ECE276A: Sensing & Estimation in Robotics
Lecture 13: Visual-Inertial SLAM

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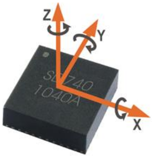
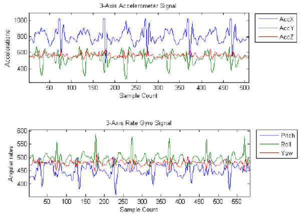
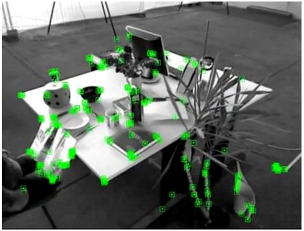
Visual-Inertial Localization and Mapping

- **Input**:
  - IMU: linear acceleration $\mathbf{a}_t \in \mathbb{R}^3$ and rotational velocity $\boldsymbol{\omega}_t \in \mathbb{R}^3$
  - Camera: features $\mathbf{z}_{t,i} \in \mathbb{R}^4$ (left and right image pixels) for $i = 1, \ldots, N_t$







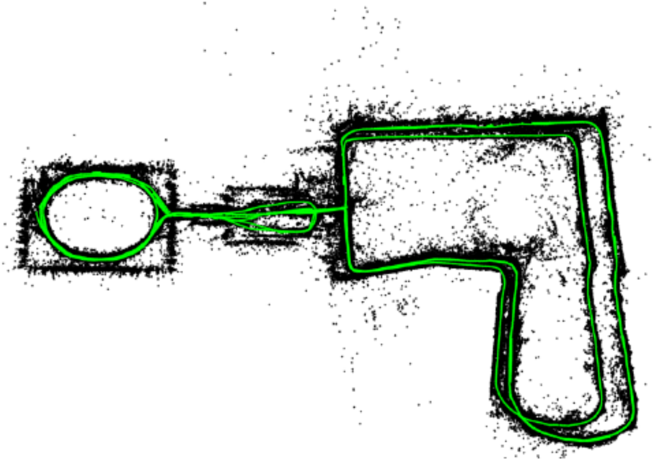
- **Assumption**: The transformation ${}_O T_I \in SE(3)$ from the IMU to the camera optical frame (extrinsic parameters) and the stereo camera calibration matrix K_s (intrinsic parameters) are known.

$$K_s := \begin{bmatrix} fs_u & 0 & c_u & 0 \\ 0 & fs_v & c_v & 0 \\ fs_u & 0 & c_u & -fs_u b \\ 0 & fs_v & c_v & 0 \end{bmatrix}$$

$$\begin{aligned} f &= \text{focal length } [m] \\ s_u, s_v &= \text{pixel scaling } [pixels/m] \\ c_u, c_v &= \text{principal point } [pixels] \\ b &= \text{stereo baseline } [m] \end{aligned}$$

Visual-Inertial Localization and Mapping

- **Output**:
  - World-frame IMU pose ${}_W T_I \in SE(3)$ over time (green)
  - World-frame coordinates $\mathbf{m}_j \in \mathbb{R}^3$ of the $j = 1, \ldots, M$ point landmarks (black) that generated the visual features $\mathbf{z}_{t,i} \in \mathbb{R}^4$



Visual Mapping

- Consider the mapping-only problem first
- **Assumption**: the IMU pose $T_t := {}_W T_{I,t} \in SE(3)$ is known
- **Objective**: given the observations $\mathbf{z}_t := \begin{bmatrix} \mathbf{z}_{t,1}^\top & \cdots & \mathbf{z}_{t,N_t}^\top \end{bmatrix}^\top \in \mathbb{R}^{4N_t}$ for $t = 0, \ldots, T$, estimate the coordinates $\mathbf{m} := \begin{bmatrix} \mathbf{m}_1^\top & \cdots & \mathbf{m}_M^\top \end{bmatrix}^\top \in \mathbb{R}^{3M}$ of the landmarks that generated them
- **Assumption**: the data association $\Delta_t : \{1, \ldots, M\} \to \{1, \ldots, N_t\}$ stipulating that landmark j corresponds to observation $\mathbf{z}_{t,i} \in \mathbb{R}^4$ with $i = \Delta_t(j)$ at time t is known or provided by an external algorithm
- **Assumption**: the landmarks $\mathbf{m}$ are static, i.e., it is not necessary to consider a motion model or a prediction step for $\mathbf{m}$

Visual Mapping via the EKF

- **Observation Model**: with measurement noise $\mathbf{v}_{t,i} \sim \mathcal{N}(0, V)$

$$\mathbf{z}_{t,i} = h(T_t, \mathbf{m}_j) + \mathbf{v}_{t,i} := K_s \pi \left({}_O T_I T_t^{-1} \underline{\mathbf{m}}_j \right) + \mathbf{v}_{t,i}$$

- Homogeneous coordinates: $\underline{\mathbf{m}}_j := \begin{bmatrix} \mathbf{m}_j \\ 1 \end{bmatrix}$

