ECE276A: Sensing & Estimation in Robotics
Lecture 13: Visual Features

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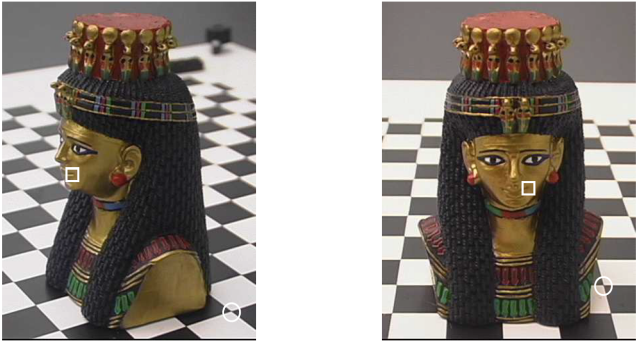
From Photometry to Geometry

- **Image**: an array of positive numbers that measure the amount of light incident on the sensor
- How do we go from measurements of light (**photometry**) to measurements of 2-D points (**features**)?
  - This lecture
- How do we go from 2-D point projections (**features**) to 3-D point positions (**landmarks**)?
  - We can use the visual-inertial SLAM approach of the previous lecture.

Correspondence



- **Corresponding points** in two views are image projections of the same geometric point in space
- **Correspondence problem**: establish which point $\mathbf{z}_2 \in \mathbb{R}^2$ in the second image corresponds to a given point $\mathbf{z}_1 \in \mathbb{R}^2$ in the first image in the sense of being the same point $\mathbf{m} \in \mathbb{R}^3$ in 3-D physical space
- **Idea**: look for a pixel $\mathbf{z}_2$ in the second image such that $I_2(\mathbf{z}_2) \approx I_1(\mathbf{z}_1)$

Correspondence

- **Matching windows**: a robust process for establishing correspondence is to compare not the brightness of individual pixels but that of small pixel windows $W(\mathbf{z}_1)$, $W(\mathbf{z}_2)$ around the points
- **Aperture problem**: the brightness profile within the selected windows may not be rich enough to allow us to recover the transformation of the pixel $\mathbf{z}_1$ uniquely (e.g., blank wall)
- **Features**: points whose local window regions are rich enough to allow solving the correspondence problem. Features establish a link between photometric measurements and geometric primitives.
- The window shape $W(\mathbf{z}_1)$ and image values $I_1(\mathbf{y})$, $\mathbf{y} \in W(\mathbf{z}_1)$, associated with a pixel $\mathbf{z}_1$ in the first image undergo a *nonlinear transformation* as a consequence of the change of viewpoint

Brightness Constancy Constraint

- Suppose we are imaging a point $\mathbf{m} \in \mathbb{R}^3$ that emits light with the same energy in all directions (Lambertian) and radiance distribution $\mathcal{R}(\mathbf{m})$
- Suppose the camera is calibrated, i.e., $K = I$, and the two camera frames are related by a rigid-body transformation $(R, \mathbf{p}) \in SE(3)$.
- Let I_1 and I_2 be two images and $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^2$ be the two pixels corresponding to $\mathbf{m}$:
$$I_2(\mathbf{z}_2) = I_1(\mathbf{z}_1) \propto \mathcal{R}(\mathbf{m})$$
- From the projection equations, the point $\mathbf{z}_1$ in image I_1 corresponds to the point $\mathbf{z}_2$ in image I_2 if:
$$\mathbf{z}_2 = g(\mathbf{z}_1) := \frac{1}{\lambda_2}\left(\lambda_1 R \mathbf{z}_1 + \mathbf{p}\right)$$
where λ_1, λ_2 are the **unknown** depths of the observed point $\mathbf{m}$
- **Brightness constancy constraint**: $\boxed{I_1(\mathbf{z}_1) = I_2(g(\mathbf{z}_1))}$

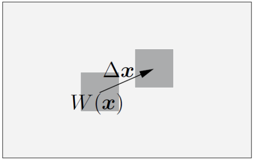
Local Deformation Models

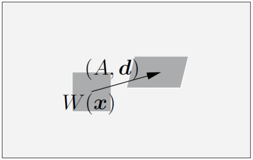
- The transformation g undergone by the entire image is determined by the depths λ_1, λ_2 of the visible surface and hence estimating g is as difficult as estimating the shape of the visible objects
- Instead, model the transformation $g(\mathbf{z})$ only locally in a region $W(\mathbf{z})$:
  - **Translation model**: each point in the window $W(\mathbf{z})$ undergoes the exact same translational motion $\mathbf{d} \in \mathbb{R}^2$:

    $$g(\mathbf{y}) \approx \mathbf{y} + \mathbf{d}, \quad \forall \mathbf{y} \in W(\mathbf{z})$$

