ECE276A: Sensing & Estimation in Robotics
Lecture 15: Localization and Odometry from Point Features

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Localization and Odometry from Point Features

- **Observation model**: relates a feature observation $\mathbf{z}_i$ obtained from robot position $\mathbf{p}$ and orientation θ or R with the position $\mathbf{m}_i$ of the point landmark that generated the feature $\mathbf{z}_i$:
  - **Position Sensor**: $\mathbf{z}_i = R^\top(\mathbf{m}_i - \mathbf{p})$
  - **Range Sensor**: $\mathbf{z}_i = \|\mathbf{m}_i - \mathbf{p}\|_2$
  - **Bearing Sensor**: $\mathbf{z}_i = \arctan\left(\frac{m_{i,y} - p_y}{m_{i,x} - p_x}\right) - \theta$
  - **Camera Sensor**: $\mathbf{z}_i = K\pi\left(R^\top(\mathbf{m}_i - \mathbf{p})\right)$
- **Localization Problem**: Given landmark positions $\{\mathbf{m}_i\}_i$ and measurements $\{\mathbf{z}_i\}_i$ at one time instance, determine the global robot position $\mathbf{p}$ and orientation θ or R
- **Odometry Problem**: Given measurements $\mathbf{z}_{t,i}$, $\mathbf{z}_{t+1,i}$ at two time instances, determine the relative position ${}_t\mathbf{p}_{t+1}$ and orientation ${}_t\theta_{t+1}$ or ${}_tR_{t+1}$ between the two robot frames at time t and $t+1$

2-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ (world frame) and **relative position** measurements (body frame):

$$\mathbf{z}_i = R^\top(\theta)(\mathbf{m}_i - \mathbf{p}) \in \mathbb{R}^2, \quad i = 1, 2$$

- Let $\delta\mathbf{z} := \mathbf{z}_1 - \mathbf{z}_2$ and $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so that:

$$\mathbf{m}_1 - \mathbf{m}_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (\mathbf{z}_1 - \mathbf{z}_2) = \begin{bmatrix} \delta\mathbf{z} & J\delta\mathbf{z} \end{bmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

- As long as $\det \begin{bmatrix} \delta\mathbf{z} & J\delta\mathbf{z} \end{bmatrix} = \|\delta\mathbf{z}\|_2^2 = \|\mathbf{m}_1 - \mathbf{m}_2\|_2^2 \neq 0$, we can compute:

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{\|\delta\mathbf{z}\|_2^2} \begin{bmatrix} \delta z_x & \delta z_y \\ -\delta z_y & \delta z_x \end{bmatrix} (\mathbf{m}_1 - \mathbf{m}_2) \qquad \boxed{\theta = \mathbf{atan2}(\sin\theta, \cos\theta)}$$

- Given the orientation θ, we can then obtain the robot position:

$$\boxed{\mathbf{p} = \frac{1}{2}\left((\mathbf{m}_1 + \mathbf{m}_2) - R(\theta)(\mathbf{z}_1 + \mathbf{z}_2)\right)}$$

3-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 \in \mathbb{R}^3$ (world frame) and **relative position** measurements (body frame):

$$\mathbf{z}_i = R^\top(\mathbf{m}_i - \mathbf{p}) \in \mathbb{R}^3, \quad i = 1, 2, 3$$

- Let $\mathbf{m}_{ij} := \mathbf{m}_i - \mathbf{m}_j$ and $\mathbf{z}_{ij} = \mathbf{z}_i - \mathbf{z}_j$ and compute:

$$\mathbf{m}_{12} \times \mathbf{m}_{13} = (R\mathbf{z}_{12}) \times (R\mathbf{z}_{13}) = R(\mathbf{z}_{12} \times \mathbf{z}_{13})$$

- The vector $\mathbf{m}_{12} \times \mathbf{m}_{13}$ provides orthogonal information to $\mathbf{m}_1$ and $\mathbf{m}_2$ and can be used to estimate the orientation R **as long as the three features are not all on the same line**:

$$\begin{bmatrix}\mathbf{m}_{12} & \mathbf{m}_{13} & \mathbf{m}_{12} \times \mathbf{m}_{13}\end{bmatrix} = R \begin{bmatrix}\mathbf{z}_{12} & \mathbf{z}_{13} & \mathbf{z}_{12} \times \mathbf{z}_{13}\end{bmatrix}$$

$$\boxed{R = \begin{bmatrix}\mathbf{m}_{12} & \mathbf{m}_{13} & \mathbf{m}_{12} \times \mathbf{m}_{13}\end{bmatrix} \begin{bmatrix}\mathbf{z}_{12} & \mathbf{z}_{13} & \mathbf{z}_{12} \times \mathbf{z}_{13}\end{bmatrix}^{-1}}$$

- Given the orientation R, we can then obtain the robot position:

$$\boxed{\mathbf{p} = \tfrac{1}{3} \textstyle\sum_{i=1}^{3} (\mathbf{m}_i - R\mathbf{z}_i)}$$

3-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: n landmark positions $\mathbf{m}_i \in \mathbb{R}^3$ (world frame) and **relative position** measurements (body frame):

$$\mathbf{z}_i = R^\top(\mathbf{m}_i - \mathbf{p}) \in \mathbb{R}^3, \quad i = 1, \ldots, n$$

- Define the landmark centroids in the world and body frames:

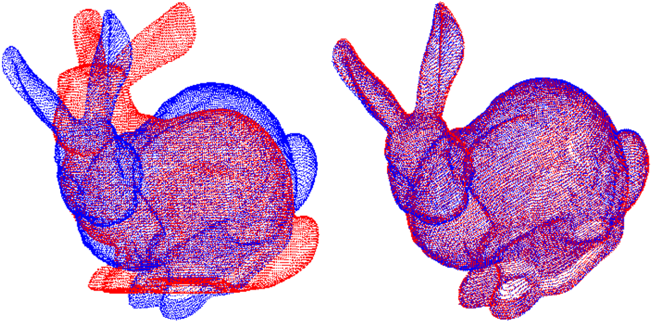
$$\bar{\mathbf{m}} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{m}_i \qquad \bar{\mathbf{z}} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \qquad \boxed{\bar{\mathbf{m}} = \mathbf{p} + R\bar{\mathbf{z}}}$$

- Let $\delta\mathbf{m}_i := \mathbf{m}_i - \bar{\mathbf{m}}$ and $\delta\mathbf{z}_i := \mathbf{z}_i - \bar{\mathbf{z}}$ so that $\delta\mathbf{m}_i = R\delta\mathbf{z}_i$ for $i = 1, \ldots, n$
- Estimate the orientation via least-squares:

$$\min_R \sum_{i=1}^{n} \|\delta\mathbf{m}_i - R\delta\mathbf{z}_i\|_2^2 = \min_R \sum_{i=1}^{n} \delta\mathbf{m}_i^\top\delta\mathbf{m}_i - 2\delta\mathbf{m}_i^\top R\delta\mathbf{z}_i - \delta\mathbf{z}_i^\top \underbrace{R^\top R}_{I_{3\times 3}}\delta\mathbf{z}_i$$

Point Cloud Registration

- 3-D localization from relative position measurements is also known as the **point cloud registration** problem
- Given two sets $\{\mathbf{m}_i\}_{i=1}^n$ and $\{\mathbf{z}_j\}_{j=1}^m$ of 3-D points, find the tranformation $\mathbf{p} \in \mathbb{R}^3$, $R \in SO(3)$ that aligns them
- The data association $\Delta : \{1, \ldots, n\} \mapsto \{0, 1, \ldots, m\}$ that specifies which point $j = \Delta(i)$ from the second set corresponds to any point i from the first set might or might not be available



Known Data Association: Kabsch Algorithm

- Find the transformation $\mathbf{p}$, R between sets $\{\mathbf{m}_i\}$ and $\{\mathbf{z}_i\}$ of associated 3-D points

- As before, define $\bar{\mathbf{m}}$ and $\bar{\mathbf{z}}$ as the point centroids and $\{\delta\mathbf{m}_i\}$ and $\{\delta\mathbf{z}_i\}$ as the centered points

- Given the rotation R, the translation is: $\boxed{\mathbf{p} = \bar{\mathbf{m}} - R\bar{\mathbf{z}}}$