- Projection function and its derivative:

$$\pi(\mathbf{q}) := \frac{1}{q_3}\mathbf{q} \in \mathbb{R}^4 \qquad \frac{d\pi}{d\mathbf{q}}(\mathbf{q}) = \frac{1}{q_3}\begin{bmatrix} 1 & 0 & -\frac{q_1}{q_3} & 0 \\ 0 & 1 & -\frac{q_2}{q_3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{q_4}{q_3} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

- All observations, stacked as a $4N_t$ vector, at time t with notation abuse:

$$\mathbf{z}_t = K_s \pi \left({}_O T_I T_t^{-1} \underline{\mathbf{m}} \right) + \mathbf{v}_t \quad \mathbf{v}_t \sim \mathcal{N}\left(\mathbf{0}, I \otimes V\right) \quad I \otimes V := \begin{bmatrix} V & & \\ & \ddots & \\ & & V \end{bmatrix}$$

Visual Mapping via the EKF

- **Prior**: $\mathbf{m} \mid \mathbf{z}_{0:t} \sim \mathcal{N}(\boldsymbol{\mu}_t, \Sigma_t)$ with $\boldsymbol{\mu}_t \in \mathbb{R}^{3M}$ and $\Sigma_t \in \mathbb{R}^{3M \times 3M}$
- **EKF Update**: given a new observation $\mathbf{z}_{t+1} \in \mathbb{R}^{4N_{t+1}}$:

$$K_{t+1} = \Sigma_t H_{t+1}^\top \left(H_{t+1} \Sigma_t H_{t+1}^\top + I \otimes V \right)^{-1}$$

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t + K_{t+1} \left(\mathbf{z}_{t+1} - \underbrace{K_s \pi \left({}_O T_I T_{t+1}^{-1} \underline{\boldsymbol{\mu}}_t \right)}_{\tilde{\mathbf{z}}_{t+1}} \right)$$

$$\Sigma_{t+1} = (I - K_{t+1} H_{t+1}) \Sigma_t$$

- $\tilde{\mathbf{z}}_{t+1} \in \mathbb{R}^{4N_{t+1}}$ is the predicted observation based on the landmark position estimates $\boldsymbol{\mu}_t$ at time t
- We need the observation model Jacobian $H_{t+1} \in \mathbb{R}^{4N_t \times 3M}$ evaluated at $\boldsymbol{\mu}_t$ with block elements $H_{t+1,i,j} \in \mathbb{R}^{4 \times 3}$:

$$H_{t+1,i,j} := \begin{cases} \frac{\partial}{\partial \mathbf{m}_j} h(T_{t+1}, \mathbf{m}_j) \Big|_{\mathbf{m}_j = \boldsymbol{\mu}_{t,j}}, & \text{if } \Delta_t(j) = i, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Stereo Camera Jacobian (by Chain Rule)

- Observation model: $h(T_{t+1}, \mathbf{m}_j) = K_s \pi \left({}_O T_I T_{t+1}^{-1} \underline{\mathbf{m}}_j \right)$

- How do we obtain $\frac{\partial}{\partial \mathbf{m}_j} h(T_{t+1}, \mathbf{m}_j) \Big|_{\mathbf{m}_j = \boldsymbol{\mu}_{t,j}}$?

- Let $\mathbf{q}_{t+1,j} = {}_O T_I T_{t+1}^{-1} \underline{\mathbf{m}}_j$ and $P = \begin{bmatrix} I & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 4}$

- Apply the chain rule:

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{m}_j} h(T_{t+1}, \mathbf{m}_j) &= K_s \frac{\partial \pi}{\partial \mathbf{q}}(\mathbf{q}_{t+1,j}) \frac{\partial \mathbf{q}_{t+1,j}}{\partial \mathbf{m}_j} \\
&= K_s \frac{\partial \pi}{\partial \mathbf{q}} \left({}_O T_I T_{t+1}^{-1} \underline{\mathbf{m}}_j \right) {}_O T_I T_{t+1}^{-1} \frac{\partial \underline{\mathbf{m}}_j}{\partial \mathbf{m}_j} \\
&= K_s \frac{\partial \pi}{\partial \mathbf{q}} \left({}_O T_I T_{t+1}^{-1} \underline{\mathbf{m}}_j \right) {}_O T_I T_{t+1}^{-1} P^\top
\end{aligned}
$$

Stereo Camera Jacobian (by Perturbation)

- The Jacobian of a function $f(\mathbf{x})$ can also be obtained using first-order Taylor series with perturbation $\delta\mathbf{x}$:

$$f(\mathbf{x} + \delta\mathbf{x}) \approx f(\mathbf{x}) + \left[\frac{\partial f}{\partial \mathbf{x}}(\mathbf{x})\right] \delta\mathbf{x}$$

- The Jacobian of $f(\mathbf{x})$ is the part that is linear in $\delta\mathbf{x}$ in the first-order Taylor series expansion

