu}_B + \Sigma_{AB}^\top \Sigma_{AA}^{-1} (\mathbf{x}_A - \boldsymbol{\mu}_A), \underbrace{\Sigma_{BB} - \Sigma_{AB}^\top \Sigma_{AA}^{-1} \Sigma_{AB}}_{\text{Schur complement of } \Sigma_{AA}} \right)$$

$$\mathbf{x}_A \mid \mathbf{x}_B \sim \mathcal{N}\left(\boldsymbol{\mu}_A + \Sigma_{AB} \Sigma_{BB}^{-1} (\mathbf{x}_B - \boldsymbol{\mu}_B), \underbrace{\Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{AB}^\top}_{\text{Schur complement of } \Sigma_{BB}} \right)$$

Gaussian Marginal

- Let $\tilde{\mathbf{x}}_A := \mathbf{x}_A - \boldsymbol{\mu}_A \in \mathbb{R}^n$ and $\tilde{\mathbf{x}}_B := \mathbf{x}_B - \boldsymbol{\mu}_B \in \mathbb{R}^m$ and consider:

$$
\begin{aligned}
p(\mathbf{x}_A) &= \int \phi\left(\begin{pmatrix}\mathbf{x}_A\\ \mathbf{x}_B\end{pmatrix}; \begin{pmatrix}\boldsymbol{\mu}_A\\ \boldsymbol{\mu}_B\end{pmatrix}, \begin{bmatrix}\Sigma_{AA} & \Sigma_{AB}\\ \Sigma_{AB}^\top & \Sigma_{BB}\end{bmatrix}\right) d\mathbf{x}_B \\
&= \frac{1}{(2\pi)^{\frac{n+m}{2}} \det(\Sigma)^{1/2}} \int \exp\left(-\frac{1}{2}\left(\tilde{\mathbf{x}}_A^\top \Omega_{AA}\tilde{\mathbf{x}}_A + 2\tilde{\mathbf{x}}_A^\top \Omega_{AB}\tilde{\mathbf{x}}_B + \tilde{\mathbf{x}}_B^\top \Omega_{BB}\tilde{\mathbf{x}}_B\right)\right) d\tilde{\mathbf{x}}_B \\
&\underline{\underline{\text{Sq. Comp.}}} \frac{1}{(2\pi)^{\frac{n+m}{2}} \det(\Sigma)^{1/2}} \int \exp\left(-\frac{1}{2}\Big[(\tilde{\mathbf{x}}_B + \Omega_{BB}^{-1}\Omega_{AB}^\top\tilde{\mathbf{x}}_A)^\top \Omega_{BB}(\tilde{\mathbf{x}}_B + \Omega_{BB}^{-1}\Omega_{AB}^\top\tilde{\mathbf{x}}_A)\right. \\
&\qquad\qquad \left. - \tilde{\mathbf{x}}_A^\top \Omega_{AB}\Omega_{BB}^{-1}\Omega_{AB}^\top \tilde{\mathbf{x}}_A + \tilde{\mathbf{x}}_A^\top \Omega_{AA}\tilde{\mathbf{x}}_A\Big]\right) d\tilde{\mathbf{x}}_B
\end{aligned}
$$

- Note that:
  - $\int \phi(\tilde{\mathbf{x}}_B; -\Omega_{BB}^{-1}\Omega_{AB}^\top\tilde{\mathbf{x}}_A, \Omega_{BB}^{-1}) d\tilde{\mathbf{x}}_B = 1$
  - $\Sigma_{AA}^{-1} = \Omega_{AA} - \Omega_{AB}\Omega_{BB}^{-1}\Omega_{AB}^\top$
  - $\det(\Sigma) = \det(\Sigma_{AA})\det(\Omega_{BB}^{-1})$

$$
= \frac{1}{(2\pi)^{\frac{n}{2}} \det(\Sigma_{AA})^{1/2}} \exp\left(-\frac{1}{2}\tilde{\mathbf{x}}_A^\top \Sigma_{AA}^{-1}\tilde{\mathbf{x}}_A\right) \underbrace{\int \phi(\tilde{\mathbf{x}}_B; -\Omega_{BB}^{-1}\Omega_{AB}^\top\tilde{\mathbf{x}}_A, \Omega_{BB}^{-1}) d\tilde{\mathbf{x}}_B}_{\to 1}
$$

$$
\Rightarrow \boxed{\mathbf{x}_A \sim \mathcal{N}(\boldsymbol{\mu}_A, \Sigma_{AA})}
$$

