ECE276A: Sensing & Estimation in Robotics
Lecture 3: Rotations

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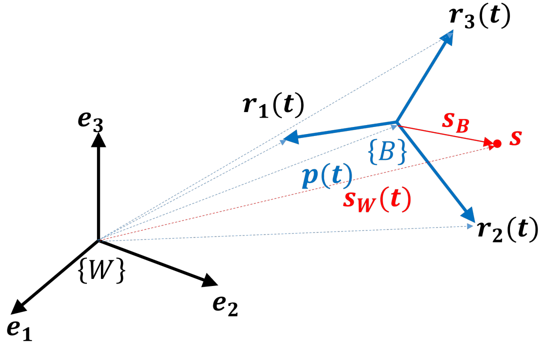
Outline

Rigid Body Motion

- Consider a rigid body moving in a fixed **world reference frame** $\{W\}$
- **Body reference frame** $\{B\}$: it is sufficient to specify the motion of one point $\mathbf{p}(t) \in \mathbb{R}^3$ and 3 coordinate axes $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, $\mathbf{r}_3(t)$ attached to the point



- A point $\mathbf{s}$ on the rigid body has fixed coordinates $\mathbf{s}_B \in \mathbb{R}^3$ in the body frame $\{B\}$ but time-varying coordinates $\mathbf{s}_W(t) \in \mathbb{R}^3$ in the world frame $\{W\}$

Rigid Body Motion

- A rigid body in 3D is free to translate (3 degrees of freedom) and rotate (3 degrees of freedom)
- The **pose** $T(t) \in SE(3)$ of a rigid body reference frame $\{B\}$ at time t in a fixed world frame $\{W\}$ is determined by:
  1. the position $\mathbf{p}(t) \in \mathbb{R}^3$ of $\{B\}$ relative to $\{W\}$,
  2. the orientation $R(t) \in SO(3)$ of $\{B\}$ relative to $\{W\}$, determined by the 3 coordinate axes $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, $\mathbf{r}_3(t)$.
- The space of positions $\mathbb{R}^3$ is familiar
- How do we describe the space of orientations $SO(3)$ and the space of poses $SE(3)$?

Special Euclidean Group

- **Rigid body motion** is described by a sequence of functions that describe how the coordinates of 3-D points on the object change with time

- Rigid body motion preserves distances (vector norms) and does not allow reflection of the coordinate system (vector cross products)

- **Euclidean Group** $E(3)$: a set of functions $g : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve the norm of any two vectors

- **Special Euclidean Group** $SE(3)$: a set of functions $g : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve the norm and the cross product of any two vectors
  1. Norm: $\|g_*(\mathbf{u}) - g_*(\mathbf{v})\| = \|\mathbf{v} - \mathbf{u}\|$, $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$
  2. Cross product: $g_*(\mathbf{u}) \times g_*(\mathbf{v}) = g_*(\mathbf{u} \times \mathbf{v})$, $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$

  where $g_*(\mathbf{x}) := g(\mathbf{x}) - g(\mathbf{0})$.

- **Corollary**: $SE(3)$ elements g also preserve:
  1. Angle: $\mathbf{u}^\top \mathbf{v} = \frac{1}{4}\left(\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2\right) \Rightarrow \mathbf{u}^\top \mathbf{v} = g_*(\mathbf{u})^\top g_*(\mathbf{v})$, $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$
  2. Volume: $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $g_*(\mathbf{u})^\top (g_*(\mathbf{v}) \times g_*(\mathbf{w})) = \mathbf{u}^\top (\mathbf{v} \times \mathbf{w})$
     (volume of parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$)

Orientation and Rotation

- Pure rotational motion is a special case of rigid body motion
- The orientation of a body frame $\{B\}$ in the world frame $\{W\}$ is determined by the coordinates of the three orthogonal vectors $\mathbf{r}_1 = g(\mathbf{e}_1)$, $\mathbf{r}_2 = g(\mathbf{e}_2)$, $\mathbf{r}_3 = g(\mathbf{e}_3)$, transformed from $\{B\}$ to $\{W\}$
- The vectors organized in a 3×3 matrix specify the orientation of $\{B\}$ in $\{W\}$:

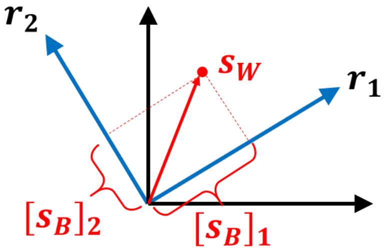
$${}_{\{W\}}R_{\{B\}} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