- Consider a perturbation $\delta\boldsymbol{\mu}_{t,j} \in \mathbb{R}^3$ for the position of landmark j:

$$\mathbf{m}_j = \boldsymbol{\mu}_{t,j} + \delta\boldsymbol{\mu}_{t,j}$$

- The first-order Taylor series approximation of the observation model:

$$\begin{aligned} K_s \pi\left({}_O T_I T_{t+1}^{-1} \underline{(\boldsymbol{\mu}_{t,j} + \delta\boldsymbol{\mu}_{t,j})}\right) &= K_s \pi\left({}_O T_I T_{t+1}^{-1}\left(\underline{\boldsymbol{\mu}}_{t,j} + P^\top \delta\boldsymbol{\mu}_{t,j}\right)\right) \\ &\approx \underbrace{K_s \pi\left({}_O T_I T_{t+1}^{-1} \underline{\boldsymbol{\mu}}_{t,j}\right)}_{\tilde{\mathbf{z}}_{t+1,i}} + \underbrace{K_s \frac{d\pi}{d\mathbf{q}}\left({}_O T_I T_{t+1}^{-1} \underline{\boldsymbol{\mu}}_{t,j}\right) {}_O T_I T_{t+1}^{-1} P^\top}_{H_{t+1,i,j}} \delta\boldsymbol{\mu}_{t,j} \end{aligned}$$

Visual Mapping via the EKF (Summary)

- Prior: $\boldsymbol{\mu}_t \in \mathbb{R}^{3M}$ and $\Sigma_t \in \mathbb{R}^{3M \times 3M}$
- Known: stereo calibration matrix K_s, extrinsics ${}_O T_I \in SE(3)$, IMU pose $T_{t+1} \in SE(3)$, new observation $\mathbf{z}_{t+1} \in \mathbb{R}^{4N_{t+1}}$
- Predicted observations based on $\boldsymbol{\mu}_t$ and known correspondences Δ_{t+1}:

$$\tilde{\mathbf{z}}_{t+1,i} := K_s \pi \left({}_O T_I T_{t+1}^{-1} \underline{\boldsymbol{\mu}}_{t,j} \right) \in \mathbb{R}^4 \qquad \text{for } i = 1, \ldots, N_{t+1}$$

- Jacobian of $\tilde{\mathbf{z}}_{t+1,i}$ with respect to $\mathbf{m}_j$ evaluated at $\boldsymbol{\mu}_{t,j}$:

$$H_{t+1,i,j} = \begin{cases} K_s \frac{d\pi}{d\mathbf{q}} \left({}_O T_I T_{t+1}^{-1} \underline{\boldsymbol{\mu}}_{t,j} \right) {}_O T_I T_{t+1}^{-1} P^\top & \text{if } \Delta_t(j) = i, \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

- EKF update:

$$K_{t+1} = \Sigma_t H_{t+1}^\top \left(H_{t+1} \Sigma_t H_{t+1}^\top + I \otimes V \right)^{-1}$$
$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t + K_{t+1} \left(\mathbf{z}_{t+1} - \tilde{\mathbf{z}}_{t+1} \right)$$
$$\Sigma_{t+1} = (I - K_{t+1} H_{t+1}) \Sigma_t$$

$$I \otimes V := \begin{bmatrix} V & & \\ & \ddots & \\ & & V \end{bmatrix}$$

Visual-Inertial Odometry

- Now, consider the localization-only problem
- We will simplify the prediction step by using kinematic rather than dynamic equations
- **Assumption**: linear velocity $\mathbf{v}_t \in \mathbb{R}^3$ instead of linear acceleration $\mathbf{a}_t \in \mathbb{R}^3$ measurements are available
- **Assumption**: known world-frame landmark coordinates $\mathbf{m} \in \mathbb{R}^{3M}$
- **Assumption**: the data association $\Delta_t : \{1, \ldots, M\} \to \{1, \ldots, N_t\}$ stipulating that landmark j corresponds to observation $\mathbf{z}_{t,i} \in \mathbb{R}^4$ with $i = \Delta_t(j)$ at time t is known or provided by an external algorithm
- **Objective**: given IMU measurements $\mathbf{u}_{0:T}$ with $\mathbf{u}_t := [\mathbf{v}_t^\top, \boldsymbol{\omega}_t^\top]^\top \in \mathbb{R}^6$ and feature observations $\mathbf{z}_{0:T}$, estimate the pose $T_t := {}_W T_{I,t} \in SE(3)$ of the IMU over time

How to deal with an $SE(3)$ state in the EKF?

- Goal: estimate $T_t \in SE(3)$ using an extended Kalman filter
- Rotations: $SO(3) := \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; R^\top R = I,\ \det(R) = 1 \right\}$
- Poses: $SE(3) := \left\{ T = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \;\middle|\; R \in SO(3), \mathbf{p} \in \mathbb{R}^3 \right\}$
- Since T_t is not a vector, we face multiple questions:
  - How do we specify a "Gaussian" distribution over T_t?
  - What is the motion model for T_t?
  - How do we find derivatives with respect to T_t?
- The **axis-angle** parametrizations of $SO(3)$ plays a key role

Exponential Map from $\mathfrak{so}(3)$ to $SO(3)$

- **Axis-Angle**: $\boldsymbol{\theta} \in \mathbb{R}^3$ specifying a rotation about an axis $\boldsymbol{\eta} := \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$ through an angle $\theta := \|\boldsymbol{\theta}\|$:

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \hat{\boldsymbol{\theta}} + \frac{1}{2!}\hat{\boldsymbol{\theta}}^2 + \frac{1}{3!}\hat{\boldsymbol{\theta}}^3 + \dots$$