- Need to solve an optimization problem in $SO(3)$ to determine R:

$$\begin{aligned} \max_{R} \; & \sum_{i=1}^{n} \delta\mathbf{m}_i^\top R \delta\mathbf{z}_i = \operatorname{tr}\left(Q^\top R\right) \\ \text{s.t.} \; & R^\top R = I, \; \det(R) = 1 \end{aligned} \qquad \text{where } Q^\top := \sum_{i=1}^{n} \delta\mathbf{z}_i \delta\mathbf{m}_i^\top$$

- This problem can be solved via the **Kabsch algorithm**

Known Data Association: Kabsch Algorithm

- **Wahba's problem**: linear optimization in $SO(3)$:

$$\max_{R \in SO(3)} \operatorname{tr}\left(Q^\top R\right)$$

- **SVD**: let $Q = U \Sigma V^\top$ be the singular value decomposition of Q
- The singular vectors U, V and singular values Σ satisfy:

$$\Sigma_{ii} \geq 0 \qquad U^\top U = I \quad \det(U) = \pm 1 \quad V^\top V = I \quad \det V = \pm 1$$

- Let $W := U^\top R V$, which is orthogonal: $W^\top W = I$ and $\det(W) = \pm 1$
- $\operatorname{tr}(Q^\top R) = \operatorname{tr}(\Sigma U^\top R V) = \operatorname{tr}(\Sigma W) = \sum_i \Sigma_{ii} W_{ii}$
- Since $\Sigma_{ii} \geq 0$ and $\det(W) = \pm 1$, the objective is maximized for:

$$W = I \quad \Rightarrow \quad U^\top R V = I \quad \underset{\text{reflection}}{\overset{\text{avoids}}{\Rightarrow}} \quad R = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(UV^\top) \end{bmatrix} V^\top$$

Unknown Data Association: Iterative Closest Point (ICP)

- Find the transformation $\mathbf{p}$, R between sets $\{\mathbf{m}_i\}_{i=1}^n$ and $\{\mathbf{z}_j\}_{j=1}^m$ of 3-D points with **unknown** data association $\Delta : \{1, \ldots, n\} \mapsto \{0, 1, \ldots, m\}$

- The ICP algorithm iterates between finding associations based on **closest points** and applying the **Kabsch algorithm** to determine $\mathbf{p}$, R

- Initialize with $\mathbf{p}_0$, R_0 (**sensitive to initial guess**) and iterate
  1. Given $\mathbf{p}_k$, R_k, find correspondences $i \leftrightarrow j$ based on **closest points**:

$$\Delta(i) = \underset{j \in \{1,\ldots,m\}}{\arg\min} \|\mathbf{m}_i - (R_k \mathbf{z}_j + \mathbf{p}_k)\|_2^2, \qquad \forall i \in \{1, \ldots, n\}$$

  2. Given correspondences $j = \Delta(i)$, find $\mathbf{p}_{k+1}$, R_{k+1} via **Kabsch**

Unknown Data Association: Probabilistic ICP

- A main challenge is determining the unknown data association. Many variations and extensions for determining correspondences in ICP exist:
  - data association via point-to-plane distance (Chen & Medioni, 1991)
  - probabilistic data association (EM-ICP, Granger & Pennec, 2002)

- Place a probability density function π (e.g., Gaussian) at each $\mathbf{m}_i$ to define a mixture distribution for the data:

$$p(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i \pi(\mathbf{x}; \mathbf{m}_i, \sigma_i^2 I) \qquad \alpha_i \geq 0 \qquad \sum_{i=1}^{n} \alpha_i = 1$$

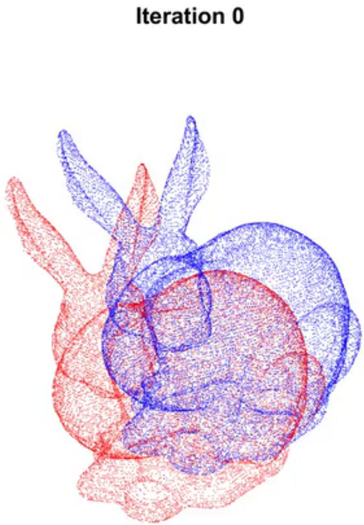
- Find parameters $\mathbf{p}$, R to maximize the likelihood of $\{R\mathbf{z}_j + \mathbf{p}\}_j$:

$$\max_{\mathbf{p},R} \sum_{j=1}^{m} \log \sum_{i=1}^{n} \alpha_i \pi(R\mathbf{z}_j + \mathbf{p}; \mathbf{m}_i, \sigma_i^2 I)$$

- Use **EM** to determine membership probabiliites (E step) and optimize the parameters $\mathbf{p}$, $\mathbf{R}$ (M step). ICP is a special case with $\sigma_i^2 \to 0$

- **Robustness**: use $\exp\left(-\frac{|\mathbf{x}-\mathbf{m}_i|^\beta}{2\sigma_i^2}\right)$ with $\beta \in (0, 2)$ instead of $\exp\left(-\frac{|\mathbf{x}-\mathbf{m}_i|^2}{2\sigma_i^2}\right)$

Iterative Closest Point (ICP)



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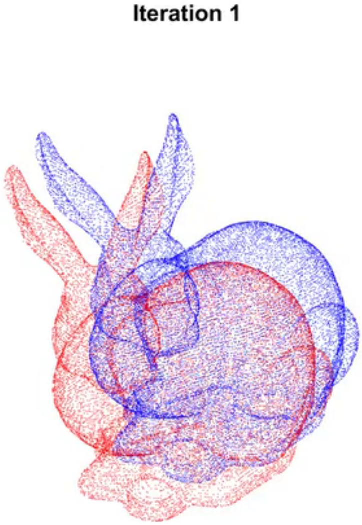

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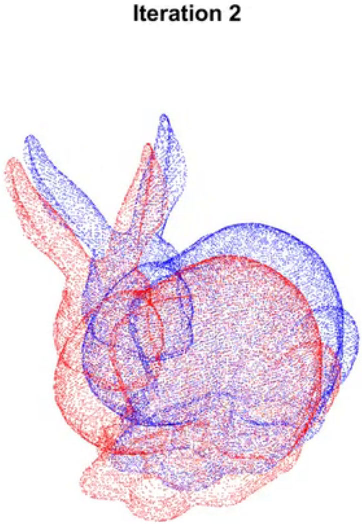

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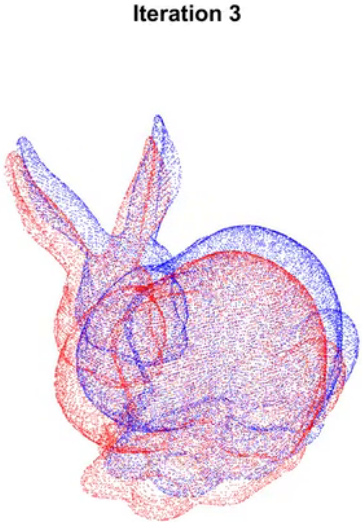

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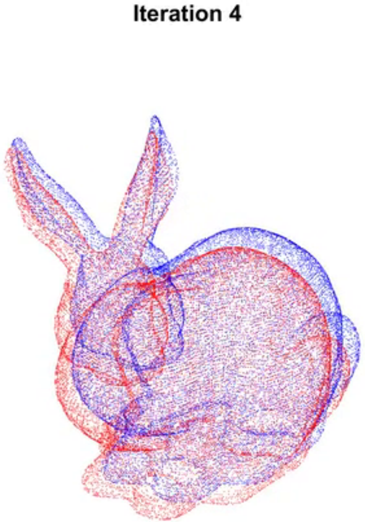

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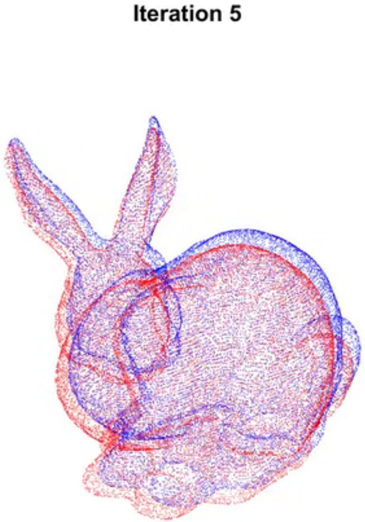