- Consider a point with coordinates $\mathbf{s}_B \in \mathbb{R}^3$ in $\{B\}$
- Its coordinates $\mathbf{s}_W$ in $\{W\}$ are:

$$\begin{aligned} \mathbf{s}_W &= [s_B]_1 \mathbf{r}_1 + [s_B]_2 \mathbf{r}_2 + [s_B]_3 \mathbf{r}_3 \\ &= R\mathbf{s}_B \end{aligned}$$

- The rotation transformation g from $\{B\}$ to $\{W\}$ is a linear function:

$$g(\mathbf{s}) = R\mathbf{s}$$



Special Orthogonal Group $SO(3)$

- $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ form an orthonormal basis: $\mathbf{r}_i^\top \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
- Since $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ form an orthonormal basis, the inverse of R is its transpose:

$$R^\top R = I \qquad\qquad R^{-1} = R^\top$$

- R belongs to the **orthogonal group**:

$$O(3) := \{R \in \mathbb{R}^{3\times3} \mid R^\top R = RR^\top = I\}$$

- Distances are preserved since $R^\top R = I$:

$$\|R(\mathbf{x} - \mathbf{y})\|_2^2 = (\mathbf{x} - \mathbf{y})^\top R^\top R(\mathbf{x} - \mathbf{y}) = (\mathbf{x} - \mathbf{y})^\top (\mathbf{x} - \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2^2$$

- Reflections are not allowed since $\det(R) = \mathbf{r}_1^\top(\mathbf{r}_2 \times \mathbf{r}_3) = 1$:

$$R(\mathbf{x} \times \mathbf{y}) = R\left(\mathbf{x} \times (R^\top R\mathbf{y})\right) = (R\hat{\mathbf{x}}R^\top)R\mathbf{y} = \frac{1}{\det(R)}(R\mathbf{x}) \times (R\mathbf{y})$$

- R belongs to the **special orthogonal group**:

$$SO(3) := \{R \in \mathbb{R}^{3\times3} \mid R^T R = I, \det(R) = 1\}$$

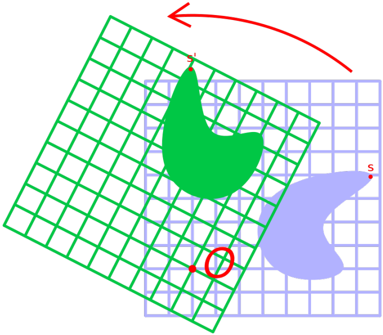
Parametrizing 2-D Rotations

- There are 2 common ways to parametrize a rotation matrix $R \in SO(2)$

- **Rotation angle**: a 2-D rotation of a point $\mathbf{s}_B \in \mathbb{R}^2$ can be parametrized by an angle θ around the z-axis:

$$\mathbf{s}_W = R(\theta)\mathbf{s}_B := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{s}_B$$

- $\theta > 0$: counterclockwise rotation



- **Unit-norm complex number**: a 2-D rotation of $[s_B]_1 + i[s_B]_2 \in \mathbb{C}$ can be parametrized by a unit-norm complex number $e^{i\theta} \in \mathbb{C}$:

$$e^{i\theta}([s_B]_1 + i[s_B]_2) = ([s_B]_1 \cos\theta - [s_B]_2 \sin\theta) + i([s_B]_1 \sin\theta + [s_B]_2 \cos\theta)$$

Parametrizing 3-D Rotations

- There are 3 common ways to parametrize a rotation matrix $R \in SO(3)$
- **Euler angles**: an extension of the rotation angle parametrization of 2-D rotations that specifies rotation angles around the three principal axes
- **Axis-Angle**: an extension of the rotation angle parametrization of 2-D rotations that allows the axis of rotation to be chosen freely instead of being a fixed principal axis
- **Unit Quaternion**: an extension of the unit-norm complex number parametrization of 2-D rotations

