ECE276A: Sensing & Estimation in Robotics
Lecture 14: SO(3) and SE(3) Geometry and Kinematics

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Special Orthogonal Group $SO(3)$

- The orientation R of a rigid body can be described by a matrix in the **special orthogonal group**:

$$SO(3) := \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; \underbrace{R^\top R = I}_{\text{distances preserved}}, \underbrace{\det(R) = 1}_{\text{no reflection}} \right\}$$

- It can be verified that $SO(3)$ satisfies all requirements of a group:
  - **Closure**: $R_1 R_2 \in SO(3)$
  - **Identity**: $I \in SO(3)$
  - **Inverse**: $R^{-1} = R^\top \in SO(3)$
  - **Associativity**: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$ for all $R_1, R_2, R_3 \in SO(3)$

Parametrizations of $SO(3)$

- **Rotation Matrix**: an element of the **Special Orthogonal Group**:

$$R \in SO(3) := \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; R^\top R = I,\ \det(R) = 1 \right\}$$

- **Euler Angles**: roll ϕ, pitch θ, yaw ψ specifying a **rzyx** rotation:

$$R = R_z(\psi) R_y(\theta) R_x(\phi)$$

- **Axis-Angle**: $\boldsymbol{\theta} \in \mathbb{R}^3$ specifying a rotation about an axis $\boldsymbol{\eta} := \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$ through an angle $\theta := \|\boldsymbol{\theta}\|$:

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \hat{\boldsymbol{\theta}} + \frac{1}{2!}\hat{\boldsymbol{\theta}}^2 + \frac{1}{3!}\hat{\boldsymbol{\theta}}^3 + \dots$$

- **Unit Quaternion**: $\mathbf{q} = [q_s,\ \mathbf{q}_v] \in \left\{ q \in \mathbb{H} \mid q_s^2 + \mathbf{q}_v^\top \mathbf{q}_v = 1 \right\}$:

$$R = E(\mathbf{q}) G(\mathbf{q})^\top \qquad \begin{aligned} E(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \\ G(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v] \end{aligned}$$

Special Euclidean Group $SE(3)$

- The pose T of a rigid body can be described by a matrix in the **special Euclidean group**:

$$SE(3) := \left\{ T := \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, R \in SO(3), \mathbf{p} \in \mathbb{R}^3 \right\}$$

- It can be verified that $SE(3)$ satisfies all requirements of a group:
  - **Closure**: $T_1 T_2 = \begin{bmatrix} R_1 & \mathbf{p}_1 \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} R_2 & \mathbf{p}_2 \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 \mathbf{p}_2 + \mathbf{p}_1 \\ \mathbf{0}^\top & 1 \end{bmatrix} \in SE(3)$
  - **Identity**: $\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in SE(3)$
  - **Inverse**: $\begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^\top & -R^\top \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in SE(3)$
  - **Associativity**: $(T_1 T_2) T_3 = T_1 (T_2 T_3)$ for all $T_1, T_2, T_3 \in SE(3)$

Matrix Lie Group

- $SO(3)$ and $SE(3)$ are **matrix Lie groups**
- A **group** is a set of elements with an operation that combines any two elements to form a third element also in the set. A group satisfies four axioms: closure, associativity, identity, and invertibility
- A **manifold** is a topological space that is locally homeomorphic to Euclidean space but globally may have more complicated structure
- A **Lie group** is a group that is also a differentiable manifold with the property that the group operations are smooth
- A **matrix Lie group** further specifies that the group elements are matrices, the combination operation is matrix multiplication, and the inversion operation is matrix inversion
- The **exponential map** relates a matrix Lie group to its **Lie algebra**

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \qquad \log(A) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (A - I)^n$$

Lie Algebra

- A **Lie algebra** is a vector space $\mathbb{V}$ over some field $\mathbb{F}$ with a binary operation, $[\cdot, \cdot]$, called a **Lie bracket**
- For all $X, Y, Z \in \mathbb{V}$ and $a, b \in \mathbb{F}$, the Lie bracket satisfies:

$$
\begin{aligned}
\text{closure :} \quad & [X, Y] \in \mathbb{V} \\
\text{bilinearity :} \quad & [aX + bY, Z] = a[X, Z] + b[Y, Z] \\
& [Z, aX + bY] = a[Z, X] + b[Z, Y] \\
\text{alternating :} \quad & [X, X] = 0 \\
\text{Jacobi identity :} \quad & [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0
\end{aligned}
$$

- A **Lie algebra** may be associated with every Lie group. The vector space of a Lie algebra forms the **tangent space** to the Lie group at the identity element of the group.

Lie Group and Lie Algebra Visualization

- **Lie Group**: free of singularities but has constraints
- **Lie Algebra**: free of constraints but has singularities

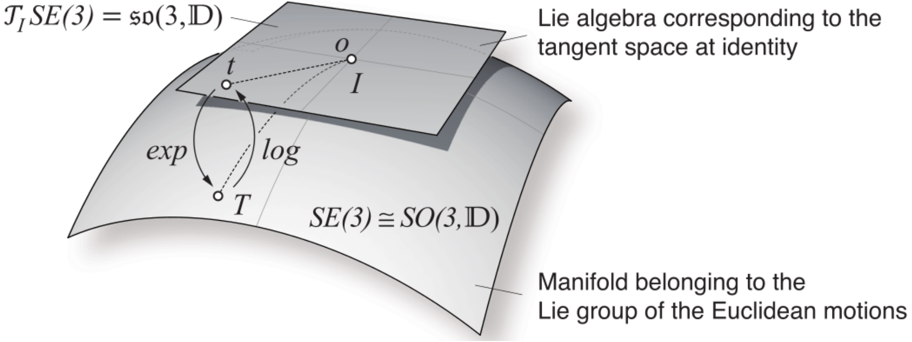

Figure: $SE(3)$ and the corresponding Lie algebra $\mathfrak{se}(3)$ as tangent space at identity

$SO(3)$ Geometry

Special Orthogonal Lie Algebra $\mathfrak{so}(3)$

- The **Lie algebra** of $SO(3)$ is the space of skew-symmetric matrices

$$\mathfrak{so}(3) := \{\hat{\boldsymbol{\theta}} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{\theta} \in \mathbb{R}^3\}$$

- The **Lie bracket** of $\mathfrak{so}(3)$ is:

$$[\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2] = \hat{\boldsymbol{\theta}}_1\hat{\boldsymbol{\theta}}_2 - \hat{\boldsymbol{\theta}}_2\hat{\boldsymbol{\theta}}_1 = \left(\hat{\boldsymbol{\theta}}_1\boldsymbol{\theta}_2\right)^{\wedge} \in \mathfrak{so}(3)$$

- **Generators of $\mathfrak{so}(3)$**: derivatives of rotations around each standard axis:

$$G_x = \frac{d}{d\phi}R_x(\phi)\bigg|_{\phi=0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad G_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad G_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The elements $\hat{\boldsymbol{\theta}} = \theta_1 G_x + \theta_2 G_y + \theta_3 G_z \in \mathfrak{so}(3)$ are linear combinations of the generators

Exponential Map from $\mathfrak{so}(3)$ to $SO(3)$

- The elements $R \in SO(3)$ are related to the elements $\hat{\boldsymbol{\theta}} \in \mathfrak{so}(3)$ through the **exponential map**:

$$R = \exp(\hat{\boldsymbol{\theta}}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\hat{\boldsymbol{\theta}})^n$$