- $\hat{\boldsymbol{\theta}} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}$ is a skew-symmetric matrix, i.e., $\hat{\boldsymbol{\theta}}^\top = -\hat{\boldsymbol{\theta}}$

- Every skew-symmetric matrix can be represented as $\hat{\boldsymbol{\theta}}$ for some $\boldsymbol{\theta} \in \mathbb{R}^3$

- Space of skew-symmetric matrices: $\mathfrak{so}(3) := \{\hat{\boldsymbol{\theta}} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{\theta} \in \mathbb{R}^3\}$

- The **exponential map** provides a mapping from the space of skew-symmetric matrices $\mathfrak{so}(3)$ to the space of rotation matrices $SO(3)$:

$$R = \exp(\hat{\boldsymbol{\theta}}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\hat{\boldsymbol{\theta}})^n$$

Rotation Kinematics

- The trajectory $R(t)$ of a continuous rotation motion should satisfy:

$$R^\top(t)R(t) = I \quad \Rightarrow \quad \dot{R}^\top(t)R(t) + R^\top(t)\dot{R}(t) = 0.$$

- The matrix $R^\top(t)\dot{R}(t)$ is **skew-symmetric**! There must exist some vector-valued function $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ such that:

$$R^\top(t)\dot{R}(t) = \hat{\boldsymbol{\omega}}(t) \quad \Rightarrow \quad \boxed{\dot{R}(t) = R(t)\hat{\boldsymbol{\omega}}(t)}$$

- A skew-symmetric matrix gives a first order approximation to a rotation matrix:

$$R(t + dt) \approx R(t) + R(t)\hat{\boldsymbol{\omega}}(t)dt$$

Rotation Kinematics

- Let $R \in SO(3)$ be the orientation of a rigid body rotating with angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ with respect to the world frame.

- **Rotation kinematic equations of motion**:

$$\boxed{\dot{R} = R\hat{\boldsymbol{\omega}}_B = \hat{\boldsymbol{\omega}}_W R}$$

  where $\boldsymbol{\omega}_B$ and $\boldsymbol{\omega}_W := R\boldsymbol{\omega}_B$ are the body-frame and world-frame coordinates of $\boldsymbol{\omega}$, respectively.

- Assuming $\boldsymbol{\omega}$ is constant over a short period τ:

$$R(t+\tau) = R(t)\exp(\tau\hat{\boldsymbol{\omega}}_B) = \exp(\tau\hat{\boldsymbol{\omega}}_W)R(t)$$

- **Discrete Rotation Kinematics**: let $R_k := R(t_k)$, $\tau_k := t_{k+1} - t_k$, and $\boldsymbol{\omega}_k := \boldsymbol{\omega}_B(t_k)$ leading to:

$$\boxed{R_{k+1} = R_k \exp(\tau_k \hat{\boldsymbol{\omega}}_k)}$$

Perturbation in $\mathbb{R}^3$, $\mathfrak{so}(3)$, and $SO(3)$

- Perturbing a vector $\mathbf{x} \in \mathbb{R}^3$ can be done by addition:
  - perturbation in $\mathbb{R}^3$: $\mathbf{x} + \delta\mathbf{x}$
- Perturbing a rotation matrix $R = \exp(\hat{\boldsymbol{\theta}}) \in SO(3)$ should be done using the exponential map:
  - perturbation in $\mathfrak{so}(3)$: $\exp\left((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^\wedge\right)$
  - perturbation in $SO(3)$: $\underbrace{\exp(\delta\hat{\boldsymbol{\psi}})R}_{\text{left perturbation}}$ or $\underbrace{R\exp(\delta\hat{\boldsymbol{\psi}})}_{\text{right perturbation}}$
- Note that the perturbations $\delta\boldsymbol{\theta}$ and $\delta\boldsymbol{\psi}$ are regular vectors in $\mathbb{R}^3$
- Infinitesimal perturbations allow us to compute derivatives and define probability distributions in $SO(3)$

How do we specify a Gaussian distribution in $SO(3)$?

- In $\mathbb{R}^3$ we can define a Gaussian distribution over a vector $\mathbf{x}$ as follows:

$$\mathbf{x} = \boldsymbol{\mu} + \boldsymbol{\epsilon} \qquad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

  where $\boldsymbol{\mu} \in \mathbb{R}^3$ is the deterministic mean and $\boldsymbol{\epsilon} \in \mathbb{R}^3$ is a zero-mean Gaussian random vector

- In $SO(3)$ we can define a Gaussian distribution over a rotation matrix R as follows:

$$R = \exp(\hat{\boldsymbol{\epsilon}})\boldsymbol{\mu} \qquad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

  where $\boldsymbol{\mu} \in SO(3)$ is the deterministic mean and $\boldsymbol{\epsilon} \in \mathbb{R}^3$ is a zero-mean Gaussian random vector

Example: Rotation of a Random Rotation Variable

- Let $Q \in SO(3)$ and $\boldsymbol{\theta} \in \mathbb{R}^3$. Then:

$$Q \exp(\hat{\boldsymbol{\theta}}) Q^\top = \exp\left(Q \hat{\boldsymbol{\theta}} Q^\top\right) = \exp\left((Q\boldsymbol{\theta})^\wedge\right)$$

- Let $R \in SO(3)$ be a random rotation with mean $\boldsymbol{\mu} \in SO(3)$ and covariance $\Sigma \in \mathbb{R}^{3\times 3}$.