Outline

Euler Angle Parametrization

- Uses three angles that specify rotations around the three principal axes
- There are 24 different ways to apply these rotations
  - **Extrinsic axes**: the rotation axes remain static
  - **Intrinsic axes**: the rotation axes move with the rotations
  - Each of the two groups (intrinsic and extrinsic) can be divided into:
    - **Euler Angles**: rotation about one axis, then a second, and then the first
    - **Tait-Bryan Angles**: rotation about all three axes
  - The Euler and Tait-Bryan Angles each have 6 possible choices for each of the extrinsic/intrinsic groups leading to $2 * 2 * 6 = 24$ possible conventions to specify a rotation sequence with three given angles
- For simplicity, we refer to all 24 conventions as **Euler Angles** and explicitly specify:
  - *r* (rotating = intrinsic) or *s* (**static = extrinsic**)
  - *xyz* or *zyx* or *zxz*, etc. (**order of rotation axes**)
- An extrinsic rotation is equivalent to an intrinsic rotation by the same angles but with inverted rotation order:

$$sxyz = rzyx$$

Principal 3-D Rotations

- A rotation by an angle ϕ around the x-axis is represented by:

$$R_x(\phi) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

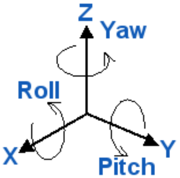
- A rotation by an angle θ around the y-axis is represented by:

$$R_y(\theta) := \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

- A rotation by an angle ψ around the z-axis is represented by:

$$R_z(\psi) := \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Roll Pitch Yaw Convention



- **Roll** (ϕ), **pitch** (θ), **yaw** (ψ) angles are used in aerospace engineering to specify rotation of an aircraft around the x, y, and z axes, respectively

- Intrinsic yaw (ψ), pitch (θ), roll (ϕ) rotation (*rzyx*):
  - A rotation ψ about the original z-axis
  - A rotation θ about the intermediate y-axis
  - A rotation ϕ about the transformed x-axis

- Extrinsic roll (ϕ), pitch (θ), yaw (ψ) rotation (*sxyz*):
  - A rotation ϕ about the global x-axis
  - A rotation θ about the global y-axis
  - A rotation ψ about the global z-axis

- Both conventions define the following body-to-world rotation:

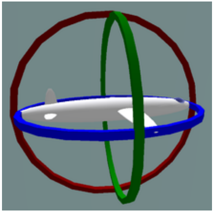
$$
\begin{aligned}
R &= R_z(\psi)R_y(\theta)R_x(\phi) \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}
\end{aligned}
$$

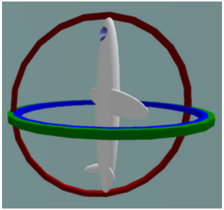
Gimbal Lock

- Angle parametrizations are widely used due to their simplicity
- Unfortunately, in 3-D, angle parametrizations are not one-to-one and lead to **singularities** known as **gimbal lock**
- Example: if the pitch becomes $\theta = 90^\circ$, the roll and yaw become associated with the same degree of freedom and cannot be uniquely determined.
- The following leads to the same rotation matrix R for any choice of δ:

$$R = R_z(\psi)R_y(\pi/2)R_x(\phi + \delta)$$

- Gimbal lock is a problem only if we want to recover the rotation angles from a rotation matrix





Outline

Cross Product and Hat Map

- The **cross product** of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is also a vector in $\mathbb{R}^3$:

$$\mathbf{x} \times \mathbf{y} := \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \hat{\mathbf{x}}\mathbf{y}$$

- The cross product $\mathbf{x} \times \mathbf{y}$ can be represented by a *linear* map $\hat{\mathbf{x}}$ called the **hat map**

- The **hat map** $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms a vector $\mathbf{x} \in \mathbb{R}^3$ to a skew-symmetric matrix:

$$\hat{\mathbf{x}} := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \qquad \qquad \hat{\mathbf{x}}^\top = -\hat{\mathbf{x}}$$

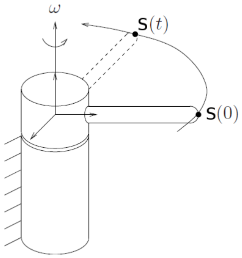
- The vector space $\mathbb{R}^3$ and the space of skew-symmetric 3×3 matrices $\mathfrak{so}(3)$ are isomorphic, i.e., there exists a one-to-one map (the hat map) that preserves their structure