- The exponential map is **surjective** but **not injective**, i.e., every element of $SO(3)$ can be generated from multiple elements of $\mathfrak{so}(3)$
- Any vector $(\|\boldsymbol{\theta}\| + 2\pi k) \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$ for integer k leads to the same $R \in SO(3)$
- The exponential map is **not commutative**, $e^{\hat{\boldsymbol{\theta}}_1} e^{\hat{\boldsymbol{\theta}}_2} \neq e^{\hat{\boldsymbol{\theta}}_2} e^{\hat{\boldsymbol{\theta}}_1} \neq e^{\hat{\boldsymbol{\theta}}_1 + \hat{\boldsymbol{\theta}}_2}$, unless $[\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2] = \hat{\boldsymbol{\theta}}_1 \hat{\boldsymbol{\theta}}_2 - \hat{\boldsymbol{\theta}}_2 \hat{\boldsymbol{\theta}}_1 = 0$

Rodrigues Formula

- A closed-from expression for the exponential map from $\mathfrak{so}(3)$ to $SO(3)$:

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2$$

- The formula is derived using that $\hat{\boldsymbol{\theta}}^{2n+1} = (-\boldsymbol{\theta}^\top\boldsymbol{\theta})^n\hat{\boldsymbol{\theta}}$:

$$\begin{aligned}
\exp(\hat{\boldsymbol{\theta}}) &= I + \sum_{n=1}^{\infty}\frac{1}{n!}\hat{\boldsymbol{\theta}}^n \\
&= I + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\hat{\boldsymbol{\theta}}^{2n+1} + \sum_{n=0}^{\infty}\frac{1}{(2n+2)!}\hat{\boldsymbol{\theta}}^{2n+2} \\
&= I + \left(\sum_{n=0}^{\infty}\frac{(-1)^n\|\boldsymbol{\theta}\|^{2n}}{(2n+1)!}\right)\hat{\boldsymbol{\theta}} + \left(\sum_{n=0}^{\infty}\frac{(-1)^n\|\boldsymbol{\theta}\|^{2n}}{(2n+2)!}\right)\hat{\boldsymbol{\theta}}^2 \\
&= I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2
\end{aligned}$$

Logarithm Map from $SO(3)$ to $\mathfrak{so}(3)$

- $\forall R \in SO(3)$, there exists a (non-unique) $\boldsymbol{\theta} \in \mathbb{R}^3$ such that $R = \exp(\hat{\boldsymbol{\theta}})$
- The **logarithm map** $\log : SO(3) \to \mathfrak{so}(3)$ is the inverse of $\exp(\hat{\boldsymbol{\theta}})$:

$$\theta = \|\boldsymbol{\theta}\| = \arccos\left(\frac{\mathrm{tr}(R) - 1}{2}\right)$$

$$\boldsymbol{\eta} = \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} = \frac{1}{2\sin(\|\boldsymbol{\theta}\|)} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

$$\hat{\boldsymbol{\theta}} = \log(R) = \frac{\|\boldsymbol{\theta}\|}{2\sin\|\boldsymbol{\theta}\|}(R - R^\top)$$

- If $R = I$, then $\theta = 0$ and $\boldsymbol{\eta}$ is undefined
- If $\mathrm{tr}(R) = -1$, then $\theta = \pi$ and for any $i \in \{1, 2, 3\}$:

$$\boldsymbol{\eta} = \frac{1}{\sqrt{2(1 + e_i^T R e_i)}}(I + R)e_i$$

- The log map has a singularity at $\theta = 0$ because there are infinite choices of rotation axes or equivalently the exponential map is many-to-one.
- The matrix exponential "integrates" $\hat{\boldsymbol{\theta}} \in \mathfrak{se}(3)$ for one second; the matrix logarithm "differentiates" $R \in SO(3)$ to obtain $\hat{\boldsymbol{\theta}} \in \mathfrak{se}(3)$

$SO(3)$ Jacobians

- The **left Jacobian** of $SO(3)$ is the matrix:

$$J_L(\boldsymbol{\theta}) := \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\hat{\boldsymbol{\theta}}\right)^n \qquad\qquad R = I + \hat{\boldsymbol{\theta}} J_L(\boldsymbol{\theta})$$

- The **right Jacobian** of $SO(3)$ is the matrix:

$$J_R(\boldsymbol{\theta}) := \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(-\hat{\boldsymbol{\theta}}\right)^n \qquad J_R(\boldsymbol{\theta}) = J_L(-\boldsymbol{\theta}) = J_L(\boldsymbol{\theta})^\top = R^\top J_L(\boldsymbol{\theta})$$

- **Baker-Campbell-Hausdorff Formulas**: the $SO(3)$ Jacobians relate small perturbations in $\mathfrak{so}(3)$ to small perturbations in $SO(3)$:

$$\begin{aligned}\exp\left((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^\wedge\right) &\approx \exp(\hat{\boldsymbol{\theta}}) \exp\left((J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta})^\wedge\right)\\ &\approx \exp\left((J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta})^\wedge\right) \exp(\hat{\boldsymbol{\theta}})\end{aligned}$$

$$\log(\exp(\hat{\boldsymbol{\theta}}_1) \exp(\hat{\boldsymbol{\theta}}_2))^\vee \approx \begin{cases} J_L(\boldsymbol{\theta}_2)^{-1}\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 & \text{if } \boldsymbol{\theta}_1 \text{ is small} \\ \boldsymbol{\theta}_1 + J_R(\boldsymbol{\theta}_1)^{-1}\boldsymbol{\theta}_2 & \text{if } \boldsymbol{\theta}_2 \text{ is small}\end{cases}$$

Closed-forms of the $SO(3)$ Jacobians

$$J_L(\boldsymbol{\theta}) = I + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\|\boldsymbol{\theta}\| - \sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\right)\hat{\boldsymbol{\theta}}^2 \approx I + \frac{1}{2}\hat{\boldsymbol{\theta}}$$

$$J_L(\boldsymbol{\theta})^{-1} = I - \frac{1}{2}\hat{\boldsymbol{\theta}} + \left(\frac{1}{\|\boldsymbol{\theta}\|^2} - \frac{1+\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}}^2 \approx I - \frac{1}{2}\hat{\boldsymbol{\theta}}$$

$$J_R(\boldsymbol{\theta}) = I - \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\|\boldsymbol{\theta}\| - \sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\right)\hat{\boldsymbol{\theta}}^2 \approx I - \frac{1}{2}\hat{\boldsymbol{\theta}}$$

$$J_R(\boldsymbol{\theta})^{-1} = I + \frac{1}{2}\hat{\boldsymbol{\theta}} + \left(\frac{1}{\|\boldsymbol{\theta}\|^2} - \frac{1+\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}}^2 \approx I + \frac{1}{2}\hat{\boldsymbol{\theta}}$$

$$J_L(\boldsymbol{\theta})J_L(\boldsymbol{\theta})^T = I + \left(1 - 2\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2 \succ 0$$

$$\left(J_L(\boldsymbol{\theta})J_L(\boldsymbol{\theta})^T\right)^{-1} = I + \left(1 - 2\frac{\|\boldsymbol{\theta}\|^2}{1-\cos\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}}^2$$

