ECE276A: Sensing & Estimation in Robotics
Lecture 5: Localization and Odometry from Point Features

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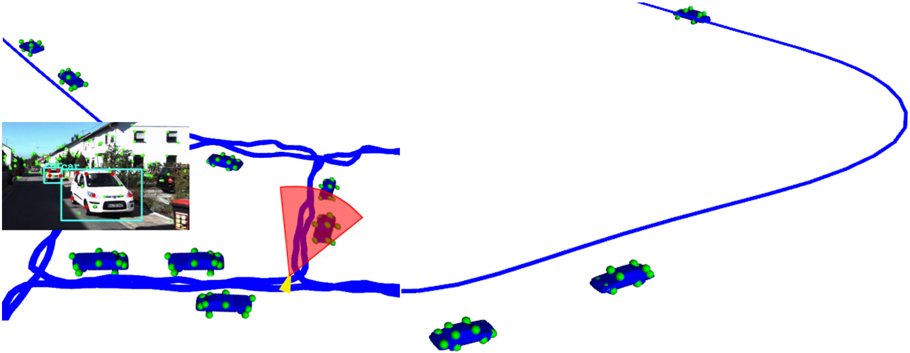
Outline

Simultaneous Localization and Mapping (SLAM)

- SLAM is a fundamental problem for mobile robot autonomy
- Basic information necessary to perform any robot task:
  - Where am I? $\Rightarrow$ **Localization**
  - What is around me? $\Rightarrow$ **Mapping**
- SLAM problem: given sensor measurements $\mathbf{z}_{0:T}$ (e.g., images) and control inputs $\mathbf{u}_{0:T-1}$ (e.g., velocity), estimate the robot state trajectory $\mathbf{x}_{0:T}$ (e.g., pose) and build a map $\mathbf{m}$ of the environment



Mathematical Formulation of SLAM Problems

- **Mapping**: given robot state trajectory $\mathbf{x}_{0:T}$ and sensor measurements $\mathbf{z}_{0:T}$ with observation model h, build a map $\mathbf{m}$ of the environment

$$\min_{\mathbf{m}} \sum_{t=0}^{T} \|\mathbf{z}_t - h(\mathbf{x}_t, \mathbf{m})\|_2^2$$

- **Localization**: given a map $\mathbf{m}$ of the environment, sensor measurements $\mathbf{z}_{0:T}$ with observation model h, and control inputs $\mathbf{u}_{0:T-1}$ with motion model f, estimate the robot state trajectory $\mathbf{x}_{0:T}$

$$\min_{\mathbf{x}_{0:T}} \sum_{t=0}^{T} \|\mathbf{z}_t - h(\mathbf{x}_t, \mathbf{m})\|_2^2 + \sum_{t=0}^{T-1} \|\mathbf{x}_{t+1} - f(\mathbf{x}_t, \mathbf{u}_t)\|_2^2$$

- **SLAM**: given initial robot state $\mathbf{x}_0$, sensor measurements $\mathbf{z}_{1:T}$ with observation model h, and control inputs $\mathbf{u}_{0:T-1}$ with motion model f, estimate the robot state trajectory $\mathbf{x}_{1:T}$ and build a map $\mathbf{m}$

$$\min_{\mathbf{x}_{1:T}, \mathbf{m}} \sum_{t=1}^{T} \|\mathbf{z}_t - h(\mathbf{x}_t, \mathbf{m})\|_2^2 + \sum_{t=0}^{T-1} \|\mathbf{x}_{t+1} - f(\mathbf{x}_t, \mathbf{u}_t)\|_2^2$$

Example: Localization with Linear Models

- State: $\mathbf{x}_t \in \mathbb{R}^n$
- Motion model: $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t) = F\mathbf{x}_t + G\mathbf{u}_t$
- Observation model: $\mathbf{z}_t = h(\mathbf{x}_t) = H\mathbf{x}_t$
- Localization: given $\mathbf{x}_0 = \mathbf{0}$, sensor measurements $\mathbf{z}_{1:T}$, and control inputs $\mathbf{u}_{0:T-1}$, estimate the state trajectory $\mathbf{x}_{1:T}$

$$\min_{\mathbf{x}_{1:T}} c(\mathbf{x}_{1:T}) := \sum_{t=1}^{T} \|\mathbf{z}_t - H\mathbf{x}_t\|_2^2 + \sum_{t=0}^{T-1} \|\mathbf{x}_{t+1} - F\mathbf{x}_t - G\mathbf{u}_t\|_2^2$$

- Gradient descent: initialize $\mathbf{x}_{1:T}^{(0)}$ and iterate:

$$\mathbf{x}_{1:T}^{(k+1)} = \mathbf{x}_{1:T}^{(k)} - \alpha^{(k)} \nabla c(\mathbf{x}_{1:T}^{(k)})$$

Example: Localization with Linear Models

- $\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_2^2 = \|x_1 - y_1\|_2^2 + \|x_2 - y_2\|_2^2$ for $x_1, y_1 \in \mathbb{R}^{d_1}$, $x_2, y_2 \in \mathbb{R}^{d_2}$
- Express the least-squares localization problem in matrix notation:

$$
\begin{aligned}
c(\mathbf{x}_{1:T}) &= \sum_{t=1}^{T} \|\mathbf{z}_t - H\mathbf{x}_t\|_2^2 + \sum_{t=0}^{T-1} \|\mathbf{x}_{t+1} - F\mathbf{x}_t - G\mathbf{u}_t\|_2^2 \\
&= \left\| \begin{bmatrix} \mathbf{z}_1 - H\mathbf{x}_1 \\ \vdots \\ \mathbf{z}_T - H\mathbf{x}_T \end{bmatrix} \right\|_2^2 + \left\| \begin{bmatrix} \mathbf{x}_1 - F\mathbf{x}_0 - G\mathbf{u}_0 \\ \vdots \\ \mathbf{x}_T - F\mathbf{x}_{T-1} - G\mathbf{u}_{T-1} \end{bmatrix} \right\|_2^2 \\
&= \left\| \begin{bmatrix} H & & \\ & \ddots & \\ & & H \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_T \end{pmatrix} - \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_T \end{bmatrix} \right\|_2^2 + \left\| \begin{bmatrix} -I & & & \\ F & \ddots & & \\ & \ddots & \ddots & \\ & & F & -I \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_T \end{pmatrix} + \begin{bmatrix} F\mathbf{x}_0 + G\mathbf{u}_0 \\ G\mathbf{u}_1 \\ \vdots \\ G\mathbf{u}_{T-1} \end{bmatrix} \right\|_2^2
\end{aligned}
$$

Example: Localization with Linear Models

- Objective:

$$c(\mathbf{x}_{1:T}) = \left\| \begin{bmatrix} H & & & \\ & \ddots & & \\ & & \ddots & \\ & & & H \\ -I & & & \\ F & \ddots & & \\ & \ddots & \ddots & \\ & & F & -I \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_T \end{pmatrix} + \begin{bmatrix} -\mathbf{z}_1 \\ \vdots \\ -\mathbf{z}_T \\ F\mathbf{x}_0 + G\mathbf{u}_0 \\ G\mathbf{u}_1 \\ \vdots \\ G\mathbf{u}_{T-1} \end{bmatrix} \right\|_2^2$$

$$= \|A\mathbf{x}_{1:T} + \mathbf{b}\|_2^2$$

- Gradient:

$$\nabla c(\mathbf{x}_{1:T}) = 2A^\top(A\mathbf{x}_{1:T} + \mathbf{b})$$

- Gradient descent: initialize $\mathbf{x}_{1:T}^{(0)}$ and iterate:

$$\mathbf{x}_{1:T}^{(k+1)} = \mathbf{x}_{1:T}^{(k)} - 2\alpha^{(k)} A^\top(A\mathbf{x}_{1:T}^{(k)} + \mathbf{b})$$