- The random variable $Y = QR \in SO(3)$ satisfies:

$$\begin{aligned} Y &= QR = Q\exp(\hat{\boldsymbol{\epsilon}})\boldsymbol{\mu} = \exp\left((Q\boldsymbol{\epsilon})^\wedge\right) Q\boldsymbol{\mu} \\ \mathbb{E}[Y] &= Q\boldsymbol{\mu} \\ \mathbf{Var}[Y] &= \mathbf{Var}[Q\boldsymbol{\epsilon}] = Q\Sigma Q^\top \end{aligned}$$

What is the motion model for a rotation matrix R?

- **Continuous-time rotation kinematics**:

$$\dot{R}(t) = R(t)\hat{\boldsymbol{\omega}}(t)$$

  where the rotation $R(t)$ is the **state** and the angular velocity $\boldsymbol{\omega}(t)$ is the **input**

- **Discrete-time rotation kinematics**:

$$R_{k+1} = R_k \exp(\tau_k \hat{\boldsymbol{\omega}}_k)$$

  where $R_k = R(t_k)$, $\tau_k = t_{k+1} - t_k$, $\boldsymbol{\omega}_k = \boldsymbol{\omega}(t_k)$, and $\boldsymbol{\omega}(t)$ is **constant** for $t \in [t_k, t_{k+1})$

How do we find derivatives with respect to a rotation R?

- In $\mathbb{R}^3$, the derivative of a function $f(\mathbf{x})$ can be obtained using first-order Taylor series with perturbation $\delta\mathbf{x} \in \mathbb{R}^3$:

$$f(\mathbf{x} + \delta\mathbf{x}) \approx f(\mathbf{x}) + \left[\frac{\partial f}{\partial \mathbf{x}}(\mathbf{x})\right] \delta\mathbf{x}$$

- In $\mathbb{R}^3$, the derivative is $\left.\frac{\partial}{\partial \delta\mathbf{x}} f(\mathbf{x} + \delta\mathbf{x})\right|_{\delta\mathbf{x}=0}$

- In $SO(3)$, the derivative of a function $f(R)$ can be obtained using first-order Taylor series with perturbation $\delta\boldsymbol{\psi} \in \mathbb{R}^3$:

$$f(R\exp(\delta\hat{\boldsymbol{\psi}})) \approx f(R) + \left[\frac{\partial f}{\partial R}(R)\right] \delta\boldsymbol{\psi}$$

- In $SO(3)$, the derivative is $\left.\frac{\partial}{\partial \delta\boldsymbol{\psi}} f(R\exp(\delta\hat{\boldsymbol{\psi}}))\right|_{\delta\boldsymbol{\psi}=0}$

Exponential Map from $\mathfrak{se}(3)$ to $SE(3)$

- In $SO(3)$, an axis-angle vector $\boldsymbol{\theta} \in \mathbb{R}^3$ is mapped to a rotation matrix $R = \exp(\hat{\boldsymbol{\theta}})$ by the exponential map
- In $SE(3)$, a position-rotation vector $\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6$ is mapped to a pose matrix $T = \exp(\hat{\boldsymbol{\xi}})$ by the exponential map
- Space of twist matrices:

$$\mathfrak{se}(3) := \left\{ \hat{\boldsymbol{\xi}} := \begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, \boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6 \right\}$$

- The **exponential map** provides a mapping from the space of twist matrices $\mathfrak{se}(3)$ to the space of pose matrices $SE(3)$:

$$T = \exp(\hat{\boldsymbol{\xi}}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\hat{\boldsymbol{\xi}})^n$$

How do we specify a Gaussian distribution in $SE(3)$?

- In $\mathbb{R}^3$ we can define a Gaussian distribution over a vector $\mathbf{x}$ as follows:

$$\mathbf{x} = \boldsymbol{\mu} + \boldsymbol{\epsilon} \qquad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

  where $\boldsymbol{\mu} \in \mathbb{R}^3$ is the deterministic mean and $\boldsymbol{\epsilon} \in \mathbb{R}^3$ is a zero-mean Gaussian random vector

- In $SE(3)$ we can define a Gaussian distribution over a pose matrix T as follows:

$$T = \exp(\hat{\boldsymbol{\epsilon}})\boldsymbol{\mu} \qquad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

  where $\boldsymbol{\mu} \in SE(3)$ is the deterministic mean and $\boldsymbol{\epsilon} \in \mathbb{R}^6$ is a zero-mean Gaussian random vector

What is the motion model for a pose matrix T?

- **Continuous-time pose kinematics**:

$$\dot{T}(t) = T(t)\hat{\zeta}(t)$$

where the pose $T(t)$ is the **state** and the generalized velocity $\zeta(t) := \begin{bmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}(t) \end{bmatrix} \in \mathbb{R}^6$ is the **input**

- **Discrete-time pose kinematics**:

$$T_{k+1} = T_k \exp(\tau_k \hat{\zeta}_k)$$

where $T_k = T(t_k)$, $\tau_k = t_{k+1} - t_k$, $\zeta_k = \zeta(t_k)$, and $\zeta(t)$ is **constant** for $t \in [t_k, t_{k+1})$

How do we find derivatives with respect to a pose T?