    This model is valid only in small windows and over short time durations but it is at the core of many feature matching and tracking algorithms.
  - **Affine model**: each point in the window $W(\mathbf{z})$ undergoes an affine transformation with parameters $A \in \mathbb{R}^{2\times 2}$ and $\mathbf{d} \in \mathbb{R}^2$:

    $$g(\mathbf{y}) \approx A\mathbf{y} + \mathbf{d}, \quad \forall \mathbf{y} \in W(\mathbf{z})$$





Matching Point Features

- Requiring that $I_1(\mathbf{z}_1) = I_2(g(\mathbf{z}_1))$ is too strict due to the approximation of g and the presence of noise and occlusions

- **Correspondence problem**: an optimization problem that aims to determine the (translation or affine) parameters of the local transformation model of $g(\mathbf{y})$ for $\mathbf{y} \in W(\mathbf{z})$:

$$\min_{\mathbf{d}} \sum_{\mathbf{y} \in W(\mathbf{z})} \|I_1(\mathbf{y}) - I_2(\mathbf{y} + \mathbf{d})\|_2^2 \quad \text{OR} \quad \min_{A,\mathbf{d}} \sum_{\mathbf{y} \in W(\mathbf{z})} \|I_1(\mathbf{y}) - I_2(A\mathbf{y} + \mathbf{d})\|_2^2$$

- Our approximations of g are valid only locally in space and **time** so consider the continuous version of the brightness constancy constraint:

$$I_1(\mathbf{z}) = I(\mathbf{z}(t), t) \underbrace{\approx}_{\text{brightness constancy}} I_2(g(\mathbf{z})) \underbrace{\approx}_{\text{approximation model}} I(A\mathbf{z}(t) + \boldsymbol{\nu}\tau, t + \tau)$$

where τ is small and $\boldsymbol{\nu} \in \mathbb{R}^2$ is the velocity of $\mathbf{z}$

Continuous-Time Brightness Constancy

- Linearizing the right-hand side around (z, t):

$$I(A\mathbf{z} + \boldsymbol{\nu}\tau, t + \tau) \approx I(\mathbf{z}, t) + \nabla_{\mathbf{z}} I(\mathbf{z}, t)^\top (A\mathbf{z} + \boldsymbol{\nu}\tau - \mathbf{z}) + \frac{\partial I}{\partial t}(\mathbf{z}, t)\tau$$

- To ensure **brightness constancy**: $I(\mathbf{z}, t) \approx I(A\mathbf{z} + \boldsymbol{\nu}\tau, t + \tau)$, choose A and $\boldsymbol{\nu}$ such that:
  - Affine model: $\min_{A,\boldsymbol{\nu}} \sum_{\mathbf{y} \in W(\mathbf{z})} \left\| \nabla_{\mathbf{z}} I(\mathbf{y}, t)^\top \left(\frac{1}{\tau}(A - I)\mathbf{y} + \boldsymbol{\nu} \right) + \frac{\partial I}{\partial t}(\mathbf{y}, t) \right\|_2^2$
  - Translation model: $\min_{\boldsymbol{\nu}} \sum_{\mathbf{y} \in W(\mathbf{z})} \left\| \nabla_{\mathbf{z}} I(\mathbf{y}, t)^\top \boldsymbol{\nu} + \frac{\partial I}{\partial t}(\mathbf{y}, t) \right\|_2^2$

- **Aperture problem**: the equation $\frac{\partial I}{\partial \mathbf{z}}\boldsymbol{\nu} + \frac{\partial I}{\partial t} = 0$ provides only one constraint for two unknowns $\boldsymbol{\nu} \in \mathbb{R}^2$.

- There are enough constraints on $\boldsymbol{\nu}$ only when the brightness constancy constraint is applied to each $\mathbf{y}$ in a region $W(\mathbf{z})$ that contains "sufficient texture" and the velocity $\boldsymbol{\nu}$ is assumed constant over the region.