Gaussian Conditional

$$p(\mathbf{x}_A \mid \mathbf{x}_B) = \frac{p(\mathbf{x}_A, \mathbf{x}_B)}{p(\mathbf{x}_B)} = \frac{\frac{1}{\sqrt{(2\pi)^{n+m}\det(\Sigma)}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}}{\frac{1}{\sqrt{(2\pi)^{m}\det(\Sigma_{BB})}} e^{-\frac{1}{2}(\mathbf{x}_B-\boldsymbol{\mu}_B)^\top \Sigma_{BB}^{-1}(\mathbf{x}_B-\boldsymbol{\mu}_B)}}$$

$$= \frac{1}{\sqrt{(2\pi)^{n}\det(\Sigma)/\det(\Sigma_{BB})}} e^{-\frac{1}{2}\left((\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) - (\mathbf{x}_B-\boldsymbol{\mu}_B)^\top \Sigma_{BB}^{-1}(\mathbf{x}_B-\boldsymbol{\mu}_B)\right)}$$

- Consider the exponent:

$$\tilde{\mathbf{x}}_A^\top \Omega_{AA}\tilde{\mathbf{x}}_A + \tilde{\mathbf{x}}_A^\top \Omega_{AB}\tilde{\mathbf{x}}_B + \tilde{\mathbf{x}}_B^\top \Omega_{AB}^\top \tilde{\mathbf{x}}_A + \tilde{\mathbf{x}}_B^\top \Omega_{BB}\tilde{\mathbf{x}}_B - \tilde{\mathbf{x}}_B^\top \Sigma_{BB}^{-1}\tilde{\mathbf{x}}_B$$

$$= \tilde{\mathbf{x}}_A^\top \Omega_{AA}\tilde{\mathbf{x}}_A - 2\tilde{\mathbf{x}}_A^\top \Omega_{AA}\Sigma_{AB}\Sigma_{BB}^{-1}\tilde{\mathbf{x}}_B + \tilde{\mathbf{x}}_B^\top \Sigma_{BB}^{-1}\Sigma_{AB}^\top \Omega_{AA}\Sigma_{AB}\Sigma_{BB}^{-1}\tilde{\mathbf{x}}_B$$

$$= (\tilde{\mathbf{x}}_A - \Sigma_{AB}\Sigma_{BB}^{-1}\tilde{\mathbf{x}}_B)^\top \Omega_{AA}(\tilde{\mathbf{x}}_A - \Sigma_{AB}\Sigma_{BB}^{-1}\tilde{\mathbf{x}}_B)$$

$$\Rightarrow \boxed{\mathbf{x}_A \mid \mathbf{x}_B \sim \mathcal{N}\left(\boldsymbol{\mu}_A + \Sigma_{AB}\Sigma_{BB}^{-1}(\mathbf{x}_B - \boldsymbol{\mu}_B), \Sigma_{AA} - \Sigma_{AB}\Sigma_{BB}^{-1}\Sigma_{AB}^\top\right)}$$

The Gaussian Distribution is Stable

- **Stable distributon**: a linear combination $aX_1 + bX_2$ of two independent copies of a random variable X has the same distribution as $cX + d$ up to location d and scale $c > 0$ parameters

- The Gaussian distribution is stable

- Since addition of random variables corresponds to convolution of their pdfs, the space of Gaussian pdfs is **closed under convolution**:

$$\int \phi(\mathbf{x}; F\mathbf{s}, W)\phi(\mathbf{s}; \boldsymbol{\mu}, \Sigma)d\mathbf{s} = \phi\left(\mathbf{x}; F\boldsymbol{\mu}, F\Sigma F^\top + W\right)$$

- The space of Gaussian pdfs is also **closed under geometric averages** (up to scaling):

$$\prod_k \phi^{\alpha_k}(\mathbf{x}; \boldsymbol{\mu}_k, \Sigma_k) \propto \phi\left(\mathbf{x}; \left(\sum_k \alpha_k \Sigma_k^{-1}\right)^{-1}\left(\sum_k \Sigma_k^{-1}\boldsymbol{\mu}_k\right), \left(\sum_k \alpha_k \Sigma_k^{-1}\right)^{-1}\right)$$

Kalman Filter

- A Bayes filter with the following **assumptions**:
  - The prior pdf $p_{t|t}$ is Gaussian
  - The motion model is linear in the state $\mathbf{x}_t$ with Gaussian noise $\mathbf{w}_t$
  - The observation model is linear in the state $\mathbf{x}_t$ with Gaussian noise $\mathbf{v}_t$
  - The motion noise $\mathbf{w}_t$ and observation noise $\mathbf{v}_t$ are independent of each other, of the state $\mathbf{x}_t$, and across time

- **Prior**: $\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1} \sim \mathcal{N}(\boldsymbol{\mu}_{t|t}, \Sigma_{t|t})$

- **Motion Model**:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t, \mathbf{w}_t) := F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim \mathcal{N}(0, W)$$

$$\mathbf{x}_{t+1} \mid \mathbf{x}_t, \mathbf{u}_t \sim \mathcal{N}(F\mathbf{x}_t + G\mathbf{u}_t, W), \quad F \in \mathbb{R}^{d_x \times d_x}, G \in \mathbb{R}^{d_x \times d_u}, \ W \in \mathbb{R}^{d_x \times d_x}$$