Project 1: Orientation Tracking

- Consider a rigid body undergoing pure rotation
- **State**: orientation $\mathbf{q}_t \in \mathbb{H}_*$ of the body frame relative to the world frame
- **Control**: body-frame angular velocity $\mathbf{u}_t \in \mathbb{R}^3$ obtained from gyroscope measurements in rad/sec during time interval τ_t
- **Motion model**: $\mathbf{q}_{t+1} = f(\mathbf{q}_t, \tau_t \mathbf{u}_t) := \mathbf{q}_t \circ \exp([0,\ \tau_t \mathbf{u}_t/2])$
- **Observation model**: body-frame acceleration $\mathbf{z}_t \in \mathbf{R}^3$ obtained from accelerometer measurements in m/sec^2 should approximately match the world-frame gravity acceleration $-g\mathbf{e}_3$:

$$\mathbf{z}_t = h(\mathbf{q}_t) := \bar{\mathbf{q}}_t \circ [0,\ -g\mathbf{e}_3] \circ \mathbf{q}_t$$

Project 1: Orientation Tracking

- Starting with $\mathbf{q}_0 = [1, \mathbf{0}] \in \mathbb{H}_*$, formulate an optimization problem to estimate $\mathbf{q}_{1:T}$ using the gyroscope inputs $\mathbf{u}_{0:T-1}$ and accelerometer measurements $\mathbf{z}_{1:T}$
- The optimization problem is **constrained** because we require that $\mathbf{q}_t$ is a valid orientation, i.e., $\mathbf{q}_t \in \mathbb{H}_*$:

$$\begin{aligned} \min_{\mathbf{q}_{1:T}} \;\; & c(\mathbf{q}_{1:T}) := \sum_{t=1}^{T} \|\mathbf{z}_t - h(\mathbf{q}_t)\|_2^2 + \sum_{t=0}^{T-1} \|2 \log\left(\mathbf{q}_{t+1}^{-1} \circ f(\mathbf{q}, \tau_t \mathbf{u}_t)\right)\|_2^2 \\ \text{s.t.} \;\; & \|\mathbf{q}_t\|_2 = 1, \;\; \forall t \end{aligned}$$

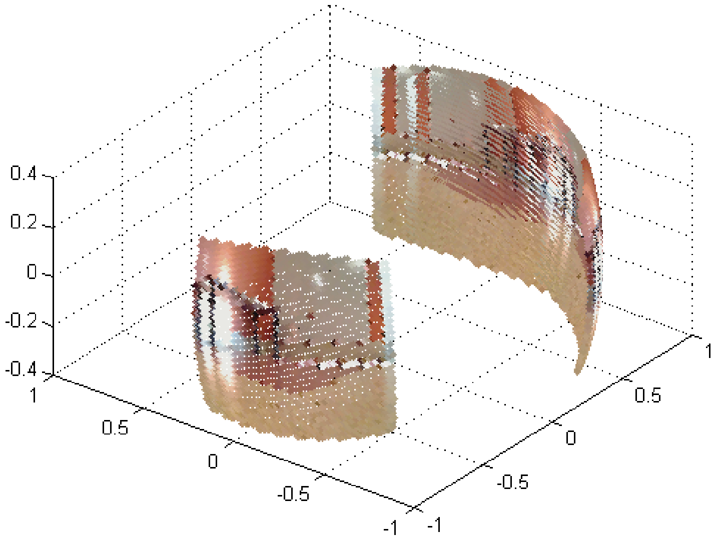
- **Possible approach**: projected gradient descent

$$\mathbf{q}_{1:T}^{(k+1)} = \Pi_{\mathbb{H}_*}\left(\mathbf{q}_{1:T}^{(k)} - \alpha^{(k)} \nabla c(\mathbf{q}_{1:T}^{(k)})\right)$$

Project 1: Panorama

- **Input**: image I and camera-to-world orientation R
- Suppose the image lies on a sphere and compute the world coordinates of each pixel:
  1. Find longitude (λ) and latitude (ϕ) of each pixel using the number of rows and columns and the horizontal (60°) and vertical (45°) fields of view
  2. Convert spherical $(\lambda, \phi, 1)$ to Cartesian coordinates assuming depth 1
  3. Rotate the Cartesian coordinates to the world frame using R
- Project world pixel coordinates to a cylinder and unwrap:
  1. Convert Cartesian to spherical coordinates
  2. Inscribe the sphere in a cylinder so that a point $(\lambda, \phi, 1)$ on the sphere has height ϕ on the cylinder and longitude λ along the cylinder circumference
  3. Unwrap the cylinder surface to a rectangular image with width 2π radians and height π radians
  4. Different options for sphere to plane projection: equidistant, equal area, Miller, etc. (see `https://en.wikipedia.org/wiki/List_of_map_projections`)

Project 1: Panorama



Outline

Localization and Odometry from Point Features

- **Point-cloud map**: suppose the map is represented as a set $\{\mathbf{m}_i\}_i$ of points $\mathbf{m}_i \in \mathbb{R}^d$
- **Observation model**: relates an observation $\mathbf{z}_i$ obtained from robot position $\mathbf{p}$ and orientation θ or R with the point $\mathbf{m}_i$ that generated it:
  - **Position Sensor**: $\mathbf{z}_i = R^\top(\mathbf{m}_i - \mathbf{p})$
  - **Range Sensor**: $\mathbf{z}_i = \|\mathbf{m}_i - \mathbf{p}\|_2$
  - **Bearing Sensor**: $\mathbf{z}_i = \arctan\left(\frac{m_{i,y} - p_y}{m_{i,x} - p_x}\right) - \theta$
  - **Camera Sensor**: $\mathbf{z}_i = K\pi\left(R^\top(\mathbf{m}_i - \mathbf{p})\right)$
- **Localization Problem**: Given landmark positions $\{\mathbf{m}_i\}_i$ and measurements $\{\mathbf{z}_i\}_i$ at one time instance, determine the global robot position $\mathbf{p}$ and orientation θ or R
- **Odometry Problem**: Given measurements $\mathbf{z}_{t,i}$, $\mathbf{z}_{t+1,i}$ at two time instances, determine the relative position ${}_t\mathbf{p}_{t+1}$ and orientation ${}_t\theta_{t+1}$ or ${}_tR_{t+1}$ between the two robot frames at time t and $t+1$

2-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ (world frame) and **relative position** measurements (body frame):

$$\mathbf{z}_i = R^\top(\theta)(\mathbf{m}_i - \mathbf{p}) \in \mathbb{R}^2, \quad i = 1, 2$$

- Let $\delta\mathbf{z} := \mathbf{z}_1 - \mathbf{z}_2$ and $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so that:

$$\mathbf{m}_1 - \mathbf{m}_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (\mathbf{z}_1 - \mathbf{z}_2) = \begin{bmatrix} \delta\mathbf{z} & J\delta\mathbf{z} \end{bmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

- As long as $\det \begin{bmatrix} \delta\mathbf{z} & J\delta\mathbf{z} \end{bmatrix} = \|\delta\mathbf{z}\|_2^2 = \|\mathbf{m}_1 - \mathbf{m}_2\|_2^2 \neq 0$, we can compute:

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{\|\delta\mathbf{z}\|_2^2} \begin{bmatrix} \delta z_x & \delta z_y \\ -\delta z_y & \delta z_x \end{bmatrix} (\mathbf{m}_1 - \mathbf{m}_2) \quad \Rightarrow \quad \theta = \mathbf{atan2}(\sin\theta, \cos\theta)$$

- Given the orientation θ, we can then obtain the robot position:

$$\mathbf{p} = \frac{1}{2}\left((\mathbf{m}_1 + \mathbf{m}_2) - R(\theta)(\mathbf{z}_1 + \mathbf{z}_2)\right)$$

3-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 \in \mathbb{R}^3$ (world frame) and **relative position** measurements (body frame):

$$\mathbf{z}_i = R^\top(\mathbf{m}_i - \mathbf{p}) \in \mathbb{R}^3, \quad i = 1, 2, 3$$

- Let $\mathbf{m}_{ij} := \mathbf{m}_i - \mathbf{m}_j$ and $\mathbf{z}_{ij} = \mathbf{z}_i - \mathbf{z}_j$ and compute:

$$\mathbf{m}_{12} \times \mathbf{m}_{13} = (R\mathbf{z}_{12}) \times (R\mathbf{z}_{13}) = R(\mathbf{z}_{12} \times \mathbf{z}_{13})$$

- The vector $\mathbf{m}_{12} \times \mathbf{m}_{13}$ provides orthogonal information to $\mathbf{m}_1$ and $\mathbf{m}_2$ and can be used to estimate the orientation R **as long as the three points are not all on the same line**:

$$\begin{bmatrix}\mathbf{m}_{12} & \mathbf{m}_{13} & \mathbf{m}_{12} \times \mathbf{m}_{13}\end{bmatrix} = R\begin{bmatrix}\mathbf{z}_{12} & \mathbf{z}_{13} & \mathbf{z}_{12} \times \mathbf{z}_{13}\end{bmatrix}$$
$$R = \begin{bmatrix}\mathbf{m}_{12} & \mathbf{m}_{13} & \mathbf{m}_{12} \times \mathbf{m}_{13}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{12} & \mathbf{z}_{13} & \mathbf{z}_{12} \times \mathbf{z}_{13}\end{bmatrix}^{-1}$$

- Given the orientation R, we can then obtain the robot position:

$$\mathbf{p} = \frac{1}{3}\sum_{i=1}^{3}(\mathbf{m}_i - R\mathbf{z}_i)$$

3-D Localization from Relative Position Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: n landmark positions $\mathbf{m}_i \in \mathbb{R}^3$ (world frame) and **relative position** measurements (body frame):

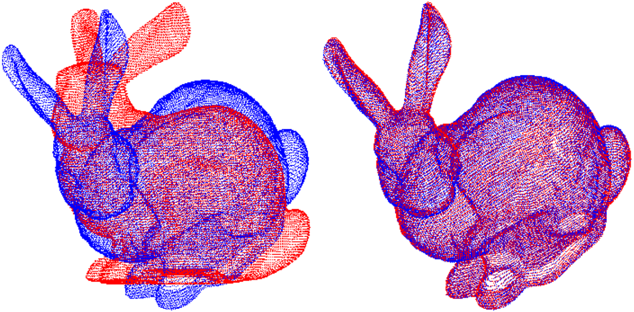
$$\mathbf{z}_i = R^\top (\mathbf{m}_i - \mathbf{p}) \in \mathbb{R}^3, \quad i = 1, \ldots, n$$

- Localization from relative position measurements is known as the **point cloud registration** problem
- Given two sets $\{\mathbf{m}_i\}$ and $\{\mathbf{z}_j\}$ of points, find the transformation $\mathbf{p}$, R that aligns them
- The **data association** $\Delta := \{(i, j) : \mathbf{m}_i \text{ corresponds to } \mathbf{z}_j\}$ that specifies which observation j corresponds to landmark i might not be available

Point Cloud Registration

- Given two sets $\{\mathbf{m}_i\}$ and $\{\mathbf{z}_j\}$ of points in $\mathbb{R}^d$, find the transformation $\mathbf{p} \in \mathbb{R}^d$, $R \in SO(d)$ and data association Δ that align them:

$$\min_{R \in SO(d), \mathbf{p} \in \mathbb{R}^d, \Delta} f(R, \mathbf{p}, \Delta) := \sum_{(i,j) \in \Delta} w_{ij} \|(R\mathbf{z}_j + \mathbf{p}) - \mathbf{m}_i\|_2^2$$



Known Data Association: Kabsch Algorithm

- Find the transformation $\mathbf{p} \in \mathbb{R}^d$, $R \in SO(d)$ between sets $\{\mathbf{m}_i\}$ and $\{\mathbf{z}_i\}$ of associated points:

$$\min_{R \in SO(d), \mathbf{p} \in \mathbb{R}^d} f(R, \mathbf{p}) := \sum_i w_i \|(R\mathbf{z}_i + \mathbf{p}) - \mathbf{m}_i\|_2^2$$

- The optimal translation is obtained by setting $\nabla_{\mathbf{p}} f(R, \mathbf{p})$ to zero:

$$\mathbf{0} = \nabla_{\mathbf{p}} f(R, \mathbf{p}) = 2 \sum_i w_i \left((R\mathbf{z}_i + \mathbf{p}) - \mathbf{m}_i\right)$$

- Let the point cloud centroids be:

$$\bar{\mathbf{m}} := \frac{\sum_i w_i \mathbf{m}_i}{\sum_i w_i} \qquad \bar{\mathbf{z}} := \frac{\sum_i w_i \mathbf{z}_i}{\sum_i w_i}$$

- Solving $\nabla_{\mathbf{p}} f(R, \mathbf{p}) = \mathbf{0}$ for $\mathbf{p}$ leads to:

$$\mathbf{p} = \bar{\mathbf{m}} - R\bar{\mathbf{z}}$$

Known Data Association: Kabsch Algorithm

- Replace $\mathbf{p} = \bar{\mathbf{m}} - R\bar{\mathbf{z}}$ in $f(R, \mathbf{p})$:

$$f(R, \bar{\mathbf{m}} - R\bar{\mathbf{z}}) = \sum_i w_i \| R(\mathbf{z}_i - \bar{\mathbf{z}}) - (\mathbf{m}_i - \bar{\mathbf{m}}) \|_2^2$$

- Define the centered point clouds:

$$\delta\mathbf{m}_i := \mathbf{m}_i - \bar{\mathbf{m}} \qquad\qquad \delta\mathbf{z}_i := \mathbf{z}_i - \bar{\mathbf{z}}$$

- Finding the optimal rotation reduces to:

$$\min_{R \in SO(d)} \sum_i w_i \| R\delta\mathbf{z}_i - \delta\mathbf{m}_i \|_2^2$$

- The objective function can be simplified further:

$$\sum_i w_i \| R\delta\mathbf{z}_i - \delta\mathbf{m}_i \|_2^2 = \sum_i w_i \left(\delta\mathbf{z}_i^\top \underbrace{R^\top R}_{I} \delta\mathbf{z}_i - 2\delta\mathbf{m}_i^\top R\delta\mathbf{z}_i + \delta\mathbf{m}_i^\top \delta\mathbf{m}_i \right)$$