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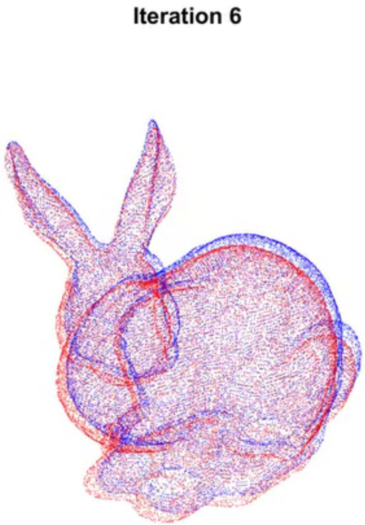

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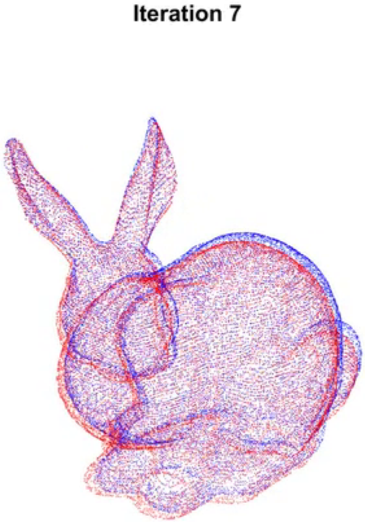

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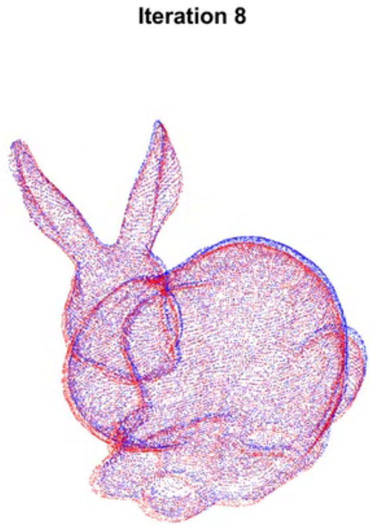

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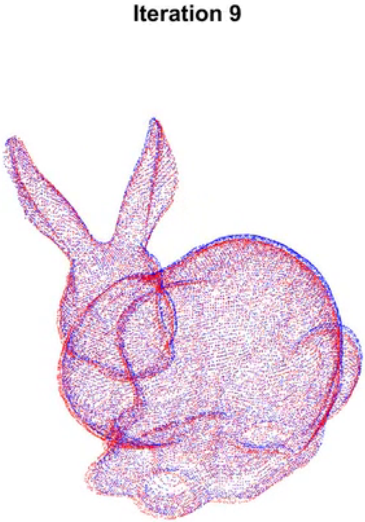

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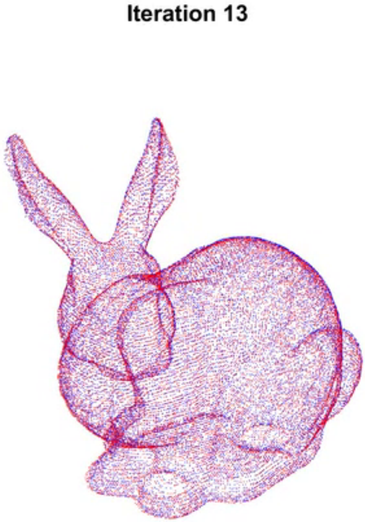

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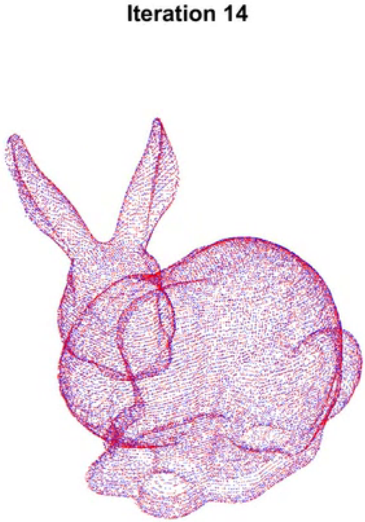

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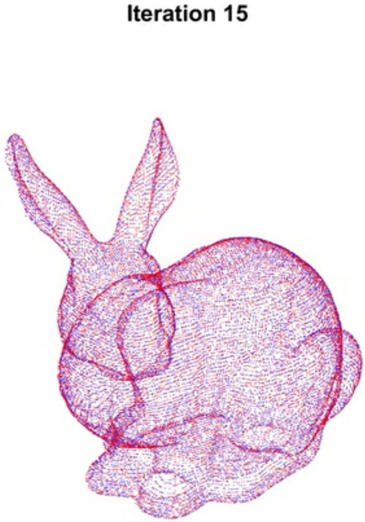

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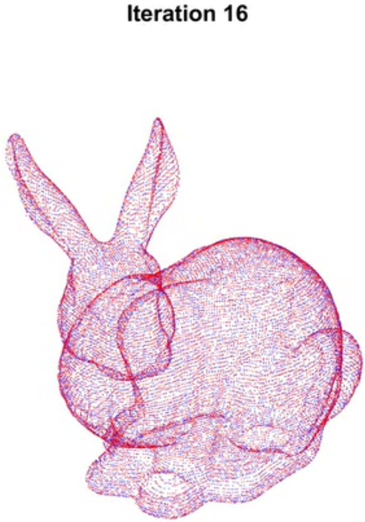


2-D Odometry from Relative Position Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: relative position measurements $\mathbf{z}_{t,1}, \mathbf{z}_{t,2} \in \mathbb{R}^2$ and $\mathbf{z}_{t+1,1}, \mathbf{z}_{t+1,2} \in \mathbb{R}^2$ at consecutive time steps to two **unknown** landmarks
- If we consider the robot frame at time t to be the "world frame", **this problem is the same as 2-D localization from relative position measurements** with $\mathbf{m}_i := \mathbf{z}_{t,i}$, $\mathbf{z}_i := \mathbf{z}_{t+1,i}$, $\mathbf{p} := {}_t\mathbf{p}_{t+1}$, $\theta := {}_t\theta_{t+1}$

3-D Odometry from Relative Position Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^3$ and ${}_tR_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t
- **Given**: relative position measurements $\mathbf{z}_{t,i} \in \mathbb{R}^3$ and $\mathbf{z}_{t+1,i} \in \mathbb{R}^3$ at consecutive time steps to n **unknown** landmarks
- If we consider the robot frame at time t to be the "world frame", **this problem is the same as 3-D localization from relative position measurements** with $\mathbf{m}_i := \mathbf{z}_{t,i}$, $\mathbf{z}_i := \mathbf{z}_{t+1,i}$, $\mathbf{p} := {}_t\mathbf{p}_{t+1}$, $R := {}_tR_{t+1}$

Summary: Rel. Position Measurements $\mathbf{z}_i = R^\top(\mathbf{m}_i - \mathbf{p})$

- **Localization**

$\mathbf{m}_1, \mathbf{m}_2, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^2$	$(\mathbf{m}_1 - \mathbf{m}_2) = R(\theta)(\mathbf{z}_1 - \mathbf{z}_2)$ $\mathbf{p} = \frac{1}{2}\sum_{i=1}^{2}(\mathbf{m}_i - R\mathbf{z}_i)$
$\mathbf{m}_1, \mathbf{z}_i \in \mathbb{R}^3,\ i = 1,2,3$ $\mathbf{m}_{ij} := \mathbf{m}_i - \mathbf{m}_j,\ \ \mathbf{z}_{ij} := \mathbf{z}_i - \mathbf{z}_j$	$\begin{bmatrix}\mathbf{m}_{12} & \mathbf{m}_{13} & \mathbf{m}_{12}\times\mathbf{m}_{13}\end{bmatrix} = R\begin{bmatrix}\mathbf{z}_{12} & \mathbf{z}_{13} & \mathbf{z}_{12}\times\mathbf{z}_{13}\end{bmatrix}$ $\mathbf{p} = \frac{1}{3}\sum_{i=1}^{3}(\mathbf{m}_i - R\mathbf{z}_i)$
$\mathbf{m}_i, \mathbf{z}_i \in \mathbb{R}^3,\ i = 1,\ldots,n$ $\delta\mathbf{m}_i := \mathbf{m}_i - \frac{1}{n}\sum_{j=1}^{n}\mathbf{m}_j,$ $\delta\mathbf{z}_i := \mathbf{z}_i - \frac{1}{n}\sum_{j=1}^{n}\mathbf{z}_j$	$R = \arg\max_{R\in SO(3)} \sum_{i=1}^{n} \delta\mathbf{m}_i^\top R\delta\mathbf{z}_i$ $\xrightarrow[SVD\left(\sum_{i=1}^{n}\delta\mathbf{m}_i\delta\mathbf{z}_i^\top\right)=U\Sigma V^\top]{\text{Kabsch algorithm}} U\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \det(UV^\top)\end{bmatrix}V^\top$ $\mathbf{p} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{m}_i - R\mathbf{z}_i)$