Distances in $SO(3)$

- There are two ways to define the difference between two rotations:

$$\boldsymbol{\theta}_{12} = \log\left(R_1^\top R_2\right)^\vee \qquad \boldsymbol{\theta}_{21} = \log\left(R_2 R_1^\top\right)^\vee \qquad R_1, R_2 \in SO(3)$$

- Inner product on $\mathfrak{so}(3)$:

$$\langle \hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2 \rangle = \frac{1}{2}\operatorname{tr}\left(\hat{\boldsymbol{\theta}}_1^\top \hat{\boldsymbol{\theta}}_2\right) = \boldsymbol{\theta}_1^\top \boldsymbol{\theta}_2$$

- The metric distance between two rotations may be defined in two ways as the magnitude of the rotation difference:

$$\theta_{12} := \sqrt{\langle \log\left(R_1^\top R_2\right), \log\left(R_1^\top R_2\right)\rangle} = \|\boldsymbol{\theta}_{12}\|_2$$
$$\theta_{21} := \sqrt{\langle \log\left(R_2 R_1^\top\right), \log\left(R_2 R_1^\top\right)\rangle} = \|\boldsymbol{\theta}_{21}\|_2$$

Integration in $SO(3)$

- The distance between a rotation $R = \exp(\hat{\boldsymbol{\theta}})$ and a small perturbation $\exp((\boldsymbol{\theta} + \delta\boldsymbol{\theta})^{\wedge})$ can be approximated using the BCH formulas:

$$\log\left(\exp(\hat{\boldsymbol{\theta}})^{\top}\exp((\boldsymbol{\theta}+\delta\boldsymbol{\theta})^{\wedge})\right)^{\vee} \approx \log\left(R^{\top}R\exp\left((J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta})^{\wedge}\right)\right)^{\vee} = J_R(\boldsymbol{\theta})\delta\boldsymbol{\theta}$$

$$\log\left(\exp((\boldsymbol{\theta}+\delta\boldsymbol{\theta})^{\wedge})\exp(\hat{\boldsymbol{\theta}})^{\top}\right)^{\vee} \approx \log\left(\exp\left((J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta})^{\wedge}\right)RR^{\top}\right)^{\vee} = J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta}$$

- Regardless of which distance metric we use, the infinitesimal volume element is the same:

$$\det(J_L(\boldsymbol{\theta})) = \det(J_R(\boldsymbol{\theta})) \qquad dR = |\det(J(\boldsymbol{\theta}))|d\boldsymbol{\theta} = 2\left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)d\boldsymbol{\theta}$$

- Integrating functions of rotations can then be carried out as follows:

$$\int_{SO(3)} f(R)dR = \int_{\|\boldsymbol{\theta}\|<\pi} f\left(\exp(\hat{\boldsymbol{\theta}})\right)|\det(J(\boldsymbol{\theta}))|d\boldsymbol{\theta}$$

Derivatives in $SO(3)$

- Consider $\mathbf{s} \in \mathbb{R}^3$ rotated by a rotation matrix $R \in SO(3)$ to a new frame
- How do we compute the derivative of $R\mathbf{s}$ with respect to the rotation R?
- Let $\boldsymbol{\theta} \in \mathbb{R}^3$ be the Lie algebra vector representing R, i.e., $R = \exp\left(\hat{\boldsymbol{\theta}}\right)$
- We can compute derivatives with respect to the elements of $\boldsymbol{\theta}$:

$$
\begin{aligned}
\frac{\partial R\mathbf{s}}{\partial \theta_i} &= \lim_{h\to 0} \frac{\exp\left((\boldsymbol{\theta} + h\mathbf{e}_i)^\wedge\right)\mathbf{s} - \exp\left(\hat{\boldsymbol{\theta}}\right)\mathbf{s}}{h} \\
&\overset{\text{BCH}}{\underset{\text{Formula}}{=\!=\!=}} \lim_{h\to 0} \frac{\exp\left((hJ_L(\boldsymbol{\theta})\mathbf{e}_i)^\wedge\right)\exp\left(\hat{\boldsymbol{\theta}}\right)\mathbf{s} - \exp\left(\hat{\boldsymbol{\theta}}\right)\mathbf{s}}{h} \\
&\overset{\exp\left(\delta\hat{\boldsymbol{\theta}}\right)\approx I + \delta\hat{\boldsymbol{\theta}}}{=\!=\!=\!=} \lim_{h\to 0} \frac{\left(I + h\left(J_L(\boldsymbol{\theta})\mathbf{e}_i\right)^\wedge\right)\exp\left(\hat{\boldsymbol{\theta}}\right)\mathbf{s} - \exp\left(\hat{\boldsymbol{\theta}}\right)\mathbf{s}}{h} \\
&= \left(J_L(\boldsymbol{\theta})\mathbf{e}_i\right)^\wedge R\mathbf{s} = -\left(R\mathbf{s}\right)^\wedge J_L(\boldsymbol{\theta})\mathbf{e}_i
\end{aligned}
$$

- Stacking the three directional derivatives: $\boxed{\frac{\partial R\mathbf{s}}{\partial \boldsymbol{\theta}} = -\left(R\mathbf{s}\right)^\wedge J_L(\boldsymbol{\theta})}$

Derivatives in $SO(3)$

- **Perturbation in $\mathfrak{so}(3)$**: the gradient can also be obtained via a small perturbation $\delta\boldsymbol{\theta}$ to the axis-angle vector $\boldsymbol{\theta}$:

$$\begin{aligned}
\exp\left((\boldsymbol{\theta}+\delta\boldsymbol{\theta})^{\wedge}\right)\mathbf{s} &\overset{\text{BCH}}{\approx} \exp\left((J_L\boldsymbol{\theta})\delta\boldsymbol{\theta})^{\wedge}\right)\exp(\hat{\boldsymbol{\theta}})\mathbf{s} \\
&\approx \left(I + (J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta})^{\wedge}\right)\exp(\hat{\boldsymbol{\theta}})\mathbf{s} \\
&= R\mathbf{s} + (J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta})^{\wedge}R\mathbf{s} = R\mathbf{s} \underbrace{- (R\mathbf{s})^{\wedge} J_L(\boldsymbol{\theta})}_{\frac{\partial R\mathbf{s}}{\partial \boldsymbol{\theta}}}\delta\boldsymbol{\theta}
\end{aligned}$$

- This is the same as using first-order Taylor series to identify the Jacobian of a function $f(\mathbf{x})$:

$$f(\mathbf{x}+\delta\mathbf{x}) \approx f(\mathbf{x}) + \left[\frac{\partial f}{\partial \mathbf{x}}(\mathbf{x})\right]\delta\mathbf{x}$$

- **Perturbation in $SO(3)$**: a small perturbation $\boldsymbol{\psi} = J_L(\boldsymbol{\theta})\delta\boldsymbol{\theta}$ may also be applied directly to R:

$$\exp(\hat{\boldsymbol{\psi}})R\mathbf{s} \approx (I + \hat{\boldsymbol{\psi}})R\mathbf{s} = R\mathbf{s} - (R\mathbf{s})^{\wedge}\boldsymbol{\psi}$$