- Note that:
  - $\delta\mathbf{z}_i^\top \delta\mathbf{z}_i$ and $\delta\mathbf{m}_i^\top \delta\mathbf{m}_i$ are constant wrt R
  - $\sum_i w_i \delta\mathbf{m}_i^\top R\delta\mathbf{z}_i = \sum_i w_i \operatorname{tr}(\delta\mathbf{m}_i^\top R\delta\mathbf{z}_i) = \operatorname{tr}\left(\left(\sum_i w_i \delta\mathbf{z}_i \delta\mathbf{m}_i^\top \right) R \right)$

Known Data Association: Kabsch Algorithm

- **Wahba's problem**: to determine the rotation R that aligns two associated centered point clouds $\{\delta\mathbf{m}_i\}$ and $\{\delta\mathbf{z}_i\}$, we need to solve a linear optimization problem in $SO(d)$:

$$\max_{R \in SO(d)} \operatorname{tr}(Q^\top R)$$

  where $Q := \sum_i w_i \delta\mathbf{m}_i \delta\mathbf{z}_i^\top$

- Wahba's problem can be solved via the **Kabsch algorithm**

Known Data Association: Kabsch Algorithm

- **Wahba's problem**: $\max_{R \in SO(d)} \operatorname{tr}\left(Q^\top R\right)$
- **SVD**: let $Q = U\Sigma V^\top$ be the singular value decomposition of Q
- The singular vectors U, V and singular values Σ satisfy:

$$\Sigma_{ii} \geq 0 \qquad U^\top U = I \qquad \det(U) = \pm 1 \qquad V^\top V = I \qquad \det(V) = \pm 1$$

- Let $W := U^\top R V$ such that $W^\top W = I$ and $\det(W) = \pm 1$
- The columns $\mathbf{w}_j$ of W are orthonormal, $\mathbf{w}_j^\top \mathbf{w}_j = 1$, and hence:

$$1 = \mathbf{w}_j^\top \mathbf{w}_j = \sum_i W_{ij}^2 \qquad \Rightarrow \qquad W_{ij}^2 \leq 1 \qquad \Rightarrow \qquad |W_{ij}| \leq 1$$

- Since Σ is diagonal with $\Sigma_{ii} \geq 0$:

$$\operatorname{tr}(Q^\top R) = \operatorname{tr}(\Sigma U^\top R V) = \operatorname{tr}(\Sigma W) = \sum_i \Sigma_{ii} W_{ii} \leq \sum_i \Sigma_{ii}$$

- The maximum is achieved with $W = I$:

$$W = I \quad \Rightarrow \quad U^\top R V = I \quad \underset{\text{reflection}}{\overset{\text{avoids}}{\Rightarrow}} \quad R = U \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \det(UV^\top) \end{bmatrix} V^\top$$

Unknown Data Association: Iterative Closest Point (ICP)

- Find the transformation $\mathbf{p}$, R between sets $\{\mathbf{m}_i\}$ and $\{\mathbf{z}_j\}$ of points with **unknown** data association Δ

- **ICP algorithm**: iterates between finding associations Δ based on **closest points** and applying the **Kabsch algorithm** to determine $\mathbf{p}$, R

- Initialize with $\mathbf{p}_0$, R_0 (**sensitive to initial guess**) and iterate
  1. Given $\mathbf{p}_k$, R_k, find correspondences $(i, j) \in \Delta$ based on **closest points**:
  $$i \quad \leftrightarrow \quad \arg\min_j \|\mathbf{m}_i - (R_k \mathbf{z}_j + \mathbf{p}_k)\|_2^2$$
  2. Given correspondences $(i, j) \in \Delta$, find $\mathbf{p}_{k+1}$, R_{k+1} via **Kabsch algorithm**

Unknown Data Association: Probabilistic ICP

- Many variations for determining the data association Δ in ICP exist:
  - data association via point-to-plane distance (Chen & Medioni, 1991)
  - probabilistic data association (EM-ICP, Granger & Pennec, 2002)

- Place a probability density function π (e.g., Gaussian) at each $\mathbf{m}_i$ to define a mixture distribution for the data:

$$p(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i \pi(\mathbf{x}; \mathbf{m}_i, \sigma_i^2 I) \qquad \alpha_i \geq 0 \qquad \sum_{i=1}^{n} \alpha_i = 1$$

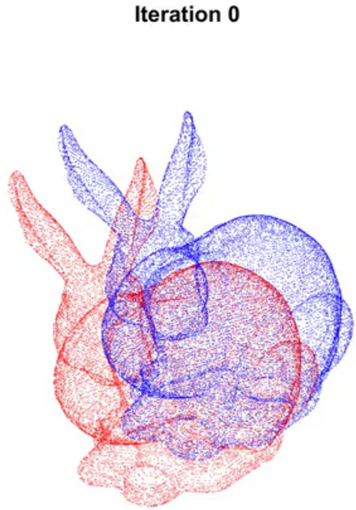
- Find parameters $\mathbf{p}$, R to maximize the likelihood of $\{R\mathbf{z}_j + \mathbf{p}\}_j$:

$$\max_{\mathbf{p},R} \sum_{j=1}^{m} \log \sum_{i=1}^{n} \alpha_i \pi(R\mathbf{z}_j + \mathbf{p}; \mathbf{m}_i, \sigma_i^2 I)$$

- Use **EM** to determine membership probabiliites (E step) and optimize the parameters $\mathbf{p}$, $\mathbf{R}$ (M step). ICP is a special case with $\sigma_i^2 \to 0$

- **Robustness**: use $\exp\left(-\frac{|\mathbf{x}-\mathbf{m}_i|^\beta}{2\sigma_i^2}\right)$ with $\beta \in (0,2)$ instead of $\exp\left(-\frac{|\mathbf{x}-\mathbf{m}_i|^2}{2\sigma_i^2}\right)$

Iterative Closest Point (ICP)



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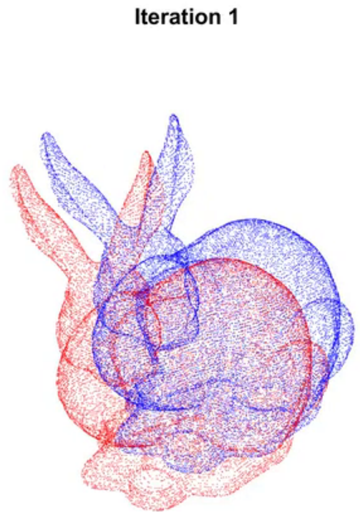

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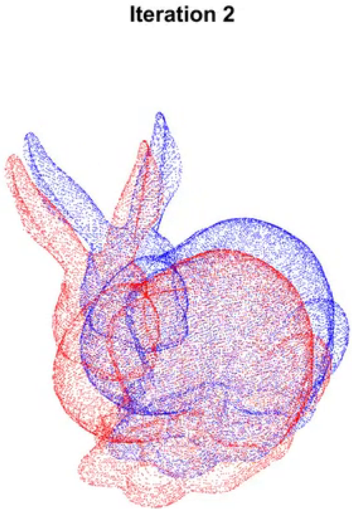

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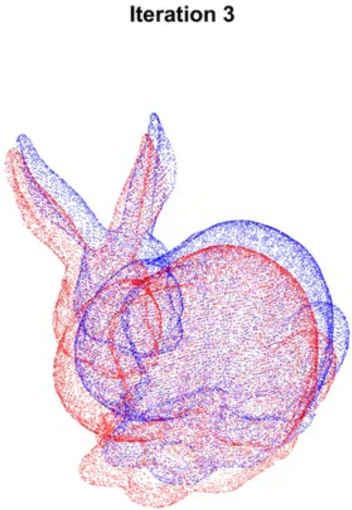