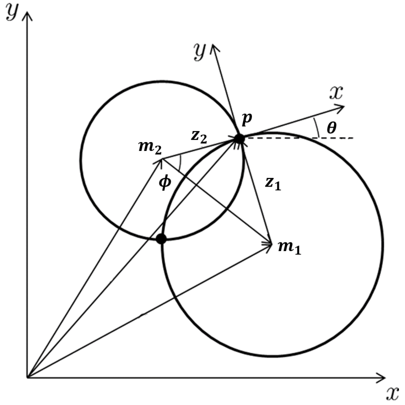
- **Odometry**: same with $\mathbf{m}_i = \mathbf{z}_{t,i}$, $\mathbf{z}_i := \mathbf{z}_{t+1,i}$, $\mathbf{p} := {}_t\mathbf{p}_{t+1}$, $R := {}_tR_{t+1}$

2-D Localization from Range Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ (world frame) and **range** measurements (body frame):

$$z_i = \|\mathbf{m}_i - \mathbf{p}\|_2 \in \mathbb{R}, \quad i = 1, 2$$

- Because all possible positions whose distance to $\mathbf{m}_1$ is z_1 is a circle, the possible robot positions are given by the intersection of two circles



2-D Localization from Range Measurements

- Applying the law of cosines to the triangle gives:

$$z_2^2 = z_1^2 + \|\mathbf{m}_2 - \mathbf{m}_1\|_2^2 - 2z_1\|\mathbf{m}_2 - \mathbf{m}_1\|_2 \cos\phi$$

- Solving for ϕ and then the circle intersection points provides the possible robot positions:

$$\mathbf{p} = \mathbf{m}_2 + z_2 R(\pm\phi)\frac{\mathbf{m}_1 - \mathbf{m}_2}{\|\mathbf{m}_1 - \mathbf{m}_2\|_2}$$

- The orientation of the robot θ is **not identifiable**

2-D Localization from Range Measurements

- **Pose disambiguation**: the robot can make a move with known translation $\mathbf{p}_\Delta$ (measured in the frame at time t) and take two new range measurements

- There are 2 possible robot positions at each time frame for a total of 4 combinations but comparing $\|\mathbf{p}_{t+1} - \mathbf{p}_t\|_2$ to the known $\|\mathbf{p}_\Delta\|_2$ leaves only <u>two valid options</u> (and we cannot distinguish between them)

- To obtain the orientation, we use geometric constraints:

$$\mathbf{p}_{t+1} - \mathbf{p}_t = R(\theta_t)\mathbf{p}_\Delta = \begin{bmatrix} p_{\Delta,x} & -p_{\Delta,y} \\ p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} \begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix}$$

- As long as $\det \begin{bmatrix} p_{\Delta,x} & -p_{\Delta,y} \\ p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} = \|\mathbf{p}_\Delta\|_2^2 \neq 0$, we can compute:

$$\begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix} = \frac{1}{\|\mathbf{p}_\Delta\|_2^2} \begin{bmatrix} p_{\Delta,x} & p_{\Delta,y} \\ -p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} (\mathbf{p}_{t+1} - \mathbf{p}_t)$$

$$\theta_t = \mathbf{atan2}(\sin\theta_t, \cos\theta_t)$$

3-D Localization from Range Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 \in \mathbb{R}^3$ (world frame) and **range** measurements (body frame):

$$z_i = \|\mathbf{m}_i - \mathbf{p}\|_2 \in \mathbb{R}, \quad i = 1, 2, 3$$

- All possible positions whose distance to $\mathbf{m}_1$ is z_1 is a sphere
- The possible robot positions are the intersections of three spheres
- To find the intersection of 3 spheres, we first find the intersection of sphere one and two (a circle) and of sphere two and three (a circle). The intersection of these two circles gives the possible robot positions.
- **Degenerate case**: all landmarks are on the same line – the intersection of the spheres is a circle with infinitely many possible robot positions

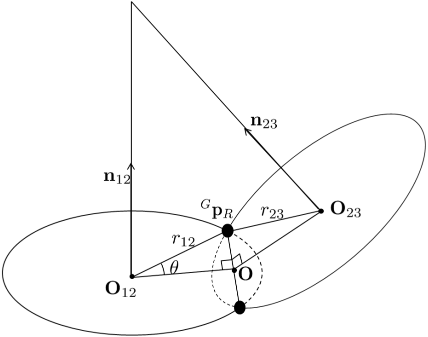
3-D Localization from Range Measurements

- **Intersecting circle of spheres with radii z_1 and z_2**: center $\mathbf{o}_{12}$, radius r_{12}, normal vector $\mathbf{n}_{12}$ (perpendicular to the circle plane)
- Law of Cosines: $z_2^2 = z_1^2 + \|\mathbf{m}_2 - \mathbf{m}_1\|_2^2 - 2z_1\|\mathbf{m}_2 - \mathbf{m}_1\|_2 \cos\theta_{12}$
- Geometric relationships:

$$\mathbf{o}_{12} = \mathbf{m}_1 + z_1 \cos\theta_{12}\mathbf{n}_{12}$$

$$r_{12} = z_1|\sin(\theta_{12})|$$

$$\mathbf{n}_{12} = \frac{\mathbf{m}_2 - \mathbf{m}_1}{\|\mathbf{m}_2 - \mathbf{m}_1\|_2}$$



- **Intersecting circle of spheres with radii z_2 and z_3**: center $\mathbf{o}_{23}$, radius r_{23}, normal vector $\mathbf{n}_{23}$ (perpendicular to the circle plane):

$$\mathbf{o}_{23} = \mathbf{m}_2 + z_2 \cos\theta_{23}\mathbf{n}_{23} \qquad r_{23} = z_2|\sin(\theta_{23})| \qquad \mathbf{n}_{23} = \frac{\mathbf{m}_3 - \mathbf{m}_2}{\|\mathbf{m}_3 - \mathbf{m}_2\|_2}$$

3-D Localization from Range Measurements

- The intersecting points of the two circles can be obtained from the geometric relationships:

$$\begin{aligned} \mathbf{n}_{12}^\top(\mathbf{o}_{12} - \mathbf{o}) &= 0 \\ \mathbf{n}_{23}^\top(\mathbf{o}_{23} - \mathbf{o}) &= 0 \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top(\mathbf{o}_{12} - \mathbf{o}) &= 0 \end{aligned} \qquad \begin{bmatrix} \mathbf{n}_{12}^\top \\ \mathbf{n}_{23}^\top \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \end{bmatrix} \mathbf{o} = \begin{bmatrix} \mathbf{n}_{12}^\top \mathbf{o}_{12} \\ \mathbf{n}_{23}^\top \mathbf{o}_{23} \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \mathbf{o}_{12} \end{bmatrix}$$

- As long as the three landmarks are not on the same line, we can uniquely solve for **o**:

$$\det \begin{bmatrix} \mathbf{n}_{12}^\top \\ \mathbf{n}_{23}^\top \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \end{bmatrix} \neq 0 \qquad \Leftrightarrow \qquad \mathbf{n}_{12} \text{ and } \mathbf{n}_{23} \text{ not colinear}$$

- The two possible robot positions are:

$$\mathbf{p} = \mathbf{o}_{12} + r_{12} R(\mathbf{n}_{12}, \pm\theta) \frac{\mathbf{o} - \mathbf{o}_{12}}{\|\mathbf{o} - \mathbf{o}_{12}\|_2} \qquad \cos\theta = \frac{\|\mathbf{o} - \mathbf{o}_{12}\|_2}{r_{12}}$$