Gradient Descent in $SO(3)$

- Consider $\min_{\mathbf{x}} f(\mathbf{x})$
- **Gradient descent in** $\mathbb{R}^d$: given an initial guess $\mathbf{x}^{(k)}$ take a step of size $\alpha^{(k)} > 0$ along the descent direction $\delta\mathbf{x}^{(k)} = -\nabla f(\mathbf{x}^{(k)})$:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha^{(k)}\delta\mathbf{x}^{(k)}$$

- Consider $\min_R f(R\mathbf{s})$
- **Gradient descent in** $SO(3)$: given an initial guess $R^{(k)}$ take a step of size $\alpha^{(k)} > 0$ along the descent direction $\boldsymbol{\psi}^{(k)} = -\boldsymbol{\delta}^{(k)}$:

$$R^{(k+1)} = \exp\left(\alpha^{(k)}\hat{\boldsymbol{\psi}}^{(k)}\right) R^{(k)}$$

where $\boldsymbol{\delta}^{(k)}$ should be the gradient of f wrt R evaluated at $R^{(k)}\mathbf{s}$

Choosing a Descent Direction in $SO(3)$

- Use a perturbation $\boldsymbol{\psi}^{(k)}$ around the initial guess $R^{(k)}$ to determine the gradient $\boldsymbol{\delta}^{(k)}$:

$$
\begin{aligned}
f\left(\exp(\hat{\boldsymbol{\psi}}^{(k)})R^{(k)}\mathbf{s}\right) &\approx f\left((I+\hat{\boldsymbol{\psi}}^{(k)})R^{(k)}\mathbf{s}\right) \\
&= f\left(R^{(k)}\mathbf{s} - \left(R^{(k)}\mathbf{s}\right)^{\wedge}\boldsymbol{\psi}^{(k)}\right) \\
&\approx f\left(R^{(k)}\mathbf{s}\right) \underbrace{-\nabla f\left(R^{(k)}\mathbf{s}\right)^{\top}\left(R^{(k)}\mathbf{s}\right)^{\wedge}}_{\boldsymbol{\delta}^{(k)\top}}\boldsymbol{\psi}^{(k)}
\end{aligned}
$$

- **Gradient descent in** $SO(3)$: given an initial guess $R^{(k)}$ take a step of size $\alpha^{(k)} > 0$ along the descent direction $\boldsymbol{\psi}^{(k)} = -\boldsymbol{\delta}^{(k)}$:

$$
\begin{aligned}
\boldsymbol{\psi}^{(k)} &= -\left(R^{(k)}\mathbf{s}\right)^{\wedge}\nabla f(R^{(k)}\mathbf{s}) \\
R^{(k+1)} &= \exp\left(\alpha^{(k)}\hat{\boldsymbol{\psi}}^{(k)}\right)R^{(k)}
\end{aligned}
$$

Gauss-Newton Optimization in $SO(3)$

- Optimization problem:

$$\min_R f(R) := \frac{1}{2} \sum_j \mathbf{e}_j(R\mathbf{s}_j)^\top \mathbf{e}_j(R\mathbf{s}_j)$$

- Linearize $f(R)$ using $\mathbf{e}_j^{(k)} := \mathbf{e}_j(R^{(k)}\mathbf{s}_j)$ and $J_j^{(k)} := -\frac{d\mathbf{e}_j}{d\mathbf{x}}(R^{(k)}\mathbf{s}_j)\left(R^{(k)}\mathbf{s}_j\right)^\wedge$

$$f(R^{(k+1)}) = f(\exp(\hat{\boldsymbol{\psi}}^{(k)})R^{(k)}) \approx \frac{1}{2}\sum_j \left(\mathbf{e}_j^{(k)} + J_j^{(k)}\boldsymbol{\psi}^{(k)}\right)^\top \left(\mathbf{e}_j^{(k)} + J_j^{(k)}\boldsymbol{\psi}^{(k)}\right)$$

- The cost is quadratic in $\boldsymbol{\psi}^{(k)}$ and setting its gradient to zero leads to:

$$\left(\sum_j J_j^{(k)}\left(J_j^{(k)}\right)^\top\right)\boldsymbol{\psi}^{(k)} = -\sum_j \left(J_j^{(k)}\right)^\top \mathbf{e}_j^{(k)}$$

- Apply the optimal perturbation $\boldsymbol{\psi}^{(k)}$ to the initial guess $R^{(k)}$ according to the left perturbation scheme:

$$R^{(k+1)} = \exp(\alpha^{(k)}\hat{\boldsymbol{\psi}}^{(k)})R^{(k)}$$

Adjoints

- The **adjoint** operator of a Lie group linearly transforms tangent vectors from one tangent space to another
- The adjoint of $R \in SO(3)$ is $Ad(R) = R$
- $Ad(SO(3)) := SO(3)$ is a matrix Lie group
- The adjoint of the Lie algebra is the derivative of Ad at identity

$$ad(\hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\theta}}$$

- $ad(\mathfrak{so}(3)) := \mathfrak{so}(3)$ is the Lie algebra associated with $Ad(SO(3))$
- Ad and ad are related through the exponential map:

$$Ad\left(\exp(\hat{\boldsymbol{\theta}})\right) = \exp\left(ad(\hat{\boldsymbol{\theta}})\right)$$

$SO(3)$ and $\mathfrak{so}(3)$ Identities

$$R = \exp\left(\hat{\boldsymbol{\theta}}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\hat{\boldsymbol{\theta}}^n = I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2 \approx I + \hat{\boldsymbol{\theta}}$$

$$R^{-1} = R^\top = \exp\left(-\hat{\boldsymbol{\theta}}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\hat{\boldsymbol{\theta}}\right)^n \approx I - \hat{\boldsymbol{\theta}}$$

$$\det(R) = 1 \qquad \hat{\boldsymbol{\theta}}^\top = -\hat{\boldsymbol{\theta}}$$

$$\operatorname{tr}(R) = 2\cos\|\boldsymbol{\theta}\| + 1 \qquad \hat{\boldsymbol{\theta}}\boldsymbol{\theta} = 0$$

$$R\boldsymbol{\theta} = \boldsymbol{\theta} \qquad (A\boldsymbol{\theta})^\wedge = \hat{\boldsymbol{\theta}}\left(\operatorname{tr}(A)I - A\right) - A^\top\hat{\boldsymbol{\theta}}, \quad A \in \mathbb{R}^{3\times 3}$$

$$R\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}R \qquad \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} = \boldsymbol{\phi}\boldsymbol{\theta}^\top - \left(\boldsymbol{\theta}^\top\boldsymbol{\phi}\right)I, \qquad \boldsymbol{\phi} \in \mathbb{R}^3$$

$$(R\mathbf{s})^\wedge = R\hat{\mathbf{s}}R^\top, \quad \mathbf{s} \in \mathbb{R}^3 \qquad \hat{\boldsymbol{\theta}}^{2k+1} = \left(-\boldsymbol{\theta}^T\boldsymbol{\theta}\right)^k\hat{\boldsymbol{\theta}}$$

$$\exp\left((R\mathbf{s})^\wedge\right) = R\exp\left(\hat{\mathbf{s}}\right)R^\top \qquad [\boldsymbol{\theta}, \boldsymbol{\phi}] = \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} = \left(\hat{\boldsymbol{\theta}}\boldsymbol{\phi}\right)^\wedge$$

$SE(3)$ Geometry

Special Euclidean Lie Algebra $\mathfrak{se}(3)$

- The Lie algebra of $SE(3)$ is the space of twist matrices:

$$\mathfrak{se}(3) := \left\{ \hat{\boldsymbol{\xi}} := \begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| \boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6 \right\}$$