- **Observation Model**:

$$\mathbf{z}_t = h(\mathbf{x}_t, \mathbf{v}_t) := H\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim \mathcal{N}(0, V)$$

$$\mathbf{z}_t \mid \mathbf{x}_t \sim \mathcal{N}(H\mathbf{x}_t, V), \quad H \in \mathbb{R}^{d_z \times d_x}, \ V \in \mathbb{R}^{d_z \times d_z}$$

Kalman Filter Prediction

$$
\begin{aligned}
p_{t+1|t}(\mathbf{x}) &= \int p_f(\mathbf{x} \mid \mathbf{s}, \mathbf{u}_t) p_{t|t}(\mathbf{s}) d\mathbf{s} = \int \phi(\mathbf{x}; F\mathbf{s} + G\mathbf{u}_t, W)\phi(\mathbf{s}; \boldsymbol{\mu}_{t|t}, \Sigma_{t|t}) d\mathbf{s} \\
&= \underbrace{\frac{1}{(2\pi)^{d_x}\sqrt{\det(W)\det(\Sigma_{t|t})}}}_{\kappa_{t|t}} \int \exp\left\{-\frac{1}{2}(\mathbf{x} - F\mathbf{s} - G\mathbf{u}_t)^\top W^{-1}(\mathbf{x} - F\mathbf{s} - G\mathbf{u}_t)\right\} \times \\
&\qquad \exp\left\{-\frac{1}{2}(\mathbf{s} - \boldsymbol{\mu}_{t|t})^\top \Sigma_{t|t}^{-1}(\mathbf{s} - \boldsymbol{\mu}_{t|t})\right\} d\mathbf{s} \\
&= \kappa_{t|t} \int \exp\left\{-\frac{1}{2}\left(\mathbf{s}^\top(F^\top W^{-1} F + \Sigma_{t|t}^{-1})\mathbf{s} - 2(\Sigma_{t|t}^{-1}\boldsymbol{\mu}_{t|t} + F^\top W^{-1}(\mathbf{x} - G\mathbf{u}_t))^\top \mathbf{s} + \dots\right)\right\} d\mathbf{s} \\
&\overset{Sq.Comp.}{\underset{Inv.Lemma}{=\!=\!=}} \phi(\mathbf{x}; F\boldsymbol{\mu}_{t|t} + G\mathbf{u}_t, F\Sigma_{t|t}F^\top + W)
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
p_{t+1|t}(\mathbf{x}) &= \int \phi(\mathbf{x}; F\mathbf{s} + G\mathbf{u}_t, W)\phi(\mathbf{s}; \boldsymbol{\mu}_{t|t}, \Sigma_{t|t}) d\mathbf{s} \\
&= \phi(\mathbf{x}; F\boldsymbol{\mu}_{t|t} + G\mathbf{u}_t, F\Sigma_{t|t}F^\top + W)
\end{aligned}
}
$$

Kalman Filter Prediction (easy version)

- Motion model with given prior:

$$\mathbf{x}_{t+1} = F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t, \qquad \mathbf{w}_t \sim \mathcal{N}(0, W), \quad \mathbf{x}_t \sim \mathcal{N}(\boldsymbol{\mu}_{t|t}, \Sigma_{t|t})$$

- Since $\mathbf{w}_t$ and $\mathbf{x}_t$ are independent and the Gaussian distribution is stable, we know that the distribution of $\mathbf{x}_{t+1}$ is Gaussian: $\mathcal{N}(\boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t})$

- We just need to compute its mean and covariance:

$$\boldsymbol{\mu}_{t+1|t} = \mathbb{E}\left[F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t\right] = F\mathbb{E}[\mathbf{x}_t] + G\mathbf{u}_t + \mathbb{E}[\mathbf{w}_t] = F\boldsymbol{\mu}_{t|t} + G\mathbf{u}_t$$

$$\begin{aligned}
\Sigma_{t+1|t} &= \mathbf{Var}\left[F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t\right] \overset{\text{independence}}{=\!=\!=\!=} \mathbf{Var}\left[F\mathbf{x}_t\right] + 0 + \mathbf{Var}\left[\mathbf{w}_t\right] \\
&= \mathbb{E}\left[F\left(\mathbf{x}_t - \boldsymbol{\mu}_{t|t}\right)\left(\mathbf{x}_t - \boldsymbol{\mu}_{t|t}\right)^\top F^\top\right] + W \\
&= F\Sigma_{t|t}F^\top + W
\end{aligned}$$