- As in the 2-D case, the robot orientation R is **not identifiable**

3-D Localization from Range Measurements

- **Pose disambiguation**: the robot can make a move with known translation $\mathbf{p}_\Delta \in \mathbb{R}^3$ and rotation $R_\Delta \in SO(3)$ and take three new range measurements

- As in the 2-D case, after eliminating the impossible robot positions, we should be left with only two options for $\mathbf{p}_t$ and $\mathbf{p}_{t+1}$

- Given $\mathbf{p}_t$, $\mathbf{p}_{t+1}$, $\mathbf{p}_\Delta$, and R_Δ, we can now obtain R_t

$$\mathbf{p}_{t+1} = \mathbf{p}_t + R_t \mathbf{p}_\Delta$$

- This is not sufficient because the rotation about $\mathbf{p}_\Delta$ is not identifiable

- The robot needs to **move a second time** to a third pose $\mathbf{p}_{t+2}, R_{t+2}$ with known translation $\mathbf{p}_{\Delta,2} \in \mathbb{R}^3$ and take three more range measurements to the three landmarks:

$$\mathbf{p}_{t+2} = \mathbf{p}_{t+1} + R_{t+1}\mathbf{p}_{\Delta,2} = \mathbf{p}_{t+1} + R_t R_\Delta \mathbf{p}_{\Delta,2}$$

3-D Localization from Range Measurements

- Putting the previous two equations together:

$$\mathbf{p}_{t+1} - \mathbf{p}_t = R_t \mathbf{p}_\Delta$$
$$\mathbf{p}_{t+2} - \mathbf{p}_{t+1} = R_t R_\Delta \mathbf{p}_{\Delta,2}$$

- Taking a cross product between the two:

$$(\mathbf{p}_{t+1} - \mathbf{p}_t) \times (\mathbf{p}_{t+2} - \mathbf{p}_{t+1}) = R_t(\mathbf{p}_\Delta \times R_\Delta \mathbf{p}_{\Delta,2})$$

- As long as $U := [\mathbf{p}_\Delta,\ R_\Delta \mathbf{p}_{\Delta,2},\ \mathbf{p}_\Delta \times R_\Delta \mathbf{p}_{\Delta,2})]$ is nonsingular, i.e., $\mathbf{p}_\Delta$ and $R_\Delta \mathbf{p}_{\Delta,2}$ are not co-linear or equivalently **the three robot positions are not on the same line**, we can determine the robot orientation:

$$\boxed{R_t = [(\mathbf{p}_{t+1} - \mathbf{p}_t),\ (\mathbf{p}_{t+2} - \mathbf{p}_{t+1}),\ (\mathbf{p}_{t+1} - \mathbf{p}_t) \times (\mathbf{p}_{t+2} - \mathbf{p}_{t+1})] U^{-1}}$$

2-D Odometry from Range Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: range measurements $z_{t,i} \in \mathbb{R}$ and $z_{t+1,i} \in \mathbb{R}$ at consecutive time steps to n **unknown** landmarks
- Let $\mathbf{m}_{t+1,i}$ be the relative position to the i-th landmark at $t+1$ so that:

$$
\begin{aligned}
z_{t+1,i} &= \|\mathbf{m}_{t+1,i}\|_2 \\
z_{t,i} &= \|{}_t\mathbf{p}_{t+1} + R({}_t\theta_{t+1})\mathbf{m}_{t+1,i}\|_2
\end{aligned}
$$

- Squaring and combining these equations, we get:

$$
[{}_t\mathbf{p}_{t+1}]^\top {}_t\mathbf{p}_{t+1} + 2\mathbf{m}_{t+1,i}^\top R^\top({}_t\theta_{t+1}){}_t\mathbf{p}_{t+1} = z_{t,i}^2 - z_{t+1,i}^2, \quad i = 1, \ldots, n
$$

- We have n equations with $n+3$ unknowns (3 for the relative pose and n for the unknown directions to the landmarks at $t+1$), which is **not solvable**.

3-D Odometry from Range Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^3$ and ${}_tR_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t
- **Given**: range measurements $z_{t,i} \in \mathbb{R}$ and $z_{t+1,i} \in \mathbb{R}$ at consecutive time steps to n **unknown** landmarks
- Following the same derivation as in the 2-D case, we obtain:

$$[{}_t\mathbf{p}_{t+1}]^\top {}_t\mathbf{p}_{t+1} + 2\mathbf{m}_{t+1,i}^\top [{}_tR_{t+1}]^\top {}_t\mathbf{p}_{t+1} = z_{t,i}^2 - z_{t+1,i}^2, \quad i = 1, \ldots, n$$

- We have n equations with $2n+6$ unknowns (6 for the relative pose and $2n$ for the unknown directions to the landmarks at $t+1$), which is **not solvable**.

Summary: Range Measurements $z_i = \|\mathbf{m}_i - \mathbf{p}\|_2$

- **2-D Localization**: given $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ and $z_1, z_2 \in \mathbb{R}$
  1. Law of Cosines: $z_2^2 = z_1^2 + \|\mathbf{m}_2 - \mathbf{m}_1\|_2^2 - 2z_1\|\mathbf{m}_2 - \mathbf{m}_1\|_2 \cos\theta$
  2. Position: $\mathbf{p} = \mathbf{m}_2 + z_2 R(\pm\theta)\frac{\mathbf{m}_1 - \mathbf{m}_2}{\|\mathbf{m}_1 - \mathbf{m}_2\|_2}$
  3. Move with known $\mathbf{p}_\Delta, \theta_\Delta$ (in frame t)
  4. Orientation: $(\mathbf{p}_{t+1} - \mathbf{p}_t) = R(\theta_t)\mathbf{p}_\Delta$
- **3-D Localization**: given $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 \in \mathbb{R}^3$ and $z_1, z_2, z_3 \in \mathbb{R}$
  1. Intersection of 2 circles with centers $\mathbf{o}_{12}, \mathbf{o}_{23}$, radii r_{12}, r_{23}, normals $\mathbf{n}_{12}, \mathbf{n}_{23}$ obtained via Law of Cosines and point $\mathbf{o}$ on intersecting line:
$$\begin{bmatrix} \mathbf{n}_{12}^\top \\ \mathbf{n}_{23}^\top \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \end{bmatrix} \mathbf{o} = \begin{bmatrix} \mathbf{n}_{12}^\top \mathbf{o}_{12} \\ \mathbf{n}_{23}^\top \mathbf{o}_{23} \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \mathbf{o}_{12} \end{bmatrix}$$
  2. Position: $\mathbf{p} = \mathbf{o}_{12} + r_{12} R(\mathbf{n}_{12}, \pm\theta)\frac{\mathbf{o} - \mathbf{o}_{12}}{\|\mathbf{o} - \mathbf{o}_{12}\|_2}$, where $\cos\theta = \frac{\|\mathbf{o} - \mathbf{o}_{12}\|_2}{r_{12}}$
  3. Move twice with known $\mathbf{p}_\Delta, R_\Delta, \mathbf{p}_{\Delta,2}, R_{\Delta,2}$
  4. Orientation: as long as $U := [\mathbf{p}_\Delta,\ R_\Delta \mathbf{p}_{\Delta,2},\ \mathbf{p}_\Delta \times R_\Delta \mathbf{p}_{\Delta,2})]$ is nonsingular:
$$R_t = [(\mathbf{p}_{t+1} - \mathbf{p}_t),\ (\mathbf{p}_{t+2} - \mathbf{p}_{t+1}),\ (\mathbf{p}_{t+1} - \mathbf{p}_t) \times (\mathbf{p}_{t+2} - \mathbf{p}_{t+1})]U^{-1}$$
- **Odometry**: not solvable

2-D Localization from Bearing Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ (world frame) and **bearing** measurements (body frame):

$$z_i = \arctan\left(\frac{m_{i,y} - p_y}{m_{i,x} - p_x}\right) - \theta \in \mathbb{R}, \qquad i = 1, 2$$

- The bearing constraints are equivalent to:

$$\frac{\mathbf{m}_i - \mathbf{p}}{\|\mathbf{m}_i - \mathbf{p}\|_2} = \begin{bmatrix}\cos(z_i + \theta)\\ \sin(z_i + \theta)\end{bmatrix} = R(z_i + \theta)\mathbf{e}_1 \quad \Rightarrow \quad R^\top(z_i)(\mathbf{m}_i - \mathbf{p}) = \|\mathbf{m}_i - \mathbf{p}\|_2 \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