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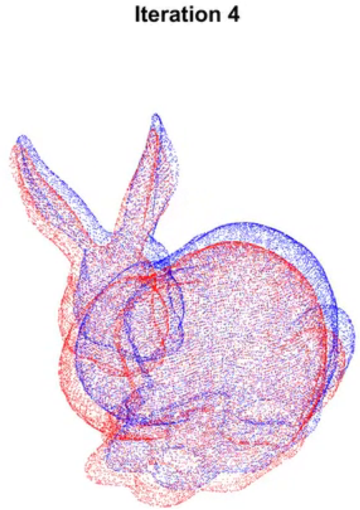

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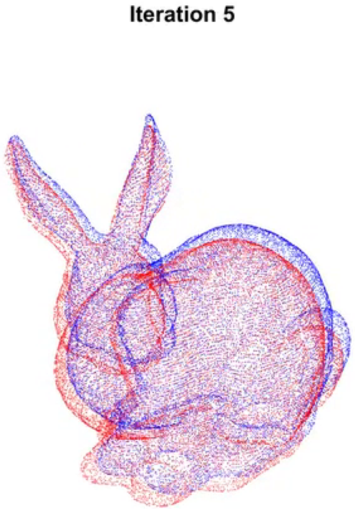

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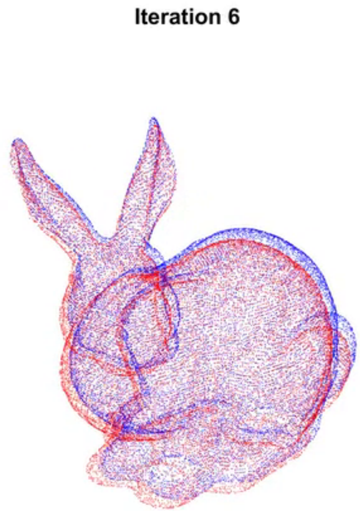

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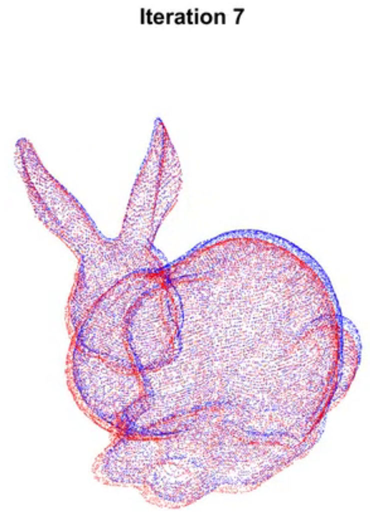

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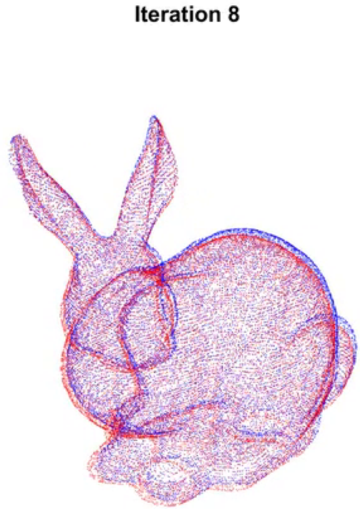

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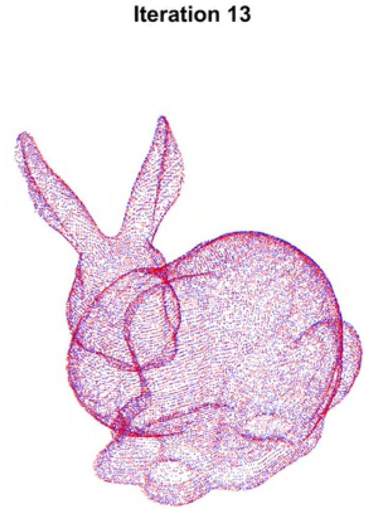

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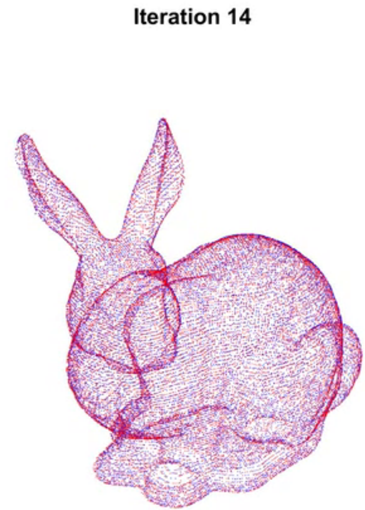

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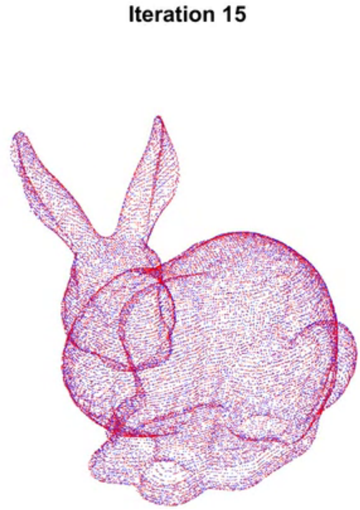

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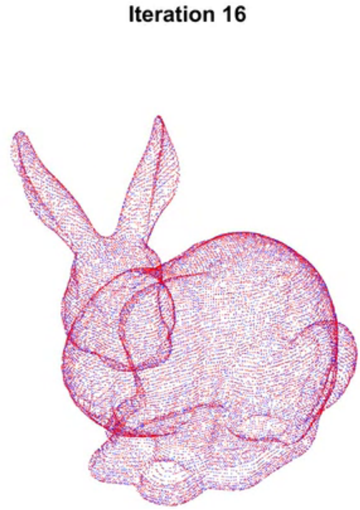


2-D Odometry from Relative Position Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: relative position measurements $\mathbf{z}_{t,1}, \mathbf{z}_{t,2} \in \mathbb{R}^2$ and $\mathbf{z}_{t+1,1}, \mathbf{z}_{t+1,2} \in \mathbb{R}^2$ at consecutive time steps to two **unknown** landmarks
- If we consider the robot frame at time t to be the "world frame", this problem is **the same as 2-D localization from relative position measurements** with $\mathbf{m}_i := \mathbf{z}_{t,i}$, $\mathbf{z}_i := \mathbf{z}_{t+1,i}$, $\mathbf{p} := {}_t\mathbf{p}_{t+1}$, $\theta := {}_t\theta_{t+1}$

3-D Odometry from Relative Position Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^3$ and ${}_tR_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t

- **Given**: relative position measurements $\mathbf{z}_{t,i} \in \mathbb{R}^3$ and $\mathbf{z}_{t+1,i} \in \mathbb{R}^3$ at consecutive time steps to n **unknown** landmarks

- If we consider the robot frame at time t to be the "world frame", this problem is **the same as 3-D localization from relative position measurements** with $\mathbf{m}_i := \mathbf{z}_{t,i}$, $\mathbf{z}_i := \mathbf{z}_{t+1,i}$, $\mathbf{p} := {}_t\mathbf{p}_{t+1}$, $R := {}_tR_{t+1}$

Summary: Rel. Position Measurements $\mathbf{z}_i = R^\top(\mathbf{m}_i - \mathbf{p})$