- In $\mathbb{R}^6$, the derivative of a function $f(\mathbf{x})$ can be obtained using first-order Taylor series with perturbation $\delta\mathbf{x} \in \mathbb{R}^6$:

$$f(\mathbf{x} + \delta\mathbf{x}) \approx f(\mathbf{x}) + \left[\frac{\partial f}{\partial \mathbf{x}}(\mathbf{x})\right] \delta\mathbf{x}$$

- In $\mathbb{R}^6$, the derivative is $\left.\frac{\partial}{\partial \delta\mathbf{x}} f(\mathbf{x} + \delta\mathbf{x})\right|_{\delta\mathbf{x}=0}$

- In $SE(3)$, the derivative of a function $f(T)$ can be obtained using first-order Taylor series with perturbation $\delta\boldsymbol{\psi} \in \mathbb{R}^6$:

$$f(T \exp(\delta\hat{\boldsymbol{\psi}})) \approx f(T) + \left[\frac{\partial f}{\partial T}(T)\right] \delta\boldsymbol{\psi}$$

- In $SE(3)$, the derivative is $\left.\frac{\partial}{\partial \delta\boldsymbol{\psi}} f(T \exp(\delta\hat{\boldsymbol{\psi}}))\right|_{\delta\boldsymbol{\psi}=0}$

Visual-Inertial Odometry

- Now, consider the localization-only problem
- We will simplify the prediction step by using kinematic rather than dynamic equations
- **Assumption**: linear velocity $\mathbf{v}_t \in \mathbb{R}^3$ instead of linear acceleration $\mathbf{a}_t \in \mathbb{R}^3$ measurements are available
- **Assumption**: known world-frame landmark coordinates $\mathbf{m} \in \mathbb{R}^{3M}$
- **Assumption**: the data association $\Delta_t : \{1, \ldots, M\} \rightarrow \{1, \ldots, N_t\}$ stipulating that landmark j corresponds to observation $\mathbf{z}_{t,i} \in \mathbb{R}^4$ with $i = \Delta_t(j)$ at time t is known or provided by an external algorithm
- **Objective**: given IMU measurements $\mathbf{u}_{0:T}$ with $\mathbf{u}_t := [\mathbf{v}_t^\top, \boldsymbol{\omega}_t^\top]^\top \in \mathbb{R}^6$ and feature observations $\mathbf{z}_{0:T}$, estimate the pose $T_t := {}_W T_{I,t} \in SE(3)$ of the IMU over time

Pose Kinematics with Perturbation

- **Motion model** for the continuous-time IMU pose $T(t)$ with noise $\mathbf{w}(t)$:

$$\dot{T} = T\left(\hat{\mathbf{u}} + \hat{\mathbf{w}}\right) \qquad\qquad \mathbf{u}(t) := \begin{bmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}(t) \end{bmatrix} \in \mathbb{R}^6$$

- To consider a Gaussian distribution over T, express it as a nominal pose $\boldsymbol{\mu} \in SE(3)$ with small perturbation $\delta\hat{\boldsymbol{\mu}} \in \mathfrak{se}(3)$:

$$T = \boldsymbol{\mu}\exp(\delta\hat{\boldsymbol{\mu}}) \approx \boldsymbol{\mu}\left(I + \delta\hat{\boldsymbol{\mu}}\right)$$

- Substitute the nominal + perturbed pose in the kinematic equations:

$$\dot{\boldsymbol{\mu}}\left(I + \delta\hat{\boldsymbol{\mu}}\right) + \boldsymbol{\mu}\left(\dot{\delta\hat{\boldsymbol{\mu}}}\right) = \boldsymbol{\mu}\left(I + \delta\hat{\boldsymbol{\mu}}\right)\left(\hat{\mathbf{u}} + \hat{\mathbf{w}}\right)$$

$$\dot{\boldsymbol{\mu}} + \dot{\boldsymbol{\mu}}\delta\hat{\boldsymbol{\mu}} + \boldsymbol{\mu}\left(\dot{\delta\hat{\boldsymbol{\mu}}}\right) = \boldsymbol{\mu}\hat{\mathbf{u}} + \boldsymbol{\mu}\hat{\mathbf{w}} + \boldsymbol{\mu}\delta\hat{\boldsymbol{\mu}}\hat{\mathbf{u}} + \cancelto{0}{\boldsymbol{\mu}\delta\hat{\boldsymbol{\mu}}\hat{\mathbf{w}}}$$

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\mu}\hat{\mathbf{u}} \qquad \boldsymbol{\mu}\hat{\mathbf{u}}\delta\hat{\boldsymbol{\mu}} + \boldsymbol{\mu}\left(\dot{\delta\hat{\boldsymbol{\mu}}}\right) = \boldsymbol{\mu}\hat{\mathbf{w}} + \boldsymbol{\mu}\delta\hat{\boldsymbol{\mu}}\hat{\mathbf{u}}$$