- To eliminate θ, the two constraints can be combined via:

$$\begin{aligned} 0 &= \|\mathbf{m}_1 - \mathbf{p}\|_2 \begin{bmatrix}\sin\theta & -\cos\theta\end{bmatrix} \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} \|\mathbf{m}_2 - \mathbf{p}\|_2 \\ &= \|\mathbf{m}_1 - \mathbf{p}\|_2 \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}^\top R\left(\frac{\pi}{2}\right) \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} \|\mathbf{m}_2 - \mathbf{p}\|_2 \end{aligned}$$

2-D Localization from Bearing Measurements

- The previous equation is quadratic in p:

$$(\mathbf{m}_1 - \mathbf{p})^\top R(z_1) R\left(\frac{\pi}{2}\right) R^\top(z_2)(\mathbf{m}_2 - \mathbf{p}) = 0$$

- Let $\eta := z_1 - z_2 + \pi/2$, so that:

$$\mathbf{p}^\top R(\eta)\mathbf{p} - \left(\mathbf{m}_1^\top R(\eta) + \mathbf{m}_2^\top R^\top(\eta)\right)\mathbf{p} + \mathbf{m}_1^\top R(\eta)\mathbf{m}_2 = 0$$

- Use the following to solve the quadratic equation:
  - $\mathbf{p}^\top R(\eta)\mathbf{p} = \cos(\eta)\mathbf{p}^\top\mathbf{p}$
  - $\mathbf{p}^\top\mathbf{p} + 2\mathbf{b}^\top\mathbf{p} + c = (\mathbf{p}+\mathbf{b})^\top(\mathbf{p}+\mathbf{b}) + c - \mathbf{b}^\top\mathbf{b}$

- As long as $\cos(\eta) \neq 0$, i.e., **the robot and the two landmarks are not on the same line**:

$$(\mathbf{p} - \mathbf{p}_0)^\top(\mathbf{p} - \mathbf{p}_0) = \left(\mathbf{p}_0^\top\mathbf{p}_0 - \frac{1}{\cos(\eta)}\mathbf{m}_1^\top R(\eta)\mathbf{m}_2\right) \qquad \mathbf{p}_0 := \frac{1}{2\cos(\eta)}\left(R^\top(\eta)\mathbf{m}_1 + R(\eta)\mathbf{m}_2\right)$$

- The position $\mathbf{p}$ lies on one of the two circles containing $\mathbf{m}_1$ and $\mathbf{m}_2$

2-D Localization from Bearing Measurements

- **Pose disambiguation**: obtain a third bearing measurement:

$$R^\top(z_i)(\mathbf{m}_i - \mathbf{p}) = \|\mathbf{m}_i - \mathbf{p}\|_2 \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \quad i = 1,2,3$$

- Find β and γ such that $R^\top(z_1) + \beta R^\top(z_2) + \gamma R^\top(z_3) = 0$. Then:

$$\underbrace{R^\top(z_1)\mathbf{m}_1 + \beta R^\top(z_2)\mathbf{m}_2 + \gamma R^\top(z_3)\mathbf{m}_3}_{:=\mathbf{u}} - \underbrace{\left[R^\top(z_1) + \beta R^\top(z_2) + \gamma R^\top(z_3)\right]}_{0}\mathbf{p}$$

$$= \left(\|\mathbf{m}_1 - \mathbf{p}\|_2 + \beta\|\mathbf{m}_2 - \mathbf{p}\|_2 + \gamma\|\mathbf{m}_3 - \mathbf{p}\|_2\right)\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

- We can compute θ as $\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} = \frac{\mathbf{u}}{\|\mathbf{u}\|_2}$ and recover $\mathbf{p}$ from:

$$R^\top(z_i)(\mathbf{m}_i - \mathbf{p}) = \|\mathbf{m}_i - \mathbf{p}\|_2 \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \quad i = 1,2,3$$

3-D Localization from Bearing Measurements (P3P)

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $\mathbf{m}_i \in \mathbb{R}^3$ (world frame) and pixel measurements $\underline{\mathbf{z}}_i \in \mathbb{R}^3$ (homogeneous coordinates, body frame) obtained from a (calibrated pinhole) camera:

$$\underline{\mathbf{z}}_i = \frac{1}{\lambda_i} R^\top (\mathbf{m}_i - \mathbf{p}) \qquad\qquad \lambda_i = \|R^\top (\mathbf{m}_i - \mathbf{p})\|_2 = \text{unknown scale}$$

- If we determine λ_i, we can transform the P3P problem to 3-D localization from relative position measurements

Find the depths λ_i via Grunert's method

- Cosines of the angles among the bearing vectors $\underline{\mathbf{z}}_1$, $\underline{\mathbf{z}}_2$, $\underline{\mathbf{z}}_3$:

$$\cos(\gamma_{ij}) = \frac{\underline{\mathbf{z}}_i^\top \underline{\mathbf{z}}_j}{\|\underline{\mathbf{z}}_i\|_2 \|\underline{\mathbf{z}}_j\|_2} \quad \Rightarrow \quad \cos(\gamma_{ij}) = \underline{\mathbf{z}}_i^\top \underline{\mathbf{z}}_j$$

- Let $\epsilon_{ij} := \|\mathbf{m}_i - \mathbf{m}_j\|_2$ be the lengths of the triangle formed in the world frame by $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. Applying the law of cosines gives:

$$\lambda_i^2 + \lambda_j^2 - 2\lambda_i \lambda_j \cos(\gamma_{ij}) = \epsilon_{ij}^2 \qquad \text{for } \lambda_i := \|\mathbf{m}_i - \mathbf{p}\|_2$$

- Let $\lambda_2 = u\lambda_1$ and $\lambda_3 = v\lambda_1$ so that:

$$\begin{aligned}
\lambda_1^2(u^2 + v^2 - 2uv\cos(\gamma_{23})) &= \epsilon_{23}^2 \\
\lambda_1^2(1 + v^2 - 2v\cos(\gamma_{13})) &= \epsilon_{13}^2 \\
\lambda_1^2(u^2 + 1 - 2u\cos(\gamma_{12})) &= \epsilon_{12}^2
\end{aligned}$$

- Equivalently

$$\lambda_1^2 = \frac{\epsilon_{23}^2}{u^2 + v^2 - 2uv\cos(\gamma_{23})} = \frac{\epsilon_{13}^2}{1 + v^2 - 2v\cos(\gamma_{13})} = \frac{\epsilon_{12}^2}{u^2 + 1 - 2u\cos(\gamma_{12})}$$

Find the depths λ_i via Grunert's method

- Cross-multiplying the second fraction, with the first and the third:

$$u^2 + \frac{\epsilon_{13}^2 - \epsilon_{23}^2}{\epsilon_{13}^2} v^2 - 2uv\cos(\gamma_{23}) + \frac{2\epsilon_{23}^2}{\epsilon_{13}^2} v\cos(\gamma_{13}) - \frac{\epsilon_{23}^2}{\epsilon_{13}^2} = 0 \quad (1)$$

$$u^2 - \frac{\epsilon_{12}^2}{\epsilon_{13}^2} v^2 + 2v\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos(\gamma_{13}) - 2u\cos(\gamma_{12}) + \frac{\epsilon_{13}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} = 0 \quad (2)$$

- Substituting (1) into (2):

$$u = \frac{\left(-1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right) v^2 - 2\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13}) v + 1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}}{2(\cos(\gamma_{12}) - v\cos(\gamma_{23}))} \quad (3)$$