Kalman Filter Update

$$p_{t+1|t+1}(\mathbf{x}) = \frac{p(\mathbf{z}_{t+1} \mid \mathbf{x})p_{t+1|t}(\mathbf{x})}{p(\mathbf{z}_{t+1} \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t})} = \frac{\phi(\mathbf{z}_{t+1}; H\mathbf{x}, V)\phi(\mathbf{x}; \boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t})}{\int \phi(\mathbf{z}_{t+1}; H\mathbf{s}, V)\phi(\mathbf{s}; \boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t})d\mathbf{s}}$$

$$= \frac{\phi(\mathbf{z}_{t+1}; H\mathbf{x}, V)\phi(\mathbf{x}; \boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t})}{\phi(\mathbf{z}_{t+1}; H\boldsymbol{\mu}_{t+1|t}, H\Sigma_{t+1|t}H^\top + V)}$$

$$= \frac{\kappa_{t+1}}{\eta_{t+1}} \exp\left\{-\frac{1}{2}(\mathbf{z}_{t+1} - H\mathbf{x})^\top V^{-1}(\mathbf{z}_{t+1} - H\mathbf{x})\right\} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_{t+1|t})^\top \Sigma_{t+1|t}^{-1}(\mathbf{x} - \boldsymbol{\mu}_{t+1|t})\right\}$$

$$= \frac{\kappa_{t+1}}{\eta_{t+1}} \exp\left\{-\frac{1}{2}\left(\mathbf{x}^\top(H^\top V^{-1}H + \Sigma_{t+1|t}^{-1})\mathbf{x} + 2(H^\top V^{-1}\mathbf{z}_{t+1} + \Sigma_{t+1|t}^{-1}\boldsymbol{\mu}_{t+1|t})^\top \mathbf{x} + \dots\right)\right\}$$

$$\overset{\text{Sq. Comp.}}{=\!=\!=} \phi\left(\mathbf{x}; (H^\top V^{-1}H + \Sigma_{t+1|t}^{-1})^{-1}(H^\top V^{-1}\mathbf{z}_{t+1} + \Sigma_{t+1|t}^{-1}\boldsymbol{\mu}_{t+1|t}), (H^\top V^{-1}H + \Sigma_{t+1|t}^{-1})^{-1}\right)$$

$$\overset{\text{Inv. Lemma}}{=\!=\!=} \boxed{\phi\left(\mathbf{x}; \boldsymbol{\mu}_{t+1|t} + K_{t+1|t}(\mathbf{z}_{t+1} - H\boldsymbol{\mu}_{t+1|t}), (I - K_{t+1|t}H)\Sigma_{t+1|t}\right)}$$

- **Kalman gain**: $K_{t+1|t} := \Sigma_{t+1|t}H^\top \left(H\Sigma_{t+1|t}H^\top + V\right)^{-1}$

Kalman Filter Update (easy version)

- Observation model with given prior:

$$\mathbf{z}_{t+1} = H\mathbf{x}_{t+1} + \mathbf{v}_{t+1}, \qquad \mathbf{v}_{t+1} \sim \mathcal{N}(0, V), \quad \mathbf{x}_{t+1} \sim \mathcal{N}(\boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t})$$

- The joint distribution of $\mathbf{x}_{t+1}$ and $\mathbf{z}_{t+1}$ is Gaussian:

$$\begin{pmatrix} \mathbf{x}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \boldsymbol{\mu}_{t+1|t} \\ H\boldsymbol{\mu}_{t+1|t} \end{pmatrix}, \begin{bmatrix} \Sigma_{t+1|t} & C_{t+1|t} \\ C_{t+1|t}^\top & H\Sigma_{t+1|t}H^\top + V \end{bmatrix} \right)$$

$$\begin{aligned} C_{t+1|t} &= \mathbb{E}\left[\left(\mathbf{x}_{t+1} - \boldsymbol{\mu}_{t+1|t}\right) \left(\mathbf{z}_{t+1} - H\boldsymbol{\mu}_{t+1|t}\right)^\top \right] \\ &= \mathbb{E}\left[\left(\mathbf{x}_{t+1} - \boldsymbol{\mu}_{t+1|t}\right) \left((\mathbf{x}_{t+1} - \boldsymbol{\mu}_{t+1|t})^\top H^\top + \mathbf{v}_{t+1}^\top \right) \right] = \Sigma_{t+1|t}H^\top \end{aligned}$$

- The conditional distribution of $\mathbf{x}_{t+1} \mid \mathbf{z}_{t+1}$ is then also Gaussian:

$$\begin{aligned} \mathbf{x}_{t+1} \mid \mathbf{z}_{t+1} \sim \mathcal{N}\Big(& \boldsymbol{\mu}_{t+1|t} + \Sigma_{t+1|t}H^\top (H\Sigma_{t+1|t}H^\top + V)^{-1} (\mathbf{z}_{t+1} - H\boldsymbol{\mu}_{t+1|t}), \\ & \Sigma_{t+1|t} - \Sigma_{t+1|t}H^\top (H\Sigma_{t+1|t}H^\top + V)^{-1} H\Sigma_{t+1|t} \Big) \end{aligned}$$