- The **Lie bracket** of $\mathfrak{se}(3)$ is:

$$[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2] = \hat{\boldsymbol{\xi}}_1 \hat{\boldsymbol{\xi}}_2 - \hat{\boldsymbol{\xi}}_2 \hat{\boldsymbol{\xi}}_1 = \left(\overset{\curlywedge}{\boldsymbol{\xi}}_1 \boldsymbol{\xi}_2 \right)^{\wedge} \in \mathfrak{se}(3) \qquad \overset{\curlywedge}{\boldsymbol{\xi}} := \begin{bmatrix} \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\rho}} \\ 0 & \hat{\boldsymbol{\theta}} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- The elements $T \in SE(3)$ are related to the elements $\hat{\boldsymbol{\xi}} \in \mathfrak{se}(3)$ through the exponential map:

$$T = \exp(\hat{\boldsymbol{\xi}}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\hat{\boldsymbol{\xi}})^n \qquad \boldsymbol{\xi} = \log(T)^{\vee}$$

Exponential Map from $\mathfrak{se}(3)$ to $SE(3)$

- The exponential map is **surjective** but **not injective**, i.e., every element of $SE(3)$ can be generated from multiple elements of $\mathfrak{se}(3)$

- **Rodrigues Formula**: obtained using $\hat{\boldsymbol{\xi}}^4 + \|\boldsymbol{\theta}\|^2 \hat{\boldsymbol{\xi}}^2 = 0$:

$$T = \exp(\hat{\boldsymbol{\xi}}) = \begin{bmatrix} \exp(\hat{\boldsymbol{\theta}}) & J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\boldsymbol{\xi}}^n =$$

$$= I + \hat{\boldsymbol{\xi}} + \left(\frac{1 - \cos \|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2} \right) \hat{\boldsymbol{\xi}}^2 + \left(\frac{\|\boldsymbol{\theta}\| - \sin \|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3} \right) \hat{\boldsymbol{\xi}}^3$$

- **Logarithm map** $\log : SE(3) \to \mathfrak{se}(3)$: for any $T \in SE(3)$, there exists a (non-unique) $\boldsymbol{\xi} \in \mathbb{R}^6$ such that:

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} = \log(T)^\vee := \begin{cases} \boldsymbol{\theta} = \log(R)^\vee, \boldsymbol{\rho} = J_L^{-1}(\boldsymbol{\theta})\mathbf{p}, & \text{if } R \neq I, \\ \boldsymbol{\theta} = 0, \boldsymbol{\rho} = \mathbf{p}, & \text{if } R = I. \end{cases}$$

$SE(3)$ Jacobians

- **Left Jacobian of** $SE(3)$: $\mathcal{J}_L(\boldsymbol{\xi}) = \begin{bmatrix} J_L(\boldsymbol{\theta}) & Q_L(\boldsymbol{\xi}) \\ 0 & J_L(\boldsymbol{\theta}) \end{bmatrix}$
- **Right Jacobian of** $SE(3)$: $\mathcal{J}_R(\boldsymbol{\xi}) = \begin{bmatrix} J_R(\boldsymbol{\theta}) & Q_R(\boldsymbol{\xi}) \\ 0 & J_R(\boldsymbol{\theta}) \end{bmatrix}$
- **Baker-Campbell-Hausdorff Formulas**: the $SE(3)$ Jacobians relate small perturbations in $\mathfrak{se}(3)$ to small perturbations in $SE(3)$:

$$\begin{aligned} \exp\left((\boldsymbol{\xi} + \delta\boldsymbol{\xi})^{\wedge}\right) &\approx \exp(\hat{\boldsymbol{\xi}}) \exp\left((\mathcal{J}_R(\boldsymbol{\xi})\delta\boldsymbol{\xi})^{\wedge}\right) \\ &\approx \exp\left((\mathcal{J}_L(\boldsymbol{\xi})\delta\boldsymbol{\xi})^{\wedge}\right) \exp(\hat{\boldsymbol{\xi}}) \end{aligned}$$

$$\log(\exp(\hat{\boldsymbol{\xi}}_1)\exp(\hat{\boldsymbol{\xi}}_2))^{\vee} \approx \begin{cases} \mathcal{J}_L(\boldsymbol{\xi}_2)^{-1}\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_1 \text{ is small} \\ \boldsymbol{\xi}_1 + \mathcal{J}_R(\boldsymbol{\xi}_1)^{-1}\boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_2 \text{ is small} \end{cases}$$

Closed-forms of the $SE(3)$ Jacobians

$$\mathcal{J}_L(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\boldsymbol{\xi}^{\curlywedge})^n = \begin{bmatrix} J_L(\boldsymbol{\theta}) & Q_L(\boldsymbol{\xi}) \\ 0 & J_L(\boldsymbol{\theta}) \end{bmatrix}$$

$$= I + \left(\frac{4 - \|\boldsymbol{\theta}\| \sin \|\boldsymbol{\theta}\| - 4 \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^2} \right) \boldsymbol{\xi}^{\curlywedge} + \left(\frac{4\|\boldsymbol{\theta}\| - 5 \sin \|\boldsymbol{\theta}\| + \|\boldsymbol{\theta}\| \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^3} \right) \boldsymbol{\xi}^{\curlywedge 2}$$

$$+ \left(\frac{2 - \|\boldsymbol{\theta}\| \sin \|\boldsymbol{\theta}\| - 2 \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^4} \right) \boldsymbol{\xi}^{\curlywedge 3} + \left(\frac{2\|\boldsymbol{\theta}\| - 3 \sin \|\boldsymbol{\theta}\| + \|\boldsymbol{\theta}\| \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^5} \right) \boldsymbol{\xi}^{\curlywedge 4}$$

$$\approx I + \frac{1}{2} \boldsymbol{\xi}^{\curlywedge}$$

$$\mathcal{J}_L(\boldsymbol{\xi})^{-1} = \begin{bmatrix} J_L(\boldsymbol{\theta})^{-1} & -J_L(\boldsymbol{\theta})^{-1} Q_L(\boldsymbol{\xi}) J_L(\boldsymbol{\theta})^{-1} \\ \mathbf{0} & J_L(\boldsymbol{\theta})^{-1} \end{bmatrix} \approx I - \frac{1}{2} \boldsymbol{\xi}^{\curlywedge}$$

$$Q_L(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(n+m+2)!} \hat{\boldsymbol{\theta}}^n \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}}^m$$

$$= \frac{1}{2} \hat{\boldsymbol{\rho}} + \left(\frac{\|\boldsymbol{\theta}\| - \sin \|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3} \right) \left(\hat{\boldsymbol{\theta}} \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}} \right) + \left(\frac{\|\boldsymbol{\theta}\|^2 + 2 \cos \|\boldsymbol{\theta}\| - 2}{2\|\boldsymbol{\theta}\|^4} \right) \left(\hat{\boldsymbol{\theta}}^2 \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}}^2 - 3 \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}} \right)$$

$$+ \left(\frac{2\|\boldsymbol{\theta}\| - 3 \sin \|\boldsymbol{\theta}\| + \|\boldsymbol{\theta}\| \cos \|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^5} \right) \left(\hat{\boldsymbol{\theta}} \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}}^2 + \hat{\boldsymbol{\theta}}^2 \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\theta}} \right)$$

$$Q_R(\boldsymbol{\xi}) = Q_L(-\boldsymbol{\xi}) = R Q_L(\boldsymbol{\xi}) + (J_L(\boldsymbol{\theta}) \boldsymbol{\rho})^{\wedge} R J_L(\boldsymbol{\theta})$$