- **Localization**

$\mathbf{m}_1, \mathbf{m}_2, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^2$	$(\mathbf{m}_1 - \mathbf{m}_2) = R(\theta)(\mathbf{z}_1 - \mathbf{z}_2)$ $\mathbf{p} = \frac{1}{2}\sum_{i=1}^{2}(\mathbf{m}_i - R\mathbf{z}_i)$
$\mathbf{m}_1, \mathbf{z}_i \in \mathbb{R}^3,\ i = 1,2,3$ $\mathbf{m}_{ij} := \mathbf{m}_i - \mathbf{m}_j,\ \ \mathbf{z}_{ij} := \mathbf{z}_i - \mathbf{z}_j$	$\begin{bmatrix}\mathbf{m}_{12} & \mathbf{m}_{13} & \mathbf{m}_{12} \times \mathbf{m}_{13}\end{bmatrix} = R\begin{bmatrix}\mathbf{z}_{12} & \mathbf{z}_{13} & \mathbf{z}_{12} \times \mathbf{z}_{13}\end{bmatrix}$ $\mathbf{p} = \frac{1}{3}\sum_{i=1}^{3}(\mathbf{m}_i - R\mathbf{z}_i)$
$\mathbf{m}_i, \mathbf{z}_i \in \mathbb{R}^3,\ i = 1,\ldots,n$ $\delta\mathbf{m}_i := \mathbf{m}_i - \frac{1}{n}\sum_{j=1}^{n}\mathbf{m}_j,$ $\delta\mathbf{z}_i := \mathbf{z}_i - \frac{1}{n}\sum_{j=1}^{n}\mathbf{z}_j$	$R = \arg\max_{R \in SO(3)} \sum_{i=1}^{n} \delta\mathbf{m}_i^\top R \delta\mathbf{z}_i$ $\xlongequal[SVD\left(\sum_{i=1}^{n}\delta\mathbf{m}_i\delta\mathbf{z}_i^\top\right)=U\Sigma V^\top]{\text{Kabsch algorithm}} U\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \det(UV^\top)\end{bmatrix}V^\top$ $\mathbf{p} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{m}_i - R\mathbf{z}_i)$

- **Odometry**: same with $\mathbf{m}_i = \mathbf{z}_{t,i}$, $\mathbf{z}_i := \mathbf{z}_{t+1,i}$, $\mathbf{p} := {}_t\mathbf{p}_{t+1}$, $R := {}_tR_{t+1}$

Outline

2-D Localization from Bearing Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ (world frame) and **bearing** measurements (body frame):

$$z_i = \arctan\left(\frac{m_{i,y} - p_y}{m_{i,x} - p_x}\right) - \theta \in \mathbb{R}, \qquad i = 1, 2$$

- The bearing constraints are equivalent to:

$$\frac{\mathbf{m}_i - \mathbf{p}}{\|\mathbf{m}_i - \mathbf{p}\|_2} = \begin{bmatrix}\cos(z_i + \theta) \\ \sin(z_i + \theta)\end{bmatrix} = R(z_i + \theta)\mathbf{e}_1 \quad \Rightarrow \quad R^\top(z_i)(\mathbf{m}_i - \mathbf{p}) = \|\mathbf{m}_i - \mathbf{p}\|_2 \begin{bmatrix}\cos(\theta) \\ \sin(\theta)\end{bmatrix}$$

- To eliminate θ, the two constraints can be combined via:

$$\begin{aligned} 0 &= \|\mathbf{m}_1 - \mathbf{p}\|_2 \begin{bmatrix}\sin\theta & -\cos\theta\end{bmatrix} \begin{bmatrix}\cos(\theta) \\ \sin(\theta)\end{bmatrix} \|\mathbf{m}_2 - \mathbf{p}\|_2 \\ &= \|\mathbf{m}_1 - \mathbf{p}\|_2 \begin{bmatrix}\cos(\theta) \\ \sin(\theta)\end{bmatrix}^\top R\left(\frac{\pi}{2}\right) \begin{bmatrix}\cos(\theta) \\ \sin(\theta)\end{bmatrix} \|\mathbf{m}_2 - \mathbf{p}\|_2 \end{aligned}$$

2-D Localization from Bearing Measurements

- The previous equation is quadratic in $\mathbf{p}$:

$$(\mathbf{m}_1 - \mathbf{p})^\top R(z_1) R\left(\frac{\pi}{2}\right) R^\top(z_2)(\mathbf{m}_2 - \mathbf{p}) = 0$$

- Let $\eta := z_1 - z_2 + \pi/2$, so that:

$$\mathbf{p}^\top R(\eta)\mathbf{p} - \left(\mathbf{m}_1^\top R(\eta) + \mathbf{m}_2^\top R^\top(\eta)\right)\mathbf{p} + \mathbf{m}_1^\top R(\eta)\mathbf{m}_2 = 0$$

- Use the following to solve the quadratic equation:
  - $\mathbf{p}^\top R(\eta)\mathbf{p} = \cos(\eta)\mathbf{p}^\top\mathbf{p}$
  - $\mathbf{p}^\top\mathbf{p} + 2\mathbf{b}^\top\mathbf{p} + c = (\mathbf{p}+\mathbf{b})^\top(\mathbf{p}+\mathbf{b}) + c - \mathbf{b}^\top\mathbf{b}$

- As long as $\cos(\eta) \neq 0$, i.e., **the robot and the two landmarks are not on the same line**:

$$(\mathbf{p} - \mathbf{p}_0)^\top(\mathbf{p} - \mathbf{p}_0) = \left(\mathbf{p}_0^\top\mathbf{p}_0 - \frac{1}{\cos(\eta)}\mathbf{m}_1^\top R(\eta)\mathbf{m}_2\right) \qquad \mathbf{p}_0 := \frac{1}{2\cos(\eta)}\left(R^\top(\eta)\mathbf{m}_1 + R(\eta)\mathbf{m}_2\right)$$

- The position $\mathbf{p}$ lies on one of the two circles containing $\mathbf{m}_1$ and $\mathbf{m}_2$

2-D Localization from Bearing Measurements

- **Pose disambiguation**: obtain a third bearing measurement:

$$R^\top(z_i)(\mathbf{m}_i - \mathbf{p}) = \|\mathbf{m}_i - \mathbf{p}\|_2 \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \quad i = 1,2,3$$

- Find β and γ such that $R^\top(z_1) + \beta R^\top(z_2) + \gamma R^\top(z_3) = 0$. Then:

$$\underbrace{R^\top(z_1)\mathbf{m}_1 + \beta R^\top(z_2)\mathbf{m}_2 + \gamma R^\top(z_3)\mathbf{m}_3}_{:=\mathbf{u}} - \underbrace{\left[R^\top(z_1) + \beta R^\top(z_2) + \gamma R^\top(z_3)\right]}_{0}\mathbf{p}$$

$$= \left(\|\mathbf{m}_1 - \mathbf{p}\|_2 + \beta\|\mathbf{m}_2 - \mathbf{p}\|_2 + \gamma\|\mathbf{m}_3 - \mathbf{p}\|_2\right)\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

- We can compute θ as $\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} = \frac{\mathbf{u}}{\|\mathbf{u}\|_2}$ and recover $\mathbf{p}$ from:

$$R^\top(z_i)(\mathbf{m}_i - \mathbf{p}) = \|\mathbf{m}_i - \mathbf{p}\|_2 \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \quad i = 1,2,3$$