Kalman Filter Summary

Motion model: $\mathbf{x}_{t+1} = F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim \mathcal{N}(0, W)$

Observation model: $\mathbf{z}_t = H\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim \mathcal{N}(0, V)$

Prior: $\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1} \sim \mathcal{N}(\boldsymbol{\mu}_{t|t}, \Sigma_{t|t})$

Prediction:

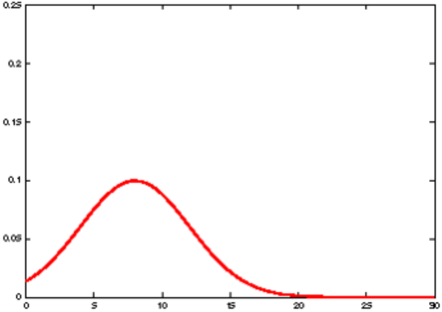
$$\boldsymbol{\mu}_{t+1|t} = F\boldsymbol{\mu}_{t|t} + G\mathbf{u}_t$$
$$\Sigma_{t+1|t} = F\Sigma_{t|t}F^\top + W$$

Update:

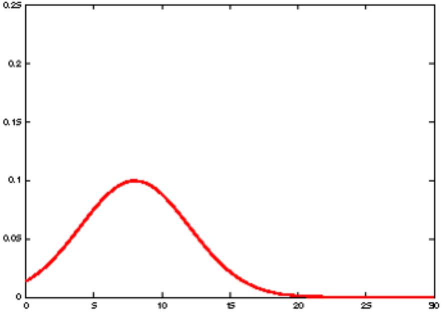
$$\boldsymbol{\mu}_{t+1|t+1} = \boldsymbol{\mu}_{t+1|t} + K_{t+1|t}(\mathbf{z}_{t+1} - H\boldsymbol{\mu}_{t+1|t})$$
$$\Sigma_{t+1|t+1} = (I - K_{t+1|t}H)\Sigma_{t+1|t}$$

Kalman Gain: $K_{t+1|t} := \Sigma_{t+1|t}H^\top \left(H\Sigma_{t+1|t}H^\top + V\right)^{-1}$