Adjoints

- The **adjoint** operator of a Lie group linearly transforms tangent vectors from one tangent space to another
- The adjoint of $T = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in SE(3)$ is: $Ad(T) := \begin{bmatrix} R & \hat{\mathbf{p}}R \\ \mathbf{0} & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$
- $Ad(SE(3)) := \{Ad(T) \in \mathbb{R}^{6\times 6} \mid T \in SE(3)\}$ is a matrix Lie group
- The adjoint of $\hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ \mathbf{0}^\top & 0 \end{bmatrix} \in \mathfrak{se}(3)$ is:

$$ad(\hat{\boldsymbol{\xi}}) := \overset{\curlywedge}{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\rho}} \\ \mathbf{0} & \hat{\boldsymbol{\theta}} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- $ad(\mathfrak{se}(3)) := \{ad(\hat{\boldsymbol{\xi}}) \in \mathbb{R}^{6\times 6} \mid \hat{\boldsymbol{\xi}} \in \mathfrak{se}(3)\}$ is the Lie algebra associated with $Ad(SE(3))$
- The relationship between $\overset{\curlywedge}{\boldsymbol{\xi}}$ and $\mathcal{T} := Ad(T)$ is specified by the exponential map:

$$\mathcal{T} = \exp(\overset{\curlywedge}{\boldsymbol{\xi}}) = I + \overset{\curlywedge}{\boldsymbol{\xi}}\mathcal{J}_L(\boldsymbol{\xi}) \qquad \mathcal{J}_L(\boldsymbol{\xi}) = \mathcal{T}\mathcal{J}_R(\boldsymbol{\xi}) = \mathcal{J}_R(-\boldsymbol{\xi})$$

Pose Lie Groups and Lie Algebras

$$
\begin{array}{cccc}
 & \text{Lie algebra} & & \text{Lie group} \\
4\times 4 & \boldsymbol{\xi}^\wedge \in \mathfrak{se}(3) & \xrightarrow{\exp} & \mathbf{T} \in SE(3) \\
 & \Big\downarrow \text{ad} & & \Big\downarrow \text{Ad} \\
6\times 6 & \boldsymbol{\xi}^\curlywedge \in \text{ad}(\mathfrak{se}(3)) & \xrightarrow{\exp} & \boldsymbol{\mathcal{T}} \in \text{Ad}(SE(3))
\end{array}
$$

$$
\mathcal{T} = Ad\underbrace{\left(\exp(\hat{\xi})\right)}_{T} = \exp\underbrace{\left(ad(\hat{\xi})\right)}_{\overset{\curlywedge}{\xi}} \qquad \xi = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6
$$

$$
= Ad\left(\exp\left(\begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}\right)\right) = \exp\left(ad\left(\begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}\right)\right)
$$

$$
= Ad\left(\begin{bmatrix} \exp(\hat{\boldsymbol{\theta}}) & J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix}\right) = \exp\left(\begin{bmatrix} \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\rho}} \\ \mathbf{0} & \hat{\boldsymbol{\theta}} \end{bmatrix}\right)
$$

$$
= \begin{bmatrix} \exp(\hat{\boldsymbol{\theta}}) & (J_L(\boldsymbol{\theta})\boldsymbol{\rho})^\wedge \exp(\hat{\boldsymbol{\theta}}) \\ \mathbf{0} & \exp(\hat{\boldsymbol{\theta}}) \end{bmatrix}
$$

Rodrigues Formula for the Adjoint of $SE(3)$

- The exponential map is **surjective** but **not injective**, i.e., every element of $Ad(SE(3))$ can be generated from multiple elements of $ad(\mathfrak{se}(3))$

- **Rodrigues Formula**: using $(\overset{\curlywedge}{\boldsymbol{\xi}})^5 + 2\|\boldsymbol{\theta}\|^2(\overset{\curlywedge}{\boldsymbol{\xi}})^3 + \|\boldsymbol{\theta}\|^4\overset{\curlywedge}{\boldsymbol{\xi}} = 0$ we can obtain a direct expression of $\mathcal{T} \in Ad(SE(3))$ in terms of $\boldsymbol{\xi} = \begin{bmatrix}\boldsymbol{\rho} \\ \boldsymbol{\theta}\end{bmatrix} \in \mathbb{R}^6$:

$$
\begin{aligned}
\mathcal{T} = Ad(T) = \exp\left(\overset{\curlywedge}{\boldsymbol{\xi}}\right) &= \begin{bmatrix}\exp(\hat{\boldsymbol{\theta}}) & (J_L(\boldsymbol{\theta})\boldsymbol{\rho})^\wedge \exp(\hat{\boldsymbol{\theta}}) \\ \mathbf{0} & \exp(\hat{\boldsymbol{\theta}})\end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!}(\overset{\curlywedge}{\boldsymbol{\xi}})^n \\
&= I + \left(\frac{3\sin\|\boldsymbol{\theta}\| - \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|}\right)\overset{\curlywedge}{\boldsymbol{\xi}} + \left(\frac{4 - \|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\| - 4\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^2}\right)(\overset{\curlywedge}{\boldsymbol{\xi}})^2 \\
&\quad + \left(\frac{\sin\|\boldsymbol{\theta}\| - \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^3}\right)(\overset{\curlywedge}{\boldsymbol{\xi}})^3 + \left(\frac{2 - \|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\| - 2\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^4}\right)(\overset{\curlywedge}{\boldsymbol{\xi}})^4
\end{aligned}
$$

Distances in $SE(3)$

- Two ways to define differences between $SE(3)$ and $Ad(SE(3))$ elements:

$$\boldsymbol{\xi}_{12} = \log\left(T_1^{-1} T_2\right)^{\vee} = \log\left(\mathcal{T}_1^{-1}\mathcal{T}_2\right)^{\curlyvee}$$

$$\boldsymbol{\xi}_{21} = \log\left(T_2 T_1^{-1}\right)^{\vee} = \log\left(\mathcal{T}_2\mathcal{T}_1^{-1}\right)^{\curlyvee}$$

- Inner product on $\mathfrak{se}(3)$ and $ad(\mathfrak{se}(3))$:

$$\langle \hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2 \rangle = \operatorname{tr}\left(\hat{\boldsymbol{\xi}}_1 \begin{bmatrix} \frac{1}{2} I & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \hat{\boldsymbol{\xi}}_2^\top\right) = \boldsymbol{\xi}_1^\top \boldsymbol{\xi}_2$$

$$\langle \boldsymbol{\xi}_1^{\curlywedge}, \boldsymbol{\xi}_2^{\curlywedge} \rangle = \operatorname{tr}\left(\boldsymbol{\xi}_1^{\curlywedge} \begin{bmatrix} \frac{1}{4} I & \mathbf{0} \\ \mathbf{0} & \frac{1}{2} I \end{bmatrix} \boldsymbol{\xi}_2^{\curlywedge\top}\right) = \boldsymbol{\xi}_1^\top \boldsymbol{\xi}_2$$