3-D Localization from Bearing Measurements (P3P)

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $\mathbf{m}_i \in \mathbb{R}^3$ (world frame) and pixel measurements $\underline{\mathbf{z}}_i \in \mathbb{R}^3$ (homogeneous coordinates, body frame) obtained from a (calibrated pinhole) camera:

$$\underline{\mathbf{z}}_i = \frac{1}{\lambda_i} R^\top (\mathbf{m}_i - \mathbf{p}) \qquad \lambda_i = \mathbf{e}_3^\top \left(R^\top (\mathbf{m}_i - \mathbf{p}) \right) = \text{unknown scale}$$

- If we determine λ_i, we can transform the P3P problem to 3-D localization from relative position measurements

Find the depths λ_i via Grunert's method

- Normalize the bearing equations:

$$\mathbf{b}_i = \frac{\underline{\mathbf{z}}_i}{\|\underline{\mathbf{z}}_i\|_2} = \frac{\lambda_i}{\lambda_i \|R^\top(\mathbf{m}_i - \mathbf{p})\|_2} R^\top(\mathbf{m}_i - \mathbf{p}) = \frac{1}{\bar{\lambda}_i} R^\top(\mathbf{m}_i - \mathbf{p})$$

where $\bar{\lambda}_i = \|R^\top(\mathbf{m}_i - \mathbf{p})\|_2 = \|\mathbf{m}_i - \mathbf{p}\|_2$

- Cosines of the angles among the bearing vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$:

$$\cos(\gamma_{ij}) = \frac{\mathbf{b}_i^\top \mathbf{b}_j}{\|\mathbf{b}_i\|_2 \|\mathbf{b}_j\|_2} = \mathbf{b}_i^\top \mathbf{b}_j$$

- Let $\epsilon_{ij} := \|\mathbf{m}_i - \mathbf{m}_j\|_2$ be the lengths of the triangle formed in the world frame by $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. Applying the law of cosines gives:

$$\bar{\lambda}_i^2 + \bar{\lambda}_j^2 - 2\bar{\lambda}_i \bar{\lambda}_j \cos(\gamma_{ij}) = \epsilon_{ij}^2$$

- Let $\bar{\lambda}_2 = u\bar{\lambda}_1$ and $\bar{\lambda}_3 = v\bar{\lambda}_1$ so that:

$$\begin{aligned}
\bar{\lambda}_1^2(u^2 + v^2 - 2uv\cos(\gamma_{23})) &= \epsilon_{23}^2 \\
\bar{\lambda}_1^2(1 + v^2 - 2v\cos(\gamma_{13})) &= \epsilon_{13}^2 \\
\bar{\lambda}_1^2(u^2 + 1 - 2u\cos(\gamma_{12})) &= \epsilon_{12}^2
\end{aligned}$$

Find the depths λ_i via Grunert's method

- Equivalently

$$\bar{\lambda}_1^2 = \frac{\epsilon_{23}^2}{u^2 + v^2 - 2uv\cos(\gamma_{23})} = \frac{\epsilon_{13}^2}{1 + v^2 - 2v\cos(\gamma_{13})} = \frac{\epsilon_{12}^2}{u^2 + 1 - 2u\cos(\gamma_{12})}$$

- Cross-multiplying the second fraction, with the first and the third:

$$u^2 + \frac{\epsilon_{13}^2 - \epsilon_{23}^2}{\epsilon_{13}^2} v^2 - 2uv\cos(\gamma_{23}) + \frac{2\epsilon_{23}^2}{\epsilon_{13}^2} v\cos(\gamma_{13}) - \frac{\epsilon_{23}^2}{\epsilon_{13}^2} = 0 \qquad (1)$$

$$u^2 - \frac{\epsilon_{12}^2}{\epsilon_{13}^2} v^2 + 2v\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos(\gamma_{13}) - 2u\cos(\gamma_{12}) + \frac{\epsilon_{13}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} = 0 \qquad (2)$$

- Substituting (1) into (2):

$$u = \frac{\left(-1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right) v^2 - 2\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13})v + 1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}}{2(\cos(\gamma_{12}) - v\cos(\gamma_{23}))} \qquad (3)$$

- Substituting (3) into (1), we get a fourth-order polynomial in v:

$$a_4 v^4 + a_3 v^3 + a_2 v^2 + a_1 v + a_0 = 0$$

Polynomial Coefficients

$$a_4 = \left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2} - 1\right)^2 - 4\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos^2(\gamma_{23})$$

$$a_3 = 4\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\left(1 - \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13}) - \left(1 - \frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{12}) + 2\frac{\epsilon_{12}^2}{\epsilon_{13}^2}\cos^2(\gamma_{23})\cos(\gamma_{13})\right)$$

$$a_2 = 2\left(\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2 - 1 + 2\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2\cos^2(\gamma_{13}) + 2\left(\frac{\epsilon_{13}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos^2(\gamma_{23}) + 2\left(\frac{\epsilon_{13}^2 - \epsilon_{23}^2}{\epsilon_{13}^2}\right)\cos^2(\gamma_{12})\right.$$
$$\left. -4\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{13})\cos(\gamma_{12})\right)$$

$$a_1 = 4\left(-\left(\frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\left(1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{13}) - \left(1 - \frac{\epsilon_{23}^2 + \epsilon_{12}^2}{\epsilon_{13}^2}\right)\cos(\gamma_{23})\cos(\gamma_{12}) + 2\frac{\epsilon_{23}^2}{\epsilon_{13}^2}\cos^2(\gamma_{12})\cos(\gamma_{13})\right)$$

$$a_0 = \left(1 + \frac{\epsilon_{23}^2 - \epsilon_{12}^2}{\epsilon_{13}^2}\right)^2 - \frac{4\epsilon_{23}^2}{\epsilon_{13}^2}\cos^2(\gamma_{12})$$

- We can obtain up to 4 real solutions for v, which we can substitute in (3) to obtain u.
- We can recover $\bar{\lambda}_1$ from u and v via the fraction relationship
- Having $\bar{\lambda}_1$, $\bar{\lambda}_2 := u\bar{\lambda}_1$, and $\bar{\lambda}_3 := v\bar{\lambda}_1$ we have converted the P3P problem into 3-D localization from relative position measurements

3-D Localization from Bearing Measurements (PnP)

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: landmark positions $\mathbf{m}_i \in \mathbb{R}^3$ (world frame) and pixel measurements $\underline{\mathbf{z}}_i \in \mathbb{R}^3$ (homogeneous coordinates) obtained from a (calibrated pinhole) camera for $i = 1, \ldots, n$:

$$\underline{\mathbf{z}}_i = \frac{1}{\lambda_i} R^\top (\mathbf{m}_i - \mathbf{p}) \qquad \lambda_i = \mathbf{e}_3^\top (R^\top (\mathbf{m}_i - \mathbf{p})) = \text{unknown scale}$$

- The PnP problem is a **constrained nonlinear least-squares** minimization:

$$\min_{\lambda_i, R, \mathbf{p}} \sum_{i=1}^{n} \left\| \underline{\mathbf{z}}_i - \frac{1}{\lambda_i} R^\top (\mathbf{m}_i - \mathbf{p}) \right\|_2^2$$

$$\text{s.t. } R^\top R = I, \quad \det R = 1, \quad \lambda_i = \mathbf{e}_3^\top (R^\top (\mathbf{m}_i - \mathbf{p}))$$