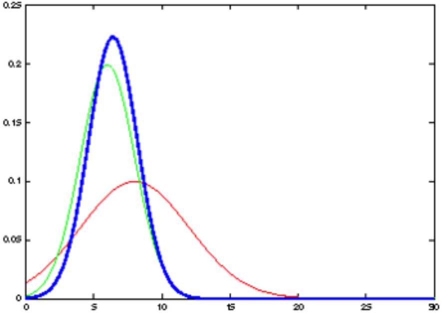
Predict | Update



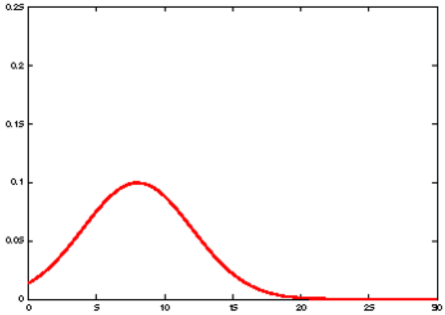
Predict

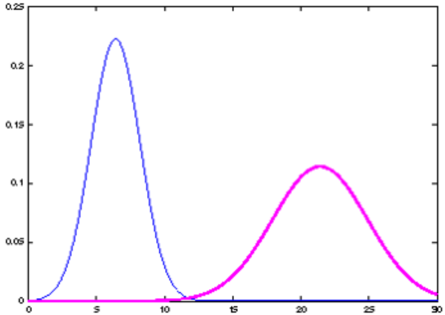


Update

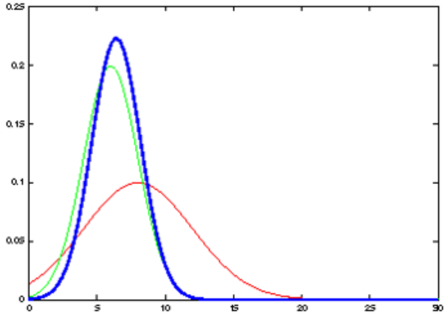


Predict

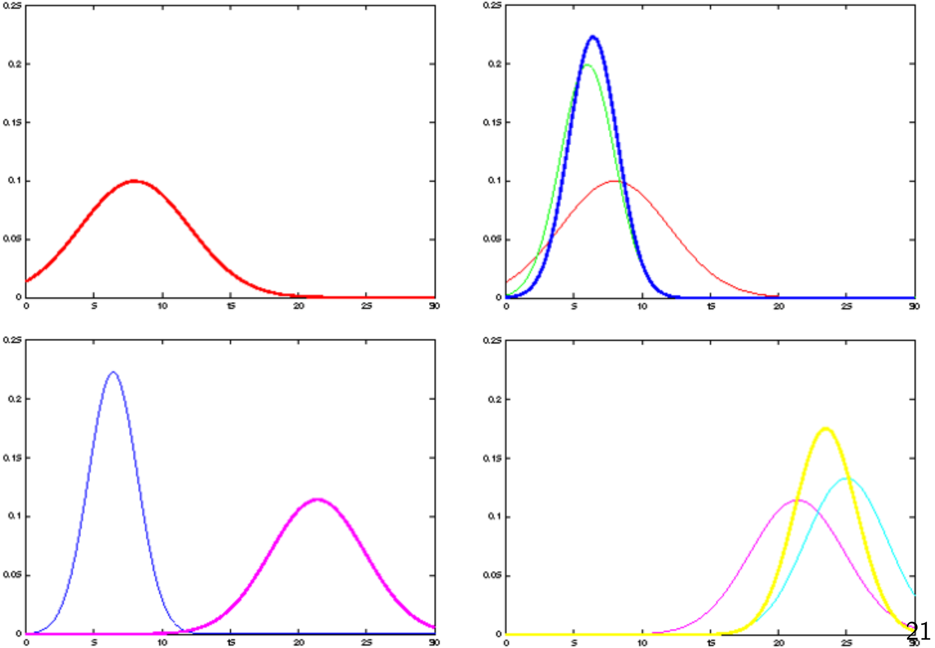




Update



Predict   Update



Kalman Filter Comments

- The normalization factor in the update step (Bayes rule) is Gaussian:

$$\begin{aligned} p(\mathbf{z}_{t+1} \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t}) &= \int p(\mathbf{z}_{t+1} \mid \mathbf{s}) p(\mathbf{s} \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t}) d\mathbf{s} \\ &= \int \phi(\mathbf{z}_{t+1}; H\mathbf{s}, V) \phi(\mathbf{s}; \boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t}) d\mathbf{s} \\ &= \phi(\mathbf{z}_{t+1}; H\boldsymbol{\mu}_{t+1|t}, H\Sigma_{t+1|t}H^\top + V) \end{aligned}$$

- **Innovation**: the term $\mathbf{r}_{t+1} := \mathbf{z}_{t+1} - H\boldsymbol{\mu}_{t+1|t}$ used to correct the mean in the update step

- The innovation $\mathbf{r}_{t+1}$ conditioned on the past information $\mathbf{z}_{0:t}, \mathbf{u}_{0:t}$ has the same covariance as $p(\mathbf{z}_{t+1} \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t})$:

$$\mathbb{E}\left[\mathbf{r}_{t+1} \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t}\right] = H\boldsymbol{\mu}_{t+1|t} - H\boldsymbol{\mu}_{t+1|t} = 0$$

$$\mathbb{E}\left[\mathbf{r}_{t+1}\mathbf{r}_{t+1}^\top \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t}\right] = H\Sigma_{t+1|t}H^\top + V$$

- **Kalman gain**: the matrix $K_{t+1|t}$ scales the innovation $\mathbf{r}_{t+1}$ by its covariance and determines how much to trust it in the update

Kalman Filter Comments

- **Efficient**: polynomial in measurement and state dim: $\boxed{O(d_z^{2.376} + d_x^2)}$
- **Optimal**: under linear, Gaussian, and independence assumptions with respect to the mean square error (MSE):

$$\mathbb{E}\left[\|\mathbf{x}_t - \boldsymbol{\mu}_{t|t}\|_2^2\right] = \operatorname{tr}(\Sigma_{t|t})$$

- To deal with **unknown models** we can use EM to learn the motion model (F, G, W) and the observation model (H, V)
- Given data $\mathcal{D} := \{\mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}\}$, apply EM with hidden variables $\mathbf{x}_{0:T}$:
  - **E step**: Given initial parameter estimates $\boldsymbol{\theta}^{(k)} := \{F^{(k)}, G^{(k)}, W^{(k)}, H^{(k)}, V^{(k)}\}$ calculate the likelihood of the hidden variables via the Kalman filter/smoother
  - **M step**: Optimize the parameters via MLE to obtain $\boldsymbol{\theta}^{(k+1)}$ which explain the posterior distribution over $\mathbf{x}_{0:T}$ better
- Most robot systems are **nonlinear**!