Hat Map Properties

- **Lemma**: A matrix $M \in \mathbb{R}^{3\times 3}$ is skew-symmetric iff $M = \hat{\mathbf{x}}$ for some $\mathbf{x} \in \mathbb{R}^3$.
- The inverse of the hat map is the **vee map**, $\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$, that extracts the components of the vector $\mathbf{x} = \hat{\mathbf{x}}^{\vee}$ from the matrix $\hat{\mathbf{x}}$.
- **Hat map properties**: for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$:
  - $\hat{\mathbf{x}}\mathbf{y} = \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x} = -\hat{\mathbf{y}}\mathbf{x}$
  - $\hat{\mathbf{x}}^2 = \mathbf{x}\mathbf{x}^\top - \mathbf{x}^\top\mathbf{x}\, I$
  - $\hat{\mathbf{x}}^{2k+1} = (-\mathbf{x}^\top\mathbf{x})^k\hat{\mathbf{x}}$
  - $-\frac{1}{2}\operatorname{tr}(\hat{\mathbf{x}}\hat{\mathbf{y}}) = \mathbf{x}^\top\mathbf{y}$
  - $\hat{\mathbf{x}}A + A^\top\hat{\mathbf{x}} = ((\operatorname{tr}(A)I - A)\mathbf{x})^\wedge$
  - $\operatorname{tr}(\hat{\mathbf{x}}A) = \frac{1}{2}\operatorname{tr}(\hat{\mathbf{x}}(A - A^\top)) = -\mathbf{x}^\top(A - A^\top)^\vee$
  - $(A\mathbf{x})^\wedge = \det(A)A^{-\top}\hat{\mathbf{x}}A^{-1}$

Axis-Angle Parametrization

- Consider a point $\mathbf{s} \in \mathbb{R}^3$ rotating about an axis $\boldsymbol{\eta} \in \mathbb{R}^3$ at constant unit velocity:



$$\dot{\mathbf{s}}(t) = \boldsymbol{\eta} \times \mathbf{s}(t) = \hat{\boldsymbol{\eta}}\mathbf{s}(t)$$

- This is a linear time-invariant (LTI) system of ordinary differential equations determined by the skew-symmetric matrix $\hat{\boldsymbol{\eta}}$

- The solution to this LTI system specifies the trajectory of the point $\mathbf{s}$:

$$\mathbf{s}(t) = \exp(t\hat{\boldsymbol{\eta}})\mathbf{s}(0)$$

- Since $\mathbf{s}$ undergoes pure rotation, we know that:

$$\mathbf{s}(t) = R(t)\mathbf{s}(0)$$

- Since the rotation is determined by constant unit velocity, the elapsed time t is equal to the angle of rotation θ:

$$R(\theta) = \exp(\theta\hat{\boldsymbol{\eta}})$$

Exponential Map from $\mathfrak{so}(3)$ to $SO(3)$

- Any rotation can be represented as a rotation about a unit-vector axis $\boldsymbol{\eta} \in \mathbb{R}^3$ through angle $\theta \in \mathbb{R}$
- The axis-angle parametrization can be combined in a single rotation vector $\boldsymbol{\theta} := \theta \boldsymbol{\eta} \in \mathbb{R}^3$
- **Exponential map** $\exp : \mathfrak{so}(3) \mapsto SO(3)$ maps a skew-symmetric matrix $\hat{\boldsymbol{\theta}}$ obtained from an axis-angle vector $\boldsymbol{\theta}$ to a rotation matrix R:

$$R = \exp(\hat{\boldsymbol{\theta}}) := \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\boldsymbol{\theta}}^n = I + \hat{\boldsymbol{\theta}} + \frac{1}{2!} \hat{\boldsymbol{\theta}}^2 + \frac{1}{3!} \hat{\boldsymbol{\theta}}^3 + \ldots$$

- The matrix exponential defines a map from the space of skew-symmetric matrices $\mathfrak{so}(3)$ to the space of rotation matrices $SO(3)$
  - The exponential map is **surjective** but **not injective**: every element of $SO(3)$ can be generated from multiple elements of $\mathfrak{so}(3)$, e.g., any vector $(\|\boldsymbol{\theta}\| + 2\pi k) \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$ for integer k leads to the same R
  - The exponential map is **not commutative**: $e^{\hat{\boldsymbol{\theta}}_1} e^{\hat{\boldsymbol{\theta}}_2} \neq e^{\hat{\boldsymbol{\theta}}_2} e^{\hat{\boldsymbol{\theta}}_1} \neq e^{\hat{\boldsymbol{\theta}}_1 + \hat{\boldsymbol{\theta}}_2}$, unless $\hat{\boldsymbol{\theta}}_1 \hat{\boldsymbol{\theta}}_2 - \hat{\boldsymbol{\theta}}_2 \hat{\boldsymbol{\theta}}_1 = 0$

Rodrigues Formula

- **Rodrigues Formula**: closed-from expression for the exponential map from $\mathfrak{so}(3)$ to $SO(3)$:

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2$$

- The formula is derived using that $\hat{\boldsymbol{\theta}}^{2n+1} = (-\boldsymbol{\theta}^\top\boldsymbol{\theta})^n\hat{\boldsymbol{\theta}}$:

$$\begin{aligned}
\exp(\hat{\boldsymbol{\theta}}) &= I + \sum_{n=1}^{\infty}\frac{1}{n!}\hat{\boldsymbol{\theta}}^n \\
&= I + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\hat{\boldsymbol{\theta}}^{2n+1} + \sum_{n=0}^{\infty}\frac{1}{(2n+2)!}\hat{\boldsymbol{\theta}}^{2n+2} \\
&= I + \left(\sum_{n=0}^{\infty}\frac{(-1)^n\|\boldsymbol{\theta}\|^{2n}}{(2n+1)!}\right)\hat{\boldsymbol{\theta}} + \left(\sum_{n=0}^{\infty}\frac{(-1)^n\|\boldsymbol{\theta}\|^{2n}}{(2n+2)!}\right)\hat{\boldsymbol{\theta}}^2 \\
&= I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2
\end{aligned}$$

Logarithm Map from $SO(3)$ to $\mathfrak{so}(3)$

- $\forall R \in SO(3)$, there exists a (non-unique) $\boldsymbol{\theta} \in \mathbb{R}^3$ such that $R = \exp(\hat{\boldsymbol{\theta}})$
- **Logarithm map** $\log : SO(3) \to \mathfrak{so}(3)$ is the inverse of $\exp(\hat{\boldsymbol{\theta}})$:

$$\theta = \|\boldsymbol{\theta}\| = \arccos\left(\frac{\operatorname{tr}(R) - 1}{2}\right)$$

$$\boldsymbol{\eta} = \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} = \frac{1}{2\sin(\|\boldsymbol{\theta}\|)}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

$$\hat{\boldsymbol{\theta}} = \log(R) = \frac{\|\boldsymbol{\theta}\|}{2\sin\|\boldsymbol{\theta}\|}(R - R^\top)$$

- If $R = I$, then $\theta = 0$ and $\boldsymbol{\eta}$ is undefined
- If $\operatorname{tr}(R) = -1$, then $\theta = \pi$ and for any $i \in \{1, 2, 3\}$:

$$\boldsymbol{\eta} = \frac{1}{\sqrt{2(1 + R_{ii})}}(I + R)e_i$$

- The matrix exponential "integrates" $\hat{\boldsymbol{\theta}} \in \mathfrak{so}(3)$ for one second; the matrix logarithm "differentiates" $R \in SO(3)$ to obtain $\hat{\boldsymbol{\theta}} \in \mathfrak{so}(3)$

Outline

Quaternions

- **Quaternions**: $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ generalize complex numbers $\mathbb{C} = \mathbb{R} + \mathbb{R}i$

$$\mathbf{q} = q_s + q_1 i + q_2 j + q_3 k = [q_s,\ \mathbf{q}_v] \qquad ij = -ji = k,\ i^2 = j^2 = k^2 = -1$$

- As in 2-D, 3-D rotations can be represented using "unit complex numbers", i.e., **unit-norm quaternions**:

$$\mathbb{H}_* := \{\mathbf{q} \in \mathbb{H} \mid q_s^2 + \mathbf{q}_v^T \mathbf{q}_v = 1\}$$

- To represent rotations without singularities, we embed a 3-D space $SO(3)$ into a 4-D space $\mathbb{H}$ and introduce a unit-norm constraint

- A rotation matrix $R \in SO(3)$ can be obtained from a unit quaternion $\mathbf{q}$:

$$R(\mathbf{q}) = E(\mathbf{q})G(\mathbf{q})^\top \qquad \begin{aligned} E(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \\ G(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v] \end{aligned}$$

- The space of quaternions $\mathbb{H}_*$ is a **double covering** of $SO(3)$ because two unit quaternions correspond to the same rotation: $R(\mathbf{q}) = R(-\mathbf{q})$

Quaternion Axis-Angle Parametrization

- A rotation around a unit axis $\boldsymbol{\eta} := \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} \in \mathbb{R}^3$ by angle $\theta := \|\boldsymbol{\theta}\|$ can be represented by a unit quaternion:

$$\mathbf{q} = \left[\cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)\boldsymbol{\eta}\right] \in \mathbb{H}_*$$