Feature Tracking and Optical Flow

- The translation model optimization is used for optical flow or feature tracking in a sequence of images
- **Optical flow**: computes the velocity $\boldsymbol{\nu}$ of a fixed image location $\mathbf{z}$
- **Feature tracking**: computes the velocity $\boldsymbol{\nu}$ of a feature $\mathbf{z}(t)$ moving in time such that: $\mathbf{z}(t+\tau) = \mathbf{z}(t) + \boldsymbol{\nu}\tau$
- The only difference between optical flow and feature tracking is at the conceptual level, whether the vector $\boldsymbol{\nu}$ is computed at fixed locations $\mathbf{z}$ in the image or at moving points $\mathbf{z}(t)$

Feature Tracking and Optical Flow

- To compute the velocity $\boldsymbol{\nu}$ we need to solve:

$$\min_{\boldsymbol{\nu}} \sum_{\mathbf{y} \in W(\mathbf{z})} \left\| \nabla_{\mathbf{z}} I(\mathbf{y}, t)^\top \boldsymbol{\nu} + \frac{\partial I}{\partial t}(\mathbf{y}, t) \right\|_2^2$$

- Letting $\mathbf{z} = (u, v)$ and setting the gradient to zero results in:

$$\begin{aligned}
0 &= 2 \sum_{\mathbf{y} \in W(\mathbf{z})} \left(\nabla_{\mathbf{z}} I(\mathbf{y}, t)^\top \boldsymbol{\nu} + \frac{\partial I}{\partial t}(\mathbf{y}, t) \right) \nabla_{\mathbf{z}} I(\mathbf{y}, t) \\
&= 2 \sum_{\mathbf{y} \in W(\mathbf{z})} \left(\begin{bmatrix} I_u^2(\mathbf{y}) & I_u(\mathbf{y}) I_v(\mathbf{y}) \\ I_u(\mathbf{y}) I_v(\mathbf{y}) & I_v(\mathbf{y})^2 \end{bmatrix} \boldsymbol{\nu} + \begin{bmatrix} I_u(\mathbf{y}) I_t(\mathbf{y}) \\ I_v(\mathbf{y}) I_t(\mathbf{y}) \end{bmatrix} \right) \\
&= 2 \left(\underbrace{\begin{bmatrix} \sum_{\mathbf{y}} I_u^2(\mathbf{y}) & \sum_{\mathbf{y}} I_u(\mathbf{y}) I_v(\mathbf{y}) \\ \sum_{\mathbf{y}} I_u(\mathbf{y}) I_v(\mathbf{y}) & \sum_{\mathbf{y}} I_v(\mathbf{y})^2 \end{bmatrix}}_{G(\mathbf{z})} \boldsymbol{\nu} + \underbrace{\begin{bmatrix} \sum_{\mathbf{y}} I_u(\mathbf{y}) I_t(\mathbf{y}) \\ \sum_{\mathbf{y}} I_v(\mathbf{y}) I_t(\mathbf{y}) \end{bmatrix}}_{b(\mathbf{z})} \right)
\end{aligned}$$

- The optimal estimate of the image velocity at $\mathbf{z}$ is $\boxed{\boldsymbol{\nu}^* = -G(\mathbf{z})^{-1} b(\mathbf{z})}$

Point Feature Selection

- For $G(\mathbf{z})$ to be invertible, the region $W(\mathbf{z})$ must have nontrivial gradients along independent directions, therefore resembling a "corner"

- **Corner**: a pixel $\mathbf{z}$ such that the smallest eigenvalue of $G(\mathbf{z})$ is larger than some threshold ρ

- **Harris corner**: a variation of the corner detector that thresholds:

$$\lambda_1\lambda_2 - k(\lambda_1 + \lambda_2)^2 = \det(G) - k\,\mathrm{tr}^2(G) \geq \rho$$

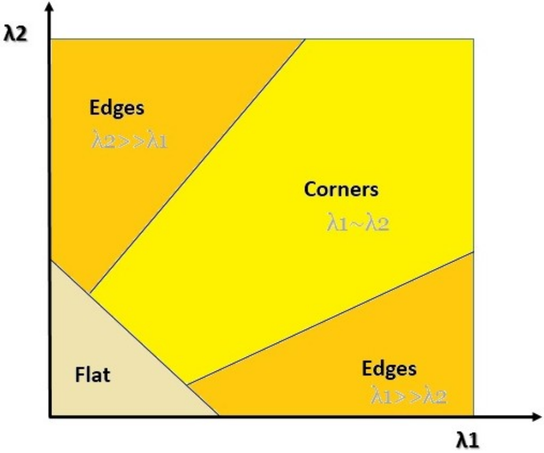
  where $k \in [0.04, 0.06]$ is a small scalar and λ_1, λ_2 are the eigenvalues of G. Since k is small, both eigenvalues of G need to be sufficiently large to pass the threshold.