- The right and left distances on $SE(3)$ and $Ad(SE(3))$ are:

$$\xi_{12} = \sqrt{\langle \hat{\boldsymbol{\xi}}_{12}, \hat{\boldsymbol{\xi}}_{12} \rangle} = \sqrt{\langle \boldsymbol{\xi}_{12}^{\curlywedge}, \boldsymbol{\xi}_{12}^{\curlywedge} \rangle} = \sqrt{\boldsymbol{\xi}_{12}^\top \boldsymbol{\xi}_{12}} = \|\boldsymbol{\xi}_{12}\|_2$$

$$\xi_{21} = \sqrt{\langle \hat{\boldsymbol{\xi}}_{21}, \hat{\boldsymbol{\xi}}_{21} \rangle} = \sqrt{\langle \boldsymbol{\xi}_{21}^{\curlywedge}, \boldsymbol{\xi}_{21}^{\curlywedge} \rangle} = \sqrt{\boldsymbol{\xi}_{21}^\top \boldsymbol{\xi}_{21}} = \|\boldsymbol{\xi}_{21}\|_2$$

Integration in $SE(3)$

- The distance between a pose $T = \exp(\hat{\boldsymbol{\xi}})$ and a small perturbation $\exp((\boldsymbol{\xi} + \delta\boldsymbol{\xi})^\wedge)$ can be approximated using the BCH formulas:

$$\log\left(\exp(\hat{\boldsymbol{\xi}})^{-1}\exp((\boldsymbol{\xi}+\delta\boldsymbol{\xi})^\wedge)\right)^\vee \approx \mathcal{J}_R(\boldsymbol{\xi})\delta\boldsymbol{\xi}$$
$$\log\left(\exp((\boldsymbol{\xi}+\delta\boldsymbol{\xi})^\wedge)\exp(\hat{\boldsymbol{\xi}})^{-1}\right)^\vee \approx \mathcal{J}_L(\boldsymbol{\xi})\delta\boldsymbol{\xi}$$

- Regardless whether the left or the right distance metric is used, the infinitesimal volume element is:

$$|\det(\mathcal{J}(\boldsymbol{\xi}))| = |\det(J(\boldsymbol{\theta}))|^2 = 4\left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)^2$$

- Integrating functions of poses can then be carried out as follows:

$$\int_{SE(3)} f(T)dT = \int_{\mathbb{R}^3, \|\boldsymbol{\theta}\|<\pi} f\left(\exp(\hat{\boldsymbol{\xi}})\right)|det(\mathcal{J}(\boldsymbol{\xi}))|d\boldsymbol{\xi}$$

Lie Algebra $\mathfrak{se}(3)$ Identities

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\boldsymbol{\theta}} & \boldsymbol{\rho} \\ \mathbf{0}^{\top} & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \qquad \overset{\curlywedge}{\boldsymbol{\xi}} = ad(\hat{\boldsymbol{\xi}}) = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}^{\curlywedge} = \begin{bmatrix} \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\rho}} \\ \mathbf{0} & \hat{\boldsymbol{\theta}} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$\overset{\curlywedge}{\boldsymbol{\zeta}}\boldsymbol{\xi} = -\overset{\curlywedge}{\boldsymbol{\xi}}\boldsymbol{\zeta} \qquad \boldsymbol{\zeta} \in \mathbb{R}^{6}$$

$$\overset{\curlywedge}{\boldsymbol{\xi}}\boldsymbol{\xi} = 0$$

$$\hat{\boldsymbol{\xi}}^{4} + \left(\mathbf{s}^{\top}\mathbf{s}\right)\hat{\boldsymbol{\xi}}^{2} = 0 \qquad \mathbf{s} \in \mathbb{R}^{3}$$

$$\left(\overset{\curlywedge}{\boldsymbol{\xi}}\right)^{5} + 2\left(\mathbf{s}^{\top}\mathbf{s}\right)\left(\overset{\curlywedge}{\boldsymbol{\xi}}\right)^{3} + \left(\mathbf{s}^{\top}\mathbf{s}\right)^{2}\overset{\curlywedge}{\boldsymbol{\xi}} = 0$$

$$\mathbf{m}^{\odot} := \begin{bmatrix} \mathbf{s} \\ \lambda \end{bmatrix}^{\odot} = \begin{bmatrix} \lambda I & -\hat{\mathbf{s}} \\ \mathbf{0}^{\top} & \mathbf{0}^{\top} \end{bmatrix} \in \mathbb{R}^{4\times 6} \qquad \mathbf{m}^{\circledcirc} := \begin{bmatrix} \mathbf{s} \\ \lambda \end{bmatrix}^{\circledcirc} = \begin{bmatrix} \mathbf{0} & \mathbf{s} \\ -\hat{\mathbf{s}} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{6\times 4}$$

$$\hat{\boldsymbol{\xi}}\mathbf{m} = \mathbf{m}^{\odot}\boldsymbol{\xi} \qquad \mathbf{m}^{\top}\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}^{\top}\mathbf{m}^{\circledcirc}$$

Lie Group $SE(3)$ Identities

$$
\begin{aligned}
T &= \exp\left(\hat{\boldsymbol{\xi}}\right) = \begin{bmatrix} \exp\left(\hat{\boldsymbol{\theta}}\right) & J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\hat{\boldsymbol{\xi}}^n = I + \hat{\boldsymbol{\xi}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\xi}}^2 + \left(\frac{\|\boldsymbol{\theta}\| - \sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^3}\right)\hat{\boldsymbol{\xi}}^3 \approx I + \hat{\boldsymbol{\xi}} \\
T^{-1} &= \exp\left(-\hat{\boldsymbol{\xi}}\right) = \begin{bmatrix} \exp\left(-\hat{\boldsymbol{\theta}}\right) & -\exp\left(-\hat{\boldsymbol{\theta}}\right) J_L(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\hat{\boldsymbol{\xi}}\right)^n \approx I - \hat{\boldsymbol{\xi}} \\
\det(T) &= 1 \\
\operatorname{tr}(T) &= 2\cos\|\boldsymbol{\theta}\| + 2 \\
T\hat{\boldsymbol{\xi}} &= \hat{\boldsymbol{\xi}}\,T
\end{aligned}
$$

Lie Group $Ad(SE(3))$ Identities

$$\mathcal{T} = Ad(T) = \exp\left(\overset{\curlywedge}{\boldsymbol{\xi}}\right) = \begin{bmatrix} \exp\left(\hat{\boldsymbol{\theta}}\right) & (J_L(\boldsymbol{\theta})\boldsymbol{\rho})^\wedge \exp\left(\hat{\boldsymbol{\theta}}\right) \\ \mathbf{0} & \exp\left(\hat{\boldsymbol{\theta}}\right) \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \overset{\curlywedge}{\boldsymbol{\xi}}^n = I + \left(\frac{3\sin\|\boldsymbol{\theta}\| - \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|}\right)\overset{\curlywedge}{\xi} + \left(\frac{4 - \|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\| - 4\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^2}\right)(\overset{\curlywedge}{\xi})^2$$