- A rotation around a unit axis $\boldsymbol{\eta} \in \mathbb{R}^3$ by angle θ can be recovered from a unit quaternion $\mathbf{q} = [q_s,\ \mathbf{q}_v] \in \mathbb{H}_*$:

$$\theta = 2\arccos(q_s) \qquad \boldsymbol{\eta} = \begin{cases} \frac{1}{\sin(\theta/2)}\mathbf{q}_v, & \text{if } \theta \neq 0 \\ 0, & \text{if } \theta = 0 \end{cases}$$

- The inverse transformation above has a singularity at $\theta = 0$ because the transformation from $\boldsymbol{\theta}$ to $\mathbf{q}$ is many-to-one and there are infinitely many rotation axes that can be used

Quaternion Operations

Addition	$\mathbf{q} + \mathbf{p} := [q_s + p_s,\ \mathbf{q}_v + \mathbf{p}_v]$
Multiplication	$\mathbf{q} \circ \mathbf{p} := \left[q_s p_s - \mathbf{q}_v^T \mathbf{p}_v,\ q_s \mathbf{p}_v + p_s \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v\right]$
Conjugation	$\bar{\mathbf{q}} := [q_s,\ -\mathbf{q}_v]$
Norm	$\|\mathbf{q}\| := \sqrt{q_s^2 + \mathbf{q}_v^T \mathbf{q}_v} \qquad \|\mathbf{q} \circ \mathbf{p}\| = \|\mathbf{q}\|\|\mathbf{p}\|$
Inverse	$\mathbf{q}^{-1} := \frac{\bar{\mathbf{q}}}{\|\mathbf{q}\|^2}$
Rotation	$[0,\ \mathbf{x}'] = \mathbf{q} \circ [0,\ \mathbf{x}] \circ \mathbf{q}^{-1} = [0,\ R(\mathbf{q})\mathbf{x}]$
Velocity	$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q} \circ [0,\ \boldsymbol{\omega}] = \frac{1}{2} G(\mathbf{q})^\top \boldsymbol{\omega}$
Exp	$\exp(\mathbf{q}) := e^{q_s} \left[\cos\|\mathbf{q}_v\|,\ \frac{\mathbf{q}_v}{\|\mathbf{q}_v\|} \sin\|\mathbf{q}_v\|\right]$
Log	$\log(\mathbf{q}) := \left[\log\|\mathbf{q}\|,\ \frac{\mathbf{q}_v}{\|\mathbf{q}_v\|} \arccos \frac{q_s}{\|\mathbf{q}\|}\right]$

- **Exp**: constructs $\mathbf{q} \in \mathbb{H}_*$ from rotation vector $\boldsymbol{\theta} \in \mathbb{R}^3$: $\mathbf{q} = \exp\left(\left[0,\ \frac{\boldsymbol{\theta}}{2}\right]\right)$
- **Log**: recovers a rotation vector $\boldsymbol{\theta} \in \mathbb{R}^3$ from $\mathbf{q} \in \mathbb{H}_*$: $[0,\ \boldsymbol{\theta}] = 2\log(\mathbf{q})$

Quaternion Multiplication and Rotation

- Quaternion multiplication: $\mathbf{q} \circ \mathbf{p} := \left[q_s p_s - \mathbf{q}_v^T \mathbf{p}_v,\ q_s \mathbf{p}_v + p_s \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v\right]$
- Quaternion multiplication $\mathbf{q} \circ \mathbf{p}$ can be represented using linear operations:

$$\begin{aligned} \mathbf{q} \circ \mathbf{p} &= [\mathbf{q}]_L\, \mathbf{p} = [\mathbf{p}]_R\, \mathbf{q} \\ [\mathbf{q}]_L &:= \begin{bmatrix} \mathbf{q} & G(\mathbf{q})^\top \end{bmatrix} \qquad G(\mathbf{q}) = [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v] \\ [\mathbf{q}]_R &:= \begin{bmatrix} \mathbf{q} & E(\mathbf{q})^\top \end{bmatrix} \qquad E(\mathbf{q}) = [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \end{aligned}$$

- Rotating a vector $\mathbf{x} \in \mathbb{R}^3$ by quaternion $\mathbf{q} \in \mathbb{H}_*$ is performed as:

$$\mathbf{q} \circ [0,\ \mathbf{x}] \circ \mathbf{q}^{-1} = [0,\ \mathbf{x}'] = [0,\ R(\mathbf{q})\mathbf{x}]$$