- More sophisticated techniques that utilize contours or edges and search for high curvature points in the detected contours are used in practice

Point Feature Selection

- Description of $W(\mathbf{z})$ as a function of the eigenvalues λ_1 and λ_2 of

$$G(\mathbf{z}) := \sum_{\mathbf{y} \in W(\mathbf{z})} \begin{bmatrix} I_u^2(\mathbf{y}) & I_u(\mathbf{y}) I_v(\mathbf{y}) \\ I_u(\mathbf{y}) I_v(\mathbf{y}) & I_v(\mathbf{y})^2 \end{bmatrix}$$



Feature Tracking and Optical Flow

Algorithm 1 Basic Feature Tracking and Optical Flow

Input: Image I at time t

Compute the image gradient (I_u, I_v)

Compute $G(\mathbf{z}) := \begin{bmatrix} \sum_{\mathbf{y}\in W(\mathbf{z})} I_u^2(\mathbf{y}) & \sum_{\mathbf{y}\in W(\mathbf{z})} I_u(\mathbf{y})I_v(\mathbf{y}) \\ \sum_{\mathbf{y}\in W(\mathbf{z})} I_u(\mathbf{y})I_v(\mathbf{y}) & \sum_{\mathbf{y}\in W(\mathbf{z})} I_v^2(\mathbf{y}) \end{bmatrix}$ at every pixel $\mathbf{z} = (u, v)$

(Feature tracking) select point features $\mathbf{z}_1, \mathbf{z}_2, \ldots$ such that $G(\mathbf{z}_i)$ is invertible

(Optical flow) select $\mathbf{z}_i$ on a fixed grid

Compute $b(\mathbf{z}) := \begin{bmatrix} \sum_{\mathbf{y}\in W(\mathbf{z})} I_u(\mathbf{y})I_t(\mathbf{y}) \\ \sum_{\mathbf{y}\in W(\mathbf{z})} I_v(\mathbf{y})I_t(\mathbf{y}) \end{bmatrix}$

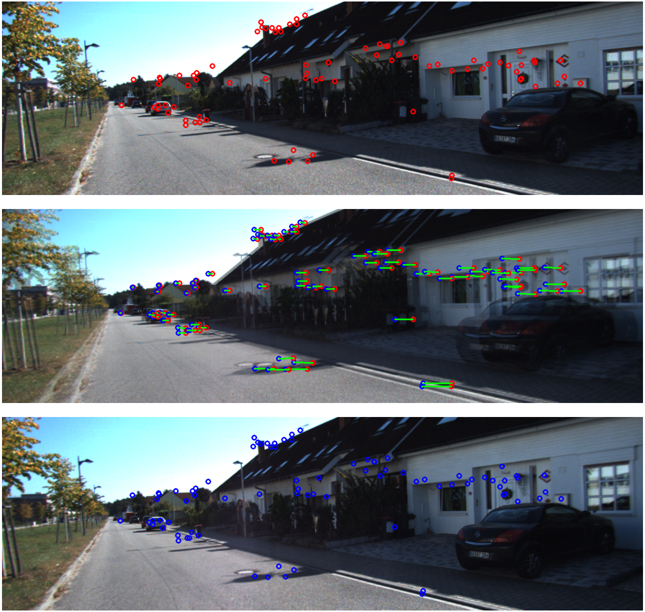
If $G(\mathbf{z})$ is invertible, compute $\boldsymbol{\nu}(\mathbf{z}) = -G(\mathbf{z})^{-1}b(\mathbf{z})$

Else $\boldsymbol{\nu}(\mathbf{z}) = 0$.

(Feature tracking) at time $t+1$, repeat the operation at $\mathbf{z} + \boldsymbol{\nu}(z)\tau$

(Optical flow) at time $t+1$, repeat the operation at $\mathbf{z}$

Feature Tracking and Optical Flow



Feature Tracking and Optical Flow

- The feature tracking/optical flow algorithm is very efficient when we use the translation model
- When features are tracked over extended periods of time, however, the approximation error accumulates
- Instead of matching image regions between adjacent frames, one could match image regions between an initial frame and the current frame
- The simple translation model is no longer accurate and we should use the affine model
- Further reading about the Kanade-Lucas-Tomasi (KLT) feature tracker:
  - B. Lucas and T. Kanade, "An Iterative Image Registration Technique with an Application to Stereo Vision," International Joint Conference on Artificial Intelligence (IJCAI), 1981.
  - C. Tomasi and T. Kanade, "Detection and Tracking of Point Features," CMU Technical Report CMU-CS-91-132, 1991.
  - J. Shi and C. Tomasi, "Good Features to Track," IEEE Conference on Computer Vision and Pattern Recognition (CVPR), 1994.