Information Filter

- Uses the information form $\mathbf{x} \sim \mathcal{G}(\boldsymbol{\nu}, \Omega)$ where $\boldsymbol{\nu} = \Sigma^{-1}\boldsymbol{\mu}$ and $\Omega = \Sigma^{-1}$
- Converts the Kalman filter equations to their information form counterparts via the matrix inversion lemma

Motion model: $\mathbf{x}_{t+1} = F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim \mathcal{G}(0, W^{-1})$

Observation model: $\mathbf{z}_t = H\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim \mathcal{G}(0, V^{-1})$

Prior: $\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1} \sim \mathcal{G}(\boldsymbol{\nu}_{t|t}, \Omega_{t|t})$

Prediction:

$$\boldsymbol{\nu}_{t+1|t} = \Omega_{t+1|t}\left(F\Omega_{t|t}^{-1}\boldsymbol{\nu}_{t|t} + G\mathbf{u}_t\right)$$
$$\Omega_{t+1|t} = \left(F\Omega_{t|t}^{-1}F^\top + W\right)^{-1}$$

Update:

$$\boldsymbol{\nu}_{t+1|t+1} = \boldsymbol{\nu}_{t+1|t} + H^\top V^{-1}\mathbf{z}_{t+1}$$
$$\Omega_{t+1|t+1} = \Omega_{t+1|t} + H^\top V^{-1}H$$

Kalman-Bucy Filter (continuous time)

Motion model:	$\dot{\mathbf{x}}(t) = F\mathbf{x}(t) + G\mathbf{u}(t) + \mathbf{w}(t)$
Observation model:	$\mathbf{z}(t) = H\mathbf{x}(t) + \mathbf{v}(t)$
Prior:	$\mathbf{x}(0) \sim \mathcal{N}(\boldsymbol{\mu}(0), \Sigma(0))$
Mean:	$\dot{\boldsymbol{\mu}}(t) = F\boldsymbol{\mu}(t) + G\mathbf{u}(t) + K(t)\left(\mathbf{z}(t) - H\boldsymbol{\mu}(t)\right)$
Covariance:	$\dot{\Sigma}(t) = F\Sigma(t) + \Sigma(t)F^\top + W - K(t)VK^\top(t)$
Kalman Gain:	$K(t) = \Sigma(t)H^\top V^{-1}$

Gaussian Mixture Filter

- **Motion model**: $\mathbf{x}_{t+1} = F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim \mathcal{N}(0, W)$
- **Observation model**: $\mathbf{z}_t = H\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim \mathcal{N}(0, V)$
- **Prior**: $\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1} \sim p_{t|t}(\mathbf{x}_t) := \sum_k \alpha_{t|t}^{(k)} \phi\left(\mathbf{x}_t; \boldsymbol{\mu}_{t|t}^{(k)}, \Sigma_{t|t}^{(k)}\right)$
- **Prediction**:

$$
\begin{aligned}
p_{t+1|t}(\mathbf{x}) &= \int p_f(\mathbf{x} \mid \mathbf{s}, \mathbf{u}_t) p_{t|t}(\mathbf{s}) d\mathbf{s} = \sum_k \alpha_{t|t}^{(k)} \int \phi(\mathbf{x}; F\mathbf{s} + G\mathbf{u}_t, W)\phi\left(\mathbf{s}; \boldsymbol{\mu}_{t|t}^{(k)}, \Sigma_{t|t}^{(k)}\right) d\mathbf{s} \\
&= \sum_k \alpha_{t|t}^{(k)} \phi\left(\mathbf{x}; F\boldsymbol{\mu}_{t|t}^{(k)} + G\mathbf{u}_t, F\Sigma_{t|t}^{(k)} F^\top + W\right)
\end{aligned}
$$

- **Update**:

$$
\begin{aligned}
p_{t+1|t+1}(\mathbf{x}) &= \frac{p_h(\mathbf{z}_{t+1} \mid \mathbf{x}) p_{t+1|t}(\mathbf{x})}{p(\mathbf{z}_{t+1} \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t})} = \frac{\phi\left(\mathbf{z}_{t+1}; H\mathbf{x}, V\right) \sum_k \alpha_{t+1|t}^{(k)} \phi\left(\mathbf{x}; \boldsymbol{\mu}_{t+1|t}^{(k)}, \Sigma_{t+1|t}^{(k)}\right)}{\int \phi\left(\mathbf{z}_{t+1}; H\mathbf{s}, V\right) \sum_j \alpha_{t+1|t}^{(j)} \phi\left(\mathbf{s}; \boldsymbol{\mu}_{t+1|t}^{(j)}, \Sigma_{t+1|t}^{(j)}\right) d\mathbf{s}} \\
&= \sum_k \left(\frac{\alpha_{t+1|t}^{(k)} \phi\left(\mathbf{z}_{t+1}; H\mathbf{x}, V\right) \phi\left(\mathbf{x}; \boldsymbol{\mu}_{t+1|t}^{(k)}, \Sigma_{t+1|t}^{(k)}\right)}{\sum_j \alpha_{t+1|t}^{(j)} \phi\left(\mathbf{z}_{t+1}; H\boldsymbol{\mu}_{t+1|t}^{(j)}, H\Sigma_{t+1|t}^{(j)} H^\top + V\right)} \times \frac{\phi\left(\mathbf{z}_{t+1}; H\boldsymbol{\mu}_{t+1|t}^{(k)}, H\Sigma_{t+1|t}^{(k)} H^\top + V\right)}{\phi\left(\mathbf{z}_{t+1}; H\boldsymbol{\mu}_{t+1|t}^{(k)}, H\Sigma_{t+1|t}^{(k)} H^\top + V\right)} \right) \\
&= \sum_k \left[\frac{\alpha_{t+1|t}^{(k)} \phi\left(\mathbf{z}_{t+1}; H\boldsymbol{\mu}_{t+1|t}^{(k)}, H\Sigma_{t+1|t}^{(k)} H^\top + V\right)}{\sum_j \alpha_{t+1|t}^{(j)} \phi\left(\mathbf{z}_{t+1}; H\boldsymbol{\mu}_{t+1|t}^{(j)}, H\Sigma_{t+1|t}^{(j)} H^\top + V\right)} \right] \phi\left(\mathbf{x}; \boldsymbol{\mu}_{t+1|t}^{(k)} + K_{t+1|t}^{(k)}(\mathbf{z}_{t+1} - H\mu_{t+1|t}^{(k)}), (I - K_{t+1|t}^{(k)} H)\Sigma_{t+1|t}^{(k)}\right)
\end{aligned}
$$