- Substituting (3) into (1), we get a fourth-order polynomial in v:

$$a_4 v^4 + a_3 v^3 + a_2 v^2 + a_1 v + a_0 = 0$$

Polynomial Coefficients

$$a_4 = \left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} - 1\right)^2 - 4\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos^2(\gamma_{23})$$

$$a_3 = 4\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\left(1 - \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13}) - \left(1 - \frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{12}) + 2\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos^2(\gamma_{23})\cos(\gamma_{13})\right)$$

$$a_2 = 2\left(\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2 - 1 + 2\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2\cos^2(\gamma_{13}) + 2\left(\frac{\epsilon_{13}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos^2(\gamma_{23}) + 2\left(\frac{\epsilon_{13}^2 - \epsilon_{23}^2}{\epsilon_{13}^2}\right)\cos^2(\gamma_{12}) \right.$$
$$\left. -4\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{13})\cos(\gamma_{12})\right)$$

$$a_1 = 4\left(-\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\left(1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13}) - \left(1 - \frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{12}) + 2\frac{\epsilon_{23}^2}{\epsilon_{13}^2}\cos^2(\gamma_{12})\cos(\gamma_{13})\right)$$

$$a_0 = \left(1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2 - \frac{4\epsilon_{23}^2}{\epsilon_{13}^2}\cos^2(\gamma_{12})$$

- We can obtain up to 4 real solutions for v, which we can substitute in (3) to obtain u.
- We can recover λ_1 from u and v via the fractions relationship
- Having λ_1, $\lambda_2 := u\lambda_1$, and $\lambda_3 := v\lambda_1$ we have converted the P3P problem into 3-D localization from relative position measurements

3-D Localization from Bearing Measurements (PnP)

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: landmark positions $\mathbf{m}_i \in \mathbb{R}^3$ (world frame) and pixel measurements $\underline{\mathbf{z}}_i \in \mathbb{R}^3$ (homogeneous coordinates) obtained from a (calibrated pinhole) camera for $i = 1, \ldots, n$:

$$\underline{\mathbf{z}}_i = \frac{1}{\lambda_i} R^\top (\mathbf{m}_i - \mathbf{p}) \qquad \lambda_i = \|R^\top (\mathbf{m}_i - \mathbf{p})\|_2 = \text{unknown scale}$$

- The PnP can be formulated as a **constrained nonlinear least-squares** minimization:

$$\begin{aligned} \min_{\lambda_i, R, \mathbf{p}} \; & \sum_{i=1}^{n} \|\underline{\mathbf{z}}_i - \frac{1}{\lambda_i} R^\top (\mathbf{m}_i - \mathbf{p})\|_2^2 \\ \text{s.t.} \; & R^\top R = I, \quad \det R = 1, \quad \lambda_i = \|R^\top (\mathbf{m}_i - \mathbf{p})\|_2 \end{aligned}$$

Reformulation into a Polynomial System

- The constraints $\lambda_i \underline{\mathbf{z}}_i = R^\top(\mathbf{m}_i - \mathbf{p})$ can be re-written in matrix form as:

$$\underbrace{\begin{bmatrix} \underline{\mathbf{z}}_1 & & & -I \\ & \ddots & & \vdots \\ & & \underline{\mathbf{z}}_n & -I \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ -R^\top \mathbf{p} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} R^\top & & \\ & \ddots & \\ & & R^\top \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_n \end{bmatrix}}_{\mathbf{d}}$$

  where A and $\mathbf{d}$ are known or measured, $\mathbf{x}$ are the unknowns we wish to eliminate, and W is a block diagonal matrix of the unknown rotation R

- We can express $\mathbf{p}$ and λ_i in terms of the other quantities as follows:

$$\mathbf{x} = (A^\top A)^{-1} A^\top W \mathbf{d} = \begin{bmatrix} U \\ V \end{bmatrix} W \mathbf{d}$$

  where $(A^\top A)^{-1} A^\top$ is partitioned so that the scale parameters are a function of U and the translation $-R^\top \mathbf{p}$ is a function of V.

Reformulation into a Polynomial System

$$\mathbf{x} = (A^\top A)^{-1} A^\top W\mathbf{d} = \begin{bmatrix} U \\ V \end{bmatrix} W\mathbf{d}$$

- Exploiting the sparse structure of A, the matrices U and V can be computed in closed form
- Both λ_i and $-R^\top\mathbf{p}$ are linear functions of the unknown $R^\top$:

$$\lambda_i = \mathbf{u}_i^\top W\mathbf{d} \qquad -R^\top\mathbf{p} = VW\mathbf{d}, \qquad i = 1, \ldots, n$$

  where $\mathbf{u}_i^\top$ is the i-th row of U.

- We can rewrite the constraints $\lambda_i \underline{\mathbf{z}}_i = R^\top(\mathbf{m}_i - \mathbf{p})$ as:

$$\underbrace{\mathbf{u}_i^\top W\mathbf{d}}_{\lambda_i} \underline{\mathbf{z}}_i = R^\top\mathbf{m}_i + \underbrace{VW\mathbf{d}}_{-R^\top\mathbf{p}}$$

- We have reduced the number of unknowns from $6 + n$ to 3

Reformulation into a Polynomial System

- **Cayley-Gibbs-Rodrigues Rotation Parameterization**

$$R^\top = \frac{\bar{C}}{1 + \mathbf{s}^\top \mathbf{s}} \qquad \bar{C} = ((1 - \mathbf{s}^\top \mathbf{s}) I_3 + 2\hat{\mathbf{s}} + 2\mathbf{s}\mathbf{s}^\top)$$

- The CGR parameters automatically satisfy the rotation matrix constraints, i.e., $R^\top R = I$ and $\det(R) = 1$ and allow us to formulate an <u>unconstrained</u> least-squares minimization in $\mathbf{s}$.

- Since $R^\top$ appears linearly in the equations, we can cancel the denominator $1 + \mathbf{s}^\top \mathbf{s}$. This leads to the following formulation of the PnP problem:

$$\min_{\mathbf{s}} J(\mathbf{s}) = \sum_{i=1}^{n} \left\| \mathbf{u}_i^\top \begin{bmatrix} \bar{C} & & \\ & \ddots & \\ & & \bar{C} \end{bmatrix} \mathbf{d}\underline{\mathbf{z}}_i - \bar{C}\mathbf{m}_i - V \begin{bmatrix} \bar{C} & & \\ & \ddots & \\ & & \bar{C} \end{bmatrix} \mathbf{d} \right\|^2$$

which contains all monomials up to degree four, i.e., $\{1, s_1, s_2, s_3, s_1 s_2, s_1 s_3, s_2 s_3, \ldots, s_1^4, s_2^4, s_3^4\}$.

Macaulay Matrix

- Since $J(\mathbf{s})$ is a fourth-order polynomial, the optimality conditions form a system of three third-order polynomials (derivatives with respect to s_1, s_2 and s_3).
- Use a **Macaulay resultant matrix** (matrix of polynomial coefficients) to find the roots of the third-order polynomials and hence compute all critical points of $J(\mathbf{s})$
- Since the polynomial system is of constant degree (independent of n), it is only necessary to compute the Macaulay matrix symbolically once.
- Online, the elements of the Macaulay matrix are formed from the data (linear operation in n) and the roots are determined via an eigen-decomposition of the Schur complement (dense 27×27 matrix) of the top block of the Macaulay matrix (sparse 120×120 matrix)

2-D Odometry from Bearing Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: bearing measurements $\mathbf{z}_{t,i} \in \mathbb{R}^2$ and $\mathbf{z}_{t+1,i} \in \mathbb{R}^2$ (unit vectors) at consecutive time steps to n **unknown** landmarks
- The measurements are related as follows:

$$d_{t,i}\mathbf{b}_{t,i} = {}_t\mathbf{p}_{t+1} + d_{t+1,i}R({}_t\theta_{t+1})\mathbf{b}_{t+1,i}, \qquad i = 1, \ldots, n$$

  where $d_{t,i}, d_{t+1,i}$ are the unknown distances to $\mathbf{m}_i$.
- There are $2n$ equations and $2n+3$ unknowns, which means that this problem is **not solvable**.