Image Gradients

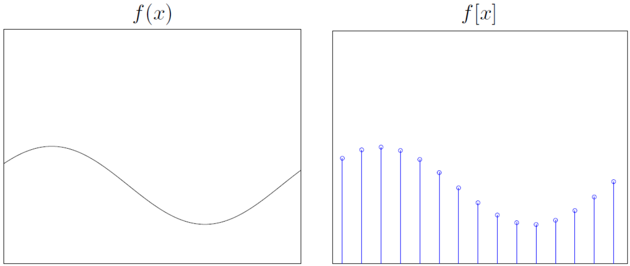
- How do we compute the gradients $I_u(u, v, t)$, $I_v(u, v, t)$, and $I_t(u, v, t)$ needed for feature tracking/optical flow?

- We could approximate the derivatives using finite differences, e.g.,:

$$I_t(u, v, t) \approx \frac{1}{\tau}\left(I(u, v, t) - I(u, v, t-1)\right) \quad \text{OR} \quad I_t(u, v, t) \approx \frac{1}{2\tau}\left(I(u, v, t+1) - I(u, v, t-1)\right)$$

- To derive a more accurate approximation, we need to understand the relationship between a continuous signal $f(x)$ and its sampled version with period τ:

$$f[x] = f(x\tau), \quad x \in \mathbb{Z}$$



Nyquist-Shannon Sampling Theorem

- If $f(x)$ is band limited, i.e., its Fourier transform satisfies $|F(\omega)| = 0$ for all $\omega > \omega_n$ (**Nyquist frequency**), it can be reconstructed exactly from a set of discrete samples at sampling frequency $\omega_s := \frac{2\pi}{\tau} > 2\omega_n$.
- The continuous signal $f(x)$ can be reconstructed by multiplying its sampled version $f[x]$ in the frequency domain with an ideal reconstruction filter $h(x)$ with Fourier transform:

$$H(\omega) = \begin{cases} 1, & \omega \in \left[-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right] \\ 0, & \text{else} \end{cases} \qquad h(x) = \textbf{sinc}\left(\frac{\pi x}{\tau}\right), \quad x \in \mathbb{R}$$

- Multiplication in the frequency domain corresponds to convolution in the spatial domain, thus as long as $\omega_n < \frac{\pi}{\tau}$:

$$f(x) = f[x] * h(x) = \sum_{k=-\infty}^{\infty} f[k]h(x-k), \qquad x \in \mathbb{R}$$

Derivative of a Sampled Signal

- Differentiating $f(x) = f[x] * h(x)$:

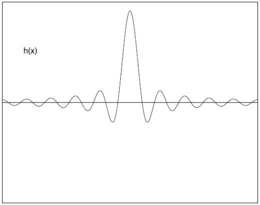
$$\frac{d}{dx} f(x) = \sum_{k=-\infty}^{\infty} f[k] \frac{d}{dx} h(x-k) = f[x] * \frac{dh}{dx}(x)$$

- Sampling the above result shows that the derivative of the sampled function $f'[x]$ can be computed as a convolution of the sampled signal $f[x]$ with the sampled derivative of the sync function $h'[x]$:

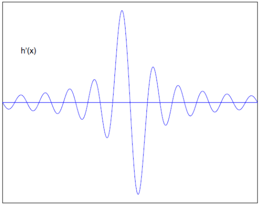
$$f'[x] = f[x] * h'[x]$$

$$h'(x) = \frac{(\pi^2 x/\tau^2)\cos(\pi x/\tau) - \pi/\tau \sin(\pi x/\tau)}{(\pi x/\tau)^2}, \quad x \in \mathbb{R}$$