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\mu}\hat{\mathbf{u}} \qquad \dot{\delta\hat{\boldsymbol{\mu}}} = \delta\hat{\boldsymbol{\mu}}\hat{\mathbf{u}} - \hat{\mathbf{u}}\delta\hat{\boldsymbol{\mu}} + \hat{\mathbf{w}} = \left(-\overset{\curlywedge}{\mathbf{u}}\delta\boldsymbol{\mu}\right)^{\wedge} + \hat{\mathbf{w}}$$

Pose Kinematics with Perturbation

- Using $T = \boldsymbol{\mu} \exp(\hat{\delta\boldsymbol{\mu}}) \approx \boldsymbol{\mu}\left(I + \hat{\delta\boldsymbol{\mu}}\right)$, the pose kinematics $\dot{T} = T\left(\hat{\mathbf{u}} + \hat{\mathbf{w}}\right)$ can be split into nominal and perturbation kinematics:

$$\begin{aligned} \text{nominal}: \quad & \dot{\boldsymbol{\mu}} = \boldsymbol{\mu}\hat{\mathbf{u}} \\ \text{perturbation}: \quad & \dot{\delta\boldsymbol{\mu}} = -\overset{\curlywedge}{\mathbf{u}}\delta\boldsymbol{\mu} + \mathbf{w} \end{aligned} \qquad \overset{\curlywedge}{\mathbf{u}} := \begin{bmatrix} \hat{\boldsymbol{\omega}} & \hat{\mathbf{v}} \\ 0 & \hat{\boldsymbol{\omega}} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- In discrete-time with discretization τ_t, the above becomes:

$$\begin{aligned} \text{nominal}: \quad & \boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t \exp\left(\tau_t \hat{\mathbf{u}}_t\right) \\ \text{perturbation}: \quad & \delta\boldsymbol{\mu}_{t+1} = \exp\left(-\tau_t \overset{\curlywedge}{\mathbf{u}}_t\right)\delta\boldsymbol{\mu}_t + \mathbf{w}_t \end{aligned}$$

- This is useful to separate the effect of the noise $\mathbf{w}_t$ from the motion of the deterministic part of T_t. See Barfoot Ch. 7.2 for details.

EKF Prediction Step

- **Prior**: $T_t | \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1} \sim \mathcal{N}(\boldsymbol{\mu}_{t|t}, \Sigma_{t|t})$ with $\boldsymbol{\mu}_{t|t} \in SE(3)$ and $\Sigma_{t|t} \in \mathbb{R}^{6\times 6}$
- This means that $T_t = \boldsymbol{\mu}_{t|t} \exp(\delta\hat{\boldsymbol{\mu}}_{t|t})$ with $\delta\boldsymbol{\mu}_{t|t} \sim \mathcal{N}(0, \Sigma_{t|t})$
- $\Sigma_{t|t}$ is 6×6 because only the 6 degrees of freedom of T_t are changing
- **Motion Model**: nominal kinematics of $\boldsymbol{\mu}_{t|t}$ and perturbation kinematics of $\delta\boldsymbol{\mu}_{t|t}$ with time discretization τ_t:

$$\boldsymbol{\mu}_{t+1|t} = \boldsymbol{\mu}_{t|t} \exp\left(\tau_t \hat{\mathbf{u}}_t\right)$$

$$\delta\boldsymbol{\mu}_{t+1|t} = \exp\left(-\tau_t \overset{\curlywedge}{\mathbf{u}}_t\right) \delta\boldsymbol{\mu}_{t|t} + \mathbf{w}_t$$

- **EKF Prediction Step** with $\mathbf{w}_t \sim \mathcal{N}(0, W)$:

$$\boldsymbol{\mu}_{t+1|t} = \boldsymbol{\mu}_{t|t} \exp\left(\tau_t \hat{\mathbf{u}}_t\right)$$

$$\Sigma_{t+1|t} = \mathbb{E}[\delta\boldsymbol{\mu}_{t+1|t} \delta\boldsymbol{\mu}_{t+1|t}^\top] = \exp\left(-\tau \overset{\curlywedge}{\mathbf{u}}_t\right) \Sigma_{t|t} \exp\left(-\tau \overset{\curlywedge}{\mathbf{u}}_t\right)^\top + W$$

where

$$\mathbf{u}_t := \begin{bmatrix} \mathbf{v}_t \\ \boldsymbol{\omega}_t \end{bmatrix} \in \mathbb{R}^6 \quad \hat{\mathbf{u}}_t := \begin{bmatrix} \hat{\boldsymbol{\omega}}_t & \mathbf{v}_t \\ \mathbf{0}^\top & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \quad \overset{\curlywedge}{\mathbf{u}}_t := \begin{bmatrix} \hat{\boldsymbol{\omega}}_t & \hat{\mathbf{v}}_t \\ 0 & \hat{\boldsymbol{\omega}}_t \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