3-D Odometry from Bearing Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^3$ and ${}_tR_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t
- **Given**: normalized pixel coordinates $\underline{\mathbf{z}}_{t,i} \in \mathbb{R}^3$ and $\underline{\mathbf{z}}_{t+1,i} \in \mathbb{R}^3$ at consecutive time steps to n **unknown** landmarks ($n \geq 5$)
- **Essential matrix**: $E := \left[{}_t\hat{\mathbf{p}}_{t+1}\right]\left[{}_tR_{t+1}\right]$
- **Epipolar constraint**: $0 = \underline{\mathbf{z}}_{t,i}^\top E \underline{\mathbf{z}}_{t+1,i}$, for $i = 1, \ldots, n$
- **Idea**: recover the essential matrix between the two views first

3-D Odometry from Bearing Measurements (8-Pt Alg)

- The epipolar constraint $0 = \underline{\mathbf{z}}_{t,i}^\top E \underline{\mathbf{z}}_{t+1,i}$ is linear in the elements of E:

$$0 = \bar{\mathbf{z}}_i^\top \mathbf{e}$$

  where $\mathbf{e} := \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{21} & E_{22} & E_{23} & E_{31} & E_{32} & E_{33} \end{bmatrix}^\top$ and $\bar{\mathbf{z}}_i := \mathbf{vec}\left(\underline{\mathbf{z}}_{t+1,i}\underline{\mathbf{z}}_{t,i}^\top\right) \in \mathbb{R}^9$ where $\mathbf{vec}(\cdot)$ is a row-wise vectorization.

- Stacking $\bar{\mathbf{z}}_i$'s from 8 point observations together, we obtain an 8×9 matrix $\bar{Z} := \begin{bmatrix} \bar{\mathbf{z}}_1 & \cdots & \bar{\mathbf{z}}_8 \end{bmatrix}^\top$ leading to the following equation for $\mathbf{e}$:

$$\bar{Z}\mathbf{e} = 0$$

- Thus, $\mathbf{e}$ is a **singular vector** of $\bar{Z}$ associated to a singular value that equals zero.

- If at least 8 linearly independent vectors $\bar{\mathbf{z}}_i$ are used to construct $\bar{Z}$, then the singular vector is unique (up to scalar multiplication) and $\mathbf{e}$ and E can be determined.

3-D Odometry from Bearing Measurements (5-Pt Alg)

- The essential matrix E can be recovered from $\bar{Z}\mathbf{e} = 0$, even if only 5 linearly independent vectors $\bar{\mathbf{z}}_i$ are available using the fact that:

$$0 = EE^\top E - \frac{1}{2}\operatorname{tr}(EE^\top)E$$

- Stacking $\bar{\mathbf{z}}_i$'s together, we obtain a 5×9 matrix $\bar{Z} := \begin{bmatrix}\bar{\mathbf{z}}_1 & \cdots & \bar{\mathbf{z}}_5\end{bmatrix}^\top$
- The right nullspace of $\bar{Z}$ has dimension 4 and the vectors that span the nullspace (obtained from SVD or QR decomposition) correspond to 3×3 matrices N_i, $i = 1, \ldots, 4$ so that

$$E = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + \alpha_4 N_4, \qquad \alpha_i \in \mathbb{R}$$

- Since the measurements are scale-invariant, we can arbitrarily fix $\alpha_4 = 1$
- Substituting $E = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + N_4$, we obtain 9 cubic-in-$\alpha_i$ equations and can recover up to 10 solutions for E

3-D Odometry from Bearing Measurements (5-Pt Alg)

- Once E is recovered, ${}_t\mathbf{p}_{t+1}$ and ${}_tR_{t+1}$ can be computed from the singular value decomposition of E

- **Pose recovery from the essential matrix**: There are exactly two relative poses corresponding to a non-zero essential matrix $E = U\mathbf{diag}(\sigma, \sigma, 0)V^\top$:

$$({}_t\hat{\mathbf{p}}_{t+1}, {}_tR_{t+1}) = \left(UR_z\left(\frac{\pi}{2}\right)\mathbf{diag}(\sigma,\sigma,0)U^\top, UR_z^\top\left(\frac{\pi}{2}\right)V^\top\right)$$
$$({}_t\hat{\mathbf{p}}_{t+1}, {}_tR_{t+1}) = \left(UR_z\left(-\frac{\pi}{2}\right)\mathbf{diag}(\sigma,\sigma,0)U^\top, UR_z^\top\left(-\frac{\pi}{2}\right)V^\top\right)$$

- Only one of these will place the points in front of both cameras

- The ambiguity can be resolved by intersecting the measurements of a single point and verifying which solution places it on the positive optical z-axis of both cameras

Summary: Bearing Measurements $\underline{\mathbf{z}}_i = \frac{1}{\lambda_i} R^\top(\mathbf{m}_i - \mathbf{p})$

- **2-D Localization**: given $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ and $z_1, z_2 \in [-\pi, \pi]$
  1. 2-D bearing: $\frac{1}{\lambda_i} R^\top(\theta)(\mathbf{m}_i - \mathbf{p}) = R(z_i)\mathbf{e}_1$
  2. Eliminate θ:
  $$0 = \lambda_1 \mathbf{e}_1^\top R(\theta) R\left(\frac{\pi}{2}\right) R(\theta)\mathbf{e}_1 \lambda_2 = (\mathbf{m}_1 - \mathbf{p})^\top R(z_1) R\left(\frac{\pi}{2}\right) R^\top(z_2)(\mathbf{m}_2 - \mathbf{p})$$
  3. The position $\mathbf{p}$ in on one of two circles containing $\mathbf{m}_1$ and $\mathbf{m}_2$ and we need a third bearing measurement z_3 to disambiguate it
  4. Find β, γ such that $R^\top(z_1) + \beta R^\top(z_2) + \gamma R^\top(z_3) = 0$ and combine $R^\top(z_i)(\mathbf{m}_i - \mathbf{p}) = \lambda_i \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ to solve for θ
  5. Orientation: $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \frac{\mathbf{u}}{\|\mathbf{u}\|_2}$ for $\mathbf{u} = R^\top(z_1)\mathbf{m}_1 + \beta R^\top(z_2)\mathbf{m}_2 + \gamma R^\top(z_3)\mathbf{m}_3$

- **3-D Localization (P3P)**: $\mathbf{m}_i \in \mathbb{R}^3$, $\underline{\mathbf{z}}_i \in \mathbb{R}^3$ (homogeneous), $i = 1, 2, 3$
  1. Convert P3P to relative position localization by determining the depths $\lambda_1, \lambda_2, \lambda_3$ via Grunert's method
  2. Define the angles γ_{ij} among $\underline{\mathbf{z}}_1, \underline{\mathbf{z}}_2, \underline{\mathbf{z}}_3$ and apply the law of cosines: $\lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j \cos(\gamma_{ij}) = \|\mathbf{m}_1 - \mathbf{m}_j\|_2^2$
  3. Let $\lambda_2 = u\lambda_1$ and $\lambda_3 = v\lambda_1$ and combine the 3 equations to get a fourth order polynomial: $a_4 v^4 + a_3 v^3 + a_2 v^2 + a_1 v + a_0 = 0$

Summary: Bearing Measurements $\underline{\mathbf{z}}_i = \frac{1}{\lambda_i} R^\top (\mathbf{m}_i - \mathbf{p})$

- **3-D Localization (PnP)**
  1. Rewrite $\lambda_i \underline{\mathbf{z}}_i = R^\top (\mathbf{m}_i - \mathbf{p})$ in matrix form and solve for $\mathbf{x} := (\lambda_1, \ldots, \lambda_n, -R^\top \mathbf{p})^\top$ in terms of R
  2. The equations for λ_i and $-R^\top \mathbf{p}$ turn out to be linear in R so we are left with one equation with 3 unknowns (the 3 degrees of freedom of R)
  3. Obtain a fourth order polynomial $J(\mathbf{s})$ in terms of the Cayley-Gibbs-Rodrigues rotation parameterization $\mathbf{s}$
  4. Compute a Macaulay matrix of the coefficients of $J(\mathbf{s})$ symbolically once. Online, determine the roots of $J(\mathbf{s})$ via an eigen-decomposition of the Schur complement of the Macaulay matrix.

- **2-D Odometry**: not solvable

- **3-D Odometry**: 5-point or 8-point algorithm:
  1. Obtain E from the epipolar constraint: $0 = \mathbf{vec}\left(\underline{\mathbf{z}}_{t+1,i} \underline{\mathbf{z}}_{t,i}^\top\right)^\top \mathbf{vec}(E)$, $i = 1, \ldots, 5$ and the property $0 = EE^\top E - \frac{1}{2} \operatorname{tr}(EE^\top) E$
  2. Recover two possible camera poses based on $SVD(E) = U \mathbf{diag}(\sigma, \sigma, 0) V^\top$ and choose the one that places the measurements in front of both cameras