$h(x)$



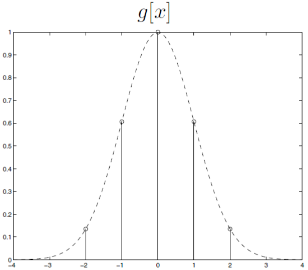
$h'(x)$

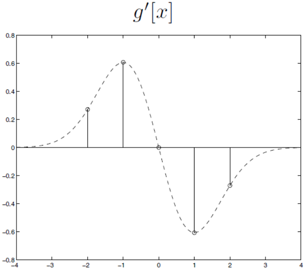


Five-tap Gaussian Filter

- The sync function has infinite support and falls off very slowly away from the origin. Hence, simple truncation of sync convolution yields undesirable artifacts and is not practically feasible
- The derivative can be approximated by convolving with a Gaussian instead of a sync since it drops to zero much faster:

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}} \qquad g'(x) = -\frac{x}{\sigma^2\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}}$$





$g[x] = \begin{bmatrix} 0.1353 & 0.6065 & 1.0000 & 0.6065 & 0.1353 \end{bmatrix} \quad g'[x] = \begin{bmatrix} 0.2707 & 0.6065 & 0 & -0.6065 & -0.2707 \end{bmatrix}$

Image Gradients

- In the case of images (2-D functions) the result is the same:

$$I(u,v) = I[u,v] * h(u,v) \quad h(u,v) = h(u)h(v) = \frac{\sin(\pi u/\tau)\sin(\pi v/\tau)}{\pi^2 uv/\tau^2}$$

- Note that $h(u,v) = h(u)h(v)$ is separable which leads to:

$$I_u[u,v] = I[u,v] * h'[u] * h[v] \qquad I_v(u,v) = I[u,v] * h[u] * h'[v]$$

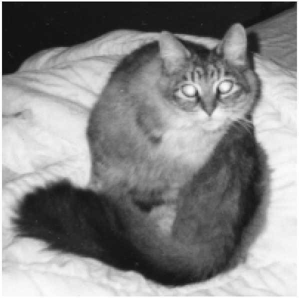
- The computation of the image derivatives is then accomplished as a pair of 1-D convolutions with filters obtained by sampling a continuous Gaussian probability density function and its derivative:

$$I_u[u,v] = I[u,v] * g'[u] * g[v] = \sum_{k=-\omega/2}^{\omega/2} \sum_{l=-\omega/2}^{\omega/2} I[k,l] g'[u-k] g[v-l]$$

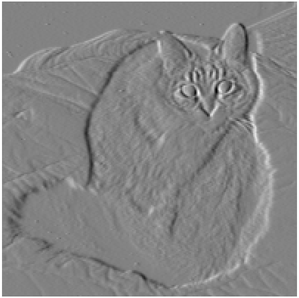
$$I_v[u,v] = I[u,v] * g[u] * g'[v] = \sum_{k=-\omega/2}^{\omega/2} \sum_{l=-\omega/2}^{\omega/2} I[k,l] g[u-k] g'[v-l]$$

- The number of samples is typically chosen as $\omega = 5\sigma$, imposing the fact that the window covers 98.76% of the area under the Gaussian curve

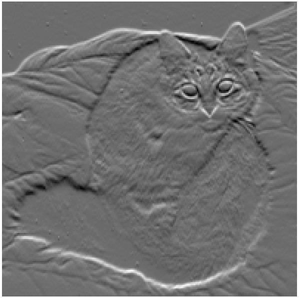
Image Gradients



I



I_u



I_v

Other Derivative Filters, Features, and Descriptors

- Other commonly used derivative filters:
  - **Interpolation filter**: $h[x] = \frac{1}{2}[1,\ 1]$ with derivative $h'[x] = \frac{1}{2}[1, -1]$
  - **Sobel filter**: $h[x] = \frac{1}{2+\sqrt{2}}[1, \sqrt{2}, 1]$ with derivative $h'[x] = \frac{1}{3}[1, 0, -1]$
  - **Gabor filter**: used for texture analysis
- Other features and descriptors (describe feature shape, color, texture):
  - **SIFT**: the Scale-Invariant Feature Transform (SIFT), introduced by David Lowe, is one of the most successful local image features/descriptors in the past decade. It makes the Harris corner scale invariant by using scale-space filtering via a Laplacian of Gaussian kernel (blob detector)
  - **SURF**: the Speeded-Up Robust Feature is a speeded-up version of SIFT which applies an approximate 2^{nd} derivative Gaussian filter at many scales along the axes and at 45° (more robust to rotation than Harris corners)
  - **FAST**: a Feature from Accelerated Segment Test detects corners by considering 16 pixels around the pixel y being tested and is several times faster than other corner detectors
  - **BRIEF**: a Binary Robust Independent Elementary Features speed up descriptor calculation and matching
  - **ORB**: Oriented FAST and Rotated BRIEF