EKF Update Step

- **Prior**: $T_{t+1}|z_{0:t}, u_{0:t} \sim \mathcal{N}(\boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t})$ with $\boldsymbol{\mu}_{t+1|t} \in SE(3)$ and $\Sigma_{t+1|t} \in \mathbb{R}^{6\times 6}$
- **Observation Model**: with measurement noise $\mathbf{v}_t \sim \mathcal{N}(0, V)$

$$\mathbf{z}_{t+1,i} = h(T_{t+1}, \mathbf{m}_j) + \mathbf{v}_{t+1,i} := K_s \pi\left({}_O T_I T_{t+1}^{-1} \underline{\mathbf{m}}_j\right) + \mathbf{v}_{t+1,i}$$

- The observation model is the same as in the visual mapping problem but this time the variable of interest is the IMU pose $T_{t+1} \in SE(3)$ instead of the landmark positions $\mathbf{m} \in \mathbb{R}^{3M}$
- We need the observation model Jacobian $H_{t+1} \in \mathbb{R}^{4N_{t+1}\times 6}$ with respect to the IMU pose T_{t+1}, evaluated at $\boldsymbol{\mu}_{t+1|t}$

EKF Update Step

- Let the elements of $H_{t+1} \in \mathbb{R}^{4N_{t+1} \times 6}$ corresponding to different observations i be $H_{t+1,i} \in \mathbb{R}^{4\times 6}$
- The first-order Taylor series approximation of observation i at time $t+1$ using an IMU pose perturbation $\delta\boldsymbol{\mu}$ is:

$$
\begin{aligned}
\mathbf{z}_{t+1,i} &= K_s \pi \left({}_O T_I \left(\boldsymbol{\mu}_{t+1|t} \exp\left(\hat{\delta\boldsymbol{\mu}} \right) \right)^{-1} \underline{\mathbf{m}}_j \right) + \mathbf{v}_{t+1,i} \\
&\approx K_s \pi \left({}_O T_I \left(I - \hat{\delta\boldsymbol{\mu}} \right) \boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j \right) + \mathbf{v}_{t+1,i} \\
&= K_s \pi \left({}_O T_I \boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j - {}_O T_I \left(\boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j \right)^{\odot} \delta\boldsymbol{\mu} \right) + \mathbf{v}_{t+1,i} \\
&\approx \underbrace{K_s \pi \left({}_O T_I \boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j \right)}_{\tilde{\mathbf{z}}_{t+1,i}} \underbrace{- K_s \frac{d\pi}{d\mathbf{q}} \left({}_O T_I \boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j \right) {}_O T_I \left(\boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j \right)^{\odot}}_{H_{t+1,i}} \delta\boldsymbol{\mu} + \mathbf{v}_{t+1,i}
\end{aligned}
$$

where for homogeneous coordinates $\underline{\mathbf{s}} \in \mathbb{R}^4$ and $\hat{\boldsymbol{\xi}} \in \mathfrak{se}(3)$:

$$
\hat{\boldsymbol{\xi}} \underline{\mathbf{s}} = \underline{\mathbf{s}}^{\odot} \boldsymbol{\xi} \qquad \begin{bmatrix} \mathbf{s} \\ 1 \end{bmatrix}^{\odot} := \begin{bmatrix} I & -\hat{\mathbf{s}} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 6}
$$

EKF Update Step

- **Prior**: $\boldsymbol{\mu}_{t+1|t} \in SE(3)$ and $\Sigma_{t+1|t} \in \mathbb{R}^{6\times 6}$
- Known: stereo calibration matrix K_s, extrinsics ${}_O T_I \in SE(3)$, landmark positions $\mathbf{m} \in \mathbb{R}^{3M}$, new observations $\mathbf{z}_{t+1} \in \mathbb{R}^{4N_{t+1}}$
- Predicted observation based on $\boldsymbol{\mu}_{t+1|t}$ and known correspondences Δ_t:

$$\tilde{\mathbf{z}}_{t+1,i} := K_s \pi\left({}_O T_I \boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j\right) \qquad \text{for } i = 1, \ldots, N_{t+1}$$

- Jacobian of $\tilde{\mathbf{z}}_{t+1,i}$ with respect to T_{t+1} evaluated at $\boldsymbol{\mu}_{t+1|t}$:

$$H_{t+1,i} = -K_s \frac{d\pi}{d\mathbf{q}}\left({}_O T_I \boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j\right) {}_O T_I \left(\boldsymbol{\mu}_{t+1|t}^{-1} \underline{\mathbf{m}}_j\right)^{\odot} \in \mathbb{R}^{4\times 6}$$

- Perform the EKF update:

$$K_{t+1} = \Sigma_{t+1|t} H_{t+1}^{\top} \left(H_{t+1} \Sigma_{t+1|t} H_{t+1}^{\top} + I \otimes V\right)^{-1}$$

$$\boldsymbol{\mu}_{t+1|t+1} = \boldsymbol{\mu}_{t+1|t} \exp\left(\left(K_{t+1}(\mathbf{z}_{t+1} - \tilde{\mathbf{z}}_{t+1})\right)^{\wedge}\right)$$

$$\Sigma_{t+1|t+1} = (I - K_{t+1} H_{t+1}) \Sigma_{t+1|t}$$

$$H_{t+1} = \begin{bmatrix} H_{t+1,1} \\ \vdots \\ H_{t+1,N_{t+1}} \end{bmatrix}$$