$$+ \left(\frac{\sin\|\boldsymbol{\theta}\| - \|\boldsymbol{\theta}\|\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^3}\right)(\overset{\curlywedge}{\xi})^3 + \left(\frac{2 - \|\boldsymbol{\theta}\|\sin\|\boldsymbol{\theta}\| - 2\cos\|\boldsymbol{\theta}\|}{2\|\boldsymbol{\theta}\|^4}\right)(\overset{\curlywedge}{\xi})^4 \approx I + \overset{\curlywedge}{\boldsymbol{\xi}}$$

$$\mathcal{T}^{-1} = \exp\left(-\overset{\curlywedge}{\boldsymbol{\xi}}\right) = \begin{bmatrix} \exp\left(-\hat{\boldsymbol{\theta}}\right) & -\exp\left(-\hat{\boldsymbol{\theta}}\right)(J_L(\boldsymbol{\theta})\boldsymbol{\rho})^\wedge \\ \mathbf{0} & \exp\left(-\hat{\boldsymbol{\theta}}\right) \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\overset{\curlywedge}{\boldsymbol{\xi}}\right)^n \approx I - \overset{\curlywedge}{\boldsymbol{\xi}}$$

$$\mathcal{T}\boldsymbol{\xi} = \boldsymbol{\xi} \qquad \mathcal{T}\overset{\curlywedge}{\boldsymbol{\xi}} = \overset{\curlywedge}{\boldsymbol{\xi}}\mathcal{T}$$

$$(\mathcal{T}\boldsymbol{\zeta})^\wedge = T\hat{\boldsymbol{\zeta}}T^{-1} \qquad (\overset{\curlywedge}{\mathcal{T}\boldsymbol{\zeta}}) = \mathcal{T}\overset{\curlywedge}{\boldsymbol{\zeta}}\mathcal{T}^{-1} \qquad \boldsymbol{\zeta} \in \mathbb{R}^6$$

$$\exp\left((\mathcal{T}\boldsymbol{\zeta})^\wedge\right) = T\exp\left(\hat{\boldsymbol{\zeta}}\right)T^{-1} \qquad \exp\left((\overset{\curlywedge}{\mathcal{T}\boldsymbol{\zeta}})\right) = \mathcal{T}\exp\left(\overset{\curlywedge}{\boldsymbol{\zeta}}\right)\mathcal{T}^{-1}$$

$$(T\mathbf{m})^\odot = T\mathbf{m}^\odot\mathcal{T}^{-1} \qquad \left((T\mathbf{m})^\odot\right)^T(T\mathbf{m})^\odot = \mathcal{T}^{-T}\left(\mathbf{m}^\odot\right)^T\mathbf{m}^\odot\mathcal{T}^{-1}$$

$SO(3)$ and $SE(3)$ Kinematics

Rotation Kinematics

- The trajectory $R(t)$ of a continuous rotation motion should satisfy:

$$R^\top(t)R(t) = I \quad \Rightarrow \quad \dot{R}^\top(t)R(t) + R^\top(t)\dot{R}(t) = 0.$$

- The matrix $R^\top(t)\dot{R}(t)$ is **skew-symmetric** and there must exist some vector-valued function $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ such that:

$$R^\top(t)\dot{R}(t) = \hat{\boldsymbol{\omega}}(t) \quad \Rightarrow \quad \boxed{\dot{R}(t) = R(t)\hat{\boldsymbol{\omega}}(t)}$$

- A skew-symmetric matrix gives a first order approximation to a rotation matrix:

$$R(t+dt) \approx R(t) + R(t)\hat{\boldsymbol{\omega}}(t)dt$$

Rotation Kinematics

- Let $R \in SO(3)$ be the orientation of a rigid body rotating with angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ with respect to the world frame.

- **Rotation kinematic equations of motion**:

$$\boxed{\dot{R} = R\hat{\boldsymbol{\omega}}_B = \hat{\boldsymbol{\omega}}_W R}$$

  where $\boldsymbol{\omega}_B$ and $\boldsymbol{\omega}_W := R\boldsymbol{\omega}_B$ are the body-frame and world-frame coordinates of $\boldsymbol{\omega}$, respectively.

- Assuming $\boldsymbol{\omega}$ is constant over a short period τ:

$$R(t + \tau) = R(t)\exp(\tau\hat{\boldsymbol{\omega}}_B) = \exp(\tau\hat{\boldsymbol{\omega}}_W)R(t)$$

- **Discrete Rotation Kinematics**: let $R_k := R(t_k)$, $\tau_k := t_{k+1} - t_k$, and $\boldsymbol{\omega}_k := \boldsymbol{\omega}_B(t_k)$ leading to:

$$\boxed{R_{k+1} = R_k \exp(\tau_k\hat{\boldsymbol{\omega}}_k)}$$

Pose Kinematics

- **Angular velocity**: $R^\top(t)\dot{R}(t) = I \quad \Rightarrow \quad R^\top(t)\dot{R}(t) = \hat{\boldsymbol{\omega}}(t) \in \mathfrak{so}(3)$
- **Twist**: similarly for $T(t) \in SE(3)$ consider:

$$T^{-1}(t)\dot{T}(t) = \begin{bmatrix} R^\top(t)\dot{R}(t) & R^\top(t)\dot{\mathbf{p}}(t) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\omega}}(t) & \mathbf{v}(t) \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$$

  where $\hat{\boldsymbol{\omega}}(t) := R^\top(t)\dot{R}(t)$ and $\mathbf{v}(t) := R^\top(t)\dot{\mathbf{p}}(t)$ are the **body-frame** angular and linear velocities of the body

- **Generalized velocity**: $\boldsymbol{\zeta}(t) := \begin{bmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}(t) \end{bmatrix} \in \mathbb{R}^6$
- $\boldsymbol{\zeta}(t)$ is the velocity of the body frame moving relative to the world frame as viewed in the **body frame**
- **Continuous-time Pose Kinematics**: $\boxed{\dot{T}(t) = T(t)\hat{\boldsymbol{\zeta}}(t)}$
- **Discrete-time Pose Kinematics**: $\boxed{T_{k+1} = T_k \exp\left(\tau_k \hat{\boldsymbol{\zeta}}_k\right)}$

Pose Kinematics

- Consider a moving body frame $\{B\}$ with pose $T(t) \in SE(3)$
- Let $\mathbf{s}_B \in \mathbb{R}^3$ be a point in the body frame with homogeneous coordinates $\underline{\mathbf{s}}_B$
- The velocity of $\mathbf{s}_B$ with respect to the world frame $\{W\}$ can be determined as follows:

$$
\begin{aligned}
\underline{\mathbf{s}}_W(t) &= T(t)\underline{\mathbf{s}}_B \\
\dot{\underline{\mathbf{s}}}_W(t) &= \dot{T}(t)\underline{\mathbf{s}}_B = \dot{T}(t)T(t)^{-1}\underline{\mathbf{s}}_W(t) \\
&= T(t)\hat{\zeta}(t)T(t)^{-1}\underline{\mathbf{s}}_W(t) \\
&= \begin{bmatrix} R(t)\hat{\boldsymbol{\omega}}(t)R(t)^\top & R(t)\mathbf{v}(t) - R(t)\hat{\boldsymbol{\omega}}(t)R(t)^\top\mathbf{p}(t) \\ \mathbf{0}^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbf{s}_W(t) \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} (R(t)\boldsymbol{\omega}(t))^\wedge (\mathbf{s}_W(t) - \mathbf{p}(t)) + R(t)\mathbf{v}(t) \\ 1 \end{bmatrix}
\end{aligned}
$$