- **Kalman Gain**: $K_{t+1|t}^{(k)} := \Sigma_{t+1|t}^{(k)} H^\top \left(H\Sigma_{t+1|t}^{(k)} H^\top + V\right)^{-1}$

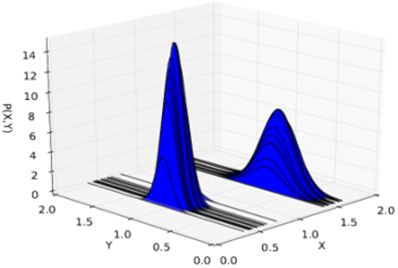
Gaussian Mixture Filter

- The GMF is just a **bank of Kalman filters** with weights $\alpha^{(k)}$ scaled by the observation model in the update step
- The GMF is also called the **Gaussian Sum Filter**
- **pdf**: $\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1} \sim p_{t|t}(\mathbf{x}) := \sum_k \alpha_{t|t}^{(k)} \phi\left(\mathbf{x}_t; \boldsymbol{\mu}_{t|t}^{(k)}, \Sigma_{t|t}^{(k)}\right)$
- **mean**: $\boldsymbol{\mu}_{t|t} := \mathbb{E}\left[\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1}\right] = \int \mathbf{x} p_{t|t}(\mathbf{x}) d\mathbf{x} = \sum_k \alpha_{t|t}^{(k)} \boldsymbol{\mu}_{t|t}^{(k)}$
- **covariance**: $\Sigma_{t|t} := \mathbb{E}\left[\mathbf{x}_t \mathbf{x}_t^\top \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1}\right] - \boldsymbol{\mu}_{t|t} \boldsymbol{\mu}_{t|t}^\top$
$= \int \mathbf{x}\mathbf{x}^\top p_{t|t}(\mathbf{x}) d\mathbf{x} - \boldsymbol{\mu}_{t|t} \boldsymbol{\mu}_{t|t}^\top = \sum_k \alpha_{t|t}^{(k)} \left(\Sigma_{t|t}^{(k)} + \boldsymbol{\mu}_{t|t}^{(k)} (\boldsymbol{\mu}_{t|t}^{(k)})^\top\right) - \boldsymbol{\mu}_{t|t} \boldsymbol{\mu}_{t|t}^\top$

Rao-Blackwellized Particle Filter

- The Rao-Blackwellized (**marginalized**) particle filter is applicable to conditionally linear-Gaussian models:

$$x^n_{t+1} = f^n_t(x^n_t) + A^n_t(x^n_t)x^l_t + G^n_t(x^n_t)w^n_t$$
$$x^l_{t+1} = f^l_t(x^n_t) + A^l_t(x^n_t)x^l_t + G^l_t(x^n_t)w^l_t$$
$$z_t = h_t(x^n_t) + C_t(x^n_t)x^l_t + v_t$$



Nonlinear states: x^n_t
Linear states: x^l_t

- **Idea**: exploit linear-Gaussian sub-structure to handle high dim. problems

$$p\left(x^l_t, x^n_{0:t} \mid z_{0:t}, u_{0:t-1}\right) = \underbrace{p\left(x^l_t \mid z_{0:t}, u_{0:t-1}, x^n_{0:t}\right)}_{\text{Kalman Filter}} \underbrace{p\left(x^n_{0:t} \mid z_{0:t}, u_{0:t-1}\right)}_{\text{Particle Filter}}$$
$$= \sum_{k=1}^{N_{t|t}} \alpha^{(k)}_{t|t} \delta\left(x^n_{0:t}; m^{(k)}_{t|t}\right) \phi\left(x^l_t; \mu^{(k)}_{t|t}, \Sigma^{(k)}_{t|t}\right)$$

- The RBPF is a combination of the particle filter and the Kalman filter, in which each particle has a Kalman filter associated to it