- This provides the relationship between a quaternion $\mathbf{q}$ and its corresponding rotation matrix $R(\mathbf{q})$:

$$\begin{aligned} \begin{bmatrix} 0 \\ R(\mathbf{q})\mathbf{x} \end{bmatrix} &= \mathbf{q} \circ [0,\ \mathbf{x}] \circ \mathbf{q}^{-1} = [\bar{\mathbf{q}}]_R\, [\mathbf{q}]_L \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \\ &= \begin{bmatrix} \bar{\mathbf{q}} & E(\bar{\mathbf{q}})^\top \end{bmatrix} \begin{bmatrix} \mathbf{q} & G(\mathbf{q})^\top \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^\top \\ E(\mathbf{q}) \end{bmatrix} \begin{bmatrix} \mathbf{q} & G(\mathbf{q})^\top \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{q}^\top \mathbf{q} & \mathbf{q}^\top G(\mathbf{q})^\top \\ E(\mathbf{q})\mathbf{q} & E(\mathbf{q}) G(\mathbf{q})^\top \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^\top G(\mathbf{q})^\top \mathbf{x} \\ E(\mathbf{q}) G(\mathbf{q})^\top \mathbf{x} \end{bmatrix} \end{aligned}$$

Example: Rotation with a Quaternion

- Let $\mathbf{x} = \mathbf{e}_2$ be a point in frame $\{A\}$
- What are the coordinates of $\mathbf{x}$ in frame $\{B\}$ which is rotated by $\theta = \pi/3$ with respect to $\{A\}$ around the x-axis?
- The quaternion corresponding to the rotation from $\{B\}$ to $\{A\}$ is:

$$ {}_A\mathbf{q}_B = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\boldsymbol{\eta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ \mathbf{e}_1 \end{bmatrix} $$

- The quaternion corresponding to the rotation from $\{A\}$ to $\{B\}$ is:

$$ {}_B\mathbf{q}_A = {}_A\mathbf{q}_B^{-1} = {}_A\bar{\mathbf{q}}_B = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -\mathbf{e}_1 \end{bmatrix} $$

- The coordinates of $\mathbf{x}$ in frame $\{B\}$ are:

$$ \begin{aligned} {}_B\mathbf{q}_A \circ [0,\ \mathbf{x}] \circ {}_B\mathbf{q}_A^{-1} &= \frac{1}{4}\begin{bmatrix} \sqrt{3} \\ -\mathbf{e}_1 \end{bmatrix} \circ \begin{bmatrix} 0 \\ \mathbf{e}_2 \end{bmatrix} \circ \begin{bmatrix} \sqrt{3} \\ \mathbf{e}_1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 0 \\ \sqrt{3}\mathbf{e}_2 - \mathbf{e}_1 \times \mathbf{e}_2 \end{bmatrix} \circ \begin{bmatrix} \sqrt{3} \\ \mathbf{e}_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ \mathbf{e}_2 - \sqrt{3}\mathbf{e}_3 \end{bmatrix} \end{aligned} $$

Representations of Orientation (Summary)

- **Rotation Matrix**: an element of the **Special Orthogonal Group**:

$$R \in SO(3) := \left\{ R \in \mathbb{R}^{3\times3} \;\middle|\; \underbrace{R^\top R = I}_{\text{distances preserved}}, \underbrace{\det(R) = 1}_{\text{no reflection}} \right\}$$

- **Euler Angles**: roll ϕ, pitch θ, yaw ψ specifying a **sxyz** or **rzyx** rotation:

$$R = R_z(\psi)R_y(\theta)R_x(\phi)$$

- **Axis-Angle**: $\boldsymbol{\theta} \in \mathbb{R}^3$ specifying rotation about axis $\boldsymbol{\eta} := \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$ through angle $\theta := \|\boldsymbol{\theta}\|$:

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \hat{\boldsymbol{\theta}} + \frac{1}{2!}\hat{\boldsymbol{\theta}}^2 + \frac{1}{3!}\hat{\boldsymbol{\theta}}^3 + \ldots = I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2$$

- **Unit Quaternion**: $\mathbf{q} = [q_s,\ \mathbf{q}_v] \in \mathbb{H}_* := \left\{\mathbf{q} \in \mathbb{H} \mid q_s^2 + \mathbf{q}_v^\top \mathbf{q}_v = 1\right\}$:

$$R = E(\mathbf{q})G(\mathbf{q})^\top \qquad \begin{aligned} E(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \\ G(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v] \end{aligned}$$

Outline

Rigid Body Pose

- Let $\{B\}$ be a body frame whose position and orientation with respect to the world frame $\{W\}$ are $\mathbf{p} \in \mathbb{R}^3$ and $R \in SO(3)$, respectively

- The coordinates of a point $\mathbf{s}_B \in \mathbb{R}^3$ can be converted to the world frame by first rotating the point and then translating it to the world frame:

$$\mathbf{s}_W = R\mathbf{s}_B + \mathbf{p}$$

- The **homogeneous coordinates** of a point $\mathbf{s} \in \mathbb{R}^3$ are

$$\underline{\mathbf{s}} := \lambda \begin{bmatrix} \mathbf{s} \\ 1 \end{bmatrix} \propto \begin{bmatrix} \mathbf{s} \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

  where the scale factor λ allows representing points arbitrarily far away from the origin as $\lambda \to 0$, e.g., $\underline{\mathbf{s}} = \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}^\top$

- Rigid-body transformations are linear in homogeneous coordinates:

$$\underline{\mathbf{s}}_W = \begin{bmatrix} \mathbf{s}_W \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{s}_B \\ 1 \end{bmatrix} = T\underline{\mathbf{s}}_B$$

Special Euclidean Group $SE(3)$

- The pose of a rigid body can be described by a matrix T in the **special Euclidean group**:

$$SE(3) := \left\{ T = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \;\middle|\; R \in SO(3), \mathbf{p} \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times4}$$

- The pose of a rigid body T specifies a transformation from the body frame $\{B\}$ to the world frame $\{W\}$:

$${}_{\{W\}}T_{\{B\}} := \begin{bmatrix} {}_{\{W\}}R_{\{B\}} & {}_{\{W\}}\mathbf{p}_{\{B\}} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- A point with body-frame coordinates $\mathbf{s}_B$, has world-frame coordinates:

$$\mathbf{s}_W = R\mathbf{s}_B + \mathbf{p} \quad \text{equivalent to} \quad \begin{bmatrix} \mathbf{s}_W \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{s}_B \\ 1 \end{bmatrix}$$

- A point with world-frame coordinates $\mathbf{s}_W$, has body-frame coordinates:

$$\begin{bmatrix} \mathbf{s}_B \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{s}_W \\ 1 \end{bmatrix} = \begin{bmatrix} R^\top & -R^\top\mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{s}_W \\ 1 \end{bmatrix}$$

Composing Transformations

- Given a robot with pose ${}_{\{W\}}T_{\{1\}}$ at time t_1 and ${}_{\{W\}}T_{\{2\}}$ at time t_2, the relative transformation from inertial frame $\{2\}$ at time t_2 to inertial frame $\{1\}$ at time t_1 is:

$$
\begin{aligned}
{}_{\{1\}}T_{\{2\}} &= {}_{\{1\}}T_{\{W\}}\ {}_{\{W\}}T_{\{2\}} = \left({}_{\{W\}}T_{\{1\}}\right)^{-1}\ {}_{\{W\}}T_{\{2\}} \\
&= \begin{bmatrix} {}_{\{W\}}R_{\{1\}}^\top & -{}_{\{W\}}R_{\{1\}}^\top \times {}_{\{W\}}\mathbf{p}_{\{1\}} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} {}_{\{W\}}R_{\{2\}} & {}_{\{W\}}\mathbf{p}_{\{2\}} \\ \mathbf{0}^\top & 1 \end{bmatrix}
\end{aligned}
$$

- The pose T_k of a robot at time t_k always specifies a transformation from the body frame at time t_k to the world frame so we will not explicitly write the world frame subscript

- The relative transformation from inertial frame $\{2\}$ with world-frame pose T_2 to an inertial frame $\{1\}$ with world-frame pose T_1 is:

$$
{}_1T_2 = T_1^{-1}T_2
$$

Summary

	Rotation $SO(3)$	**Pose** $SE(3)$
Representation	$R : \begin{cases} R^T R = I \\ \det(R) = 1 \end{cases}$	$T = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Transformation	$\mathbf{s}_W = R\mathbf{s}_B$	$\mathbf{s}_W = R\mathbf{s}_B + \mathbf{p}$
Inverse	$R^{-1} = R^\top$	$T^{-1} = \begin{bmatrix} R^\top & -R^\top \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Composition	${}_W R_B = {}_W R_A \, {}_A R_B$	${}_W T_B = {}_W T_A \, {}_A T_B$