Solving the PnP Problem

- Terzakis and Lourakis, ECCV'20:
  - Eliminate the auxiliary variables λ_i and directly optimize over $\mathbf{p}$ and R
  - The optimal translation is a function of R and can be eliminated to obtain optimization in R only
  - Sequential quadratic programming with careful initialization on the 8-sphere
- Hesch and Roumeliotis, ICCV'11:
  - Express $\mathbf{p}$ and λ_i in terms of R and eliminate them to obtain an optimization in R only
  - Use Cayley-Gibbs-Rodrigues rotation parameterization to obtain a polynomial system of equations

$$R = (I + \hat{\mathbf{g}})^{-1}(I - \hat{\mathbf{g}}) = \frac{1}{1 + \mathbf{g}^\top \mathbf{g}}((1 - \mathbf{g}^\top \mathbf{g})I + 2\mathbf{g}\mathbf{g}^\top - 2\hat{\mathbf{g}})$$

where $\mathbf{g} \in \mathbb{R}^3$ is related to the angle θ and axis $\boldsymbol{\eta}$ of rotation as: $\mathbf{g} = \boldsymbol{\eta} \tan \frac{\theta}{2}$

Solving the PnP Problem (Terzakis and Lourakis, ECCV'20)

- Re-write the PnP objective in quadratic form:

$$\min_{\mathbf{r},\mathbf{b}} \sum_{i=1}^{n} (A_i\mathbf{r} + \mathbf{b})^\top Q_i (A_i\mathbf{r} + \mathbf{b})$$

  where $A_i := I \otimes \mathbf{m}_i^\top \in \mathbb{R}^{3\times 9}$, $\mathbf{r} = \text{vec}(R^\top)$, $\mathbf{b} = -R^\top\mathbf{p}$,
  $Q_i = (\underline{\mathbf{z}}_i\mathbf{e}_3^\top - I)^\top(\underline{\mathbf{z}}_i\mathbf{e}_3^\top - I) \in \mathbb{R}^{3\times 3}$

- The optimal translation is:

$$\mathbf{b} = P\mathbf{r} \qquad P = -\left(\sum_{i=1}^{n} Q_i\right)^{-1}\left(\sum_{i=1}^{n} Q_i A_i\right)$$

- With $\Omega = \sum_{i=1}^{n}(A_i + P)^\top Q_i (A_i + P)$, we get a non-linear quadratic program:

$$\min_{\text{mat}(\mathbf{r})\in SO(3)} \mathbf{r}^\top\Omega\mathbf{r}$$

- Use sequential quadratic programming initialized from solutions of $\min_{\mathbf{r}\in\mathbb{S}^8} \mathbf{r}^\top\Omega\mathbf{r}$

Solving the PnP Problem (Hesch and Roumeliotis, ICCV'11)

- The constraints $\lambda_i \underline{\mathbf{z}}_i = R^\top(\mathbf{m}_i - \mathbf{p})$ can be re-written in matrix form as:

$$\underbrace{\begin{bmatrix} \underline{\mathbf{z}}_1 & & & -I \\ & \ddots & & \vdots \\ & & \underline{\mathbf{z}}_n & -I \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ -R^\top \mathbf{p} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} R^\top & & \\ & \ddots & \\ & & R^\top \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_n \end{bmatrix}}_{\mathbf{d}}$$

  where A and $\mathbf{d}$ are known or measured, $\mathbf{x}$ are the unknowns we wish to eliminate, and W is a block diagonal matrix of the unknown rotation R

- Express $\mathbf{p}$ and λ_i in terms of the other quantities:

$$\mathbf{x} = (A^\top A)^{-1} A^\top W \mathbf{d} = \begin{bmatrix} U \\ V \end{bmatrix} W \mathbf{d}$$

  where $(A^\top A)^{-1} A^\top$ is partitioned so that the scale parameters are a function of U and the translation $-R^\top \mathbf{p}$ is a function of V.

Solving the PnP Problem (Hesch and Roumeliotis, ICCV'11)

$$\mathbf{x} = (A^\top A)^{-1} A^\top W \mathbf{d} = \begin{bmatrix} U \\ V \end{bmatrix} W \mathbf{d}$$

- Exploiting the sparse structure of A, the matrices U and V can be computed in closed form
- Both λ_i and $-R^\top \mathbf{p}$ are linear functions of the unknown $R^\top$:

  $$\lambda_i = \mathbf{u}_i^\top W \mathbf{d} \qquad - R^\top \mathbf{p} = V W \mathbf{d}, \qquad i = 1, \ldots, n$$

  where $\mathbf{u}_i^\top$ is the i-th row of U
- We can rewrite the constraints $\lambda_i \underline{\mathbf{z}}_i = R^\top(\mathbf{m}_i - \mathbf{p})$ as:

  $$\underbrace{\mathbf{u}_i^\top W \mathbf{d}}_{\lambda_i} \underline{\mathbf{z}}_i = R^\top \mathbf{m}_i + \underbrace{V W \mathbf{d}}_{-R^\top \mathbf{p}}$$
- We have reduced the number of unknowns from $6 + n$ to 3

Solving the PnP Problem (Hesch and Roumeliotis, ICCV'11)

- **Cayley-Gibbs-Rodrigues rotation parameterization**:

$$R^\top = \frac{\bar{C}}{1+\mathbf{g}^\top\mathbf{g}} \qquad \bar{C} = (I-\hat{\mathbf{g}})^{-1}(I+\hat{\mathbf{g}}) = ((1-\mathbf{g}^\top\mathbf{g})I_3 + 2\hat{\mathbf{g}} + 2\mathbf{g}\mathbf{g}^\top)$$

- The CGR parameters automatically satisfy the rotation matrix constraints, i.e., $R^\top R = I$ and $\det(R) = 1$ and allow us to formulate an unconstrained least-squares minimization in $\mathbf{g}$.

- **Reformulation into a polynomial system**: Since $R^\top$ appears linearly in the equations, we can cancel the denominator $1+\mathbf{g}^\top\mathbf{g}$. This leads to the following formulation of the PnP problem:

$$\min_{\mathbf{g}} J(\mathbf{g}) = \sum_{i=1}^{n} \left\| \mathbf{u}_i^\top \begin{bmatrix} \bar{C} & & \\ & \ddots & \\ & & \bar{C} \end{bmatrix} \mathbf{d}\underline{\mathbf{z}}_i - \bar{C}\mathbf{m}_i - V \begin{bmatrix} \bar{C} & & \\ & \ddots & \\ & & \bar{C} \end{bmatrix} \mathbf{d} \right\|^2$$

which contains all monomials up to degree four, i.e.,
$\{1, g_1, g_2, g_3, g_1g_2, g_1g_3, g_2g_3, \ldots, g_1^4, g_2^4, g_3^4\}$.

Solving the PnP Problem (Hesch and Roumeliotis, ICCV'11)

- Since $J(\mathbf{g})$ is a fourth-order polynomial, the optimality conditions form a system of three third-order polynomials (derivatives with respect to g_1, g_2 and g_3).
- Use a **Macaulay resultant matrix** (matrix of polynomial coefficients) to find the roots of the third-order polynomials and hence compute all critical points of $J(\mathbf{g})$
- Since the polynomial system is of constant degree (independent of n), it is only necessary to compute the Macaulay matrix symbolically once
- Online, the elements of the Macaulay matrix are formed from the data (linear operation in n) and the roots are determined via an eigen-decomposition of the Schur complement (dense 27×27 matrix) of the top block of the Macaulay matrix (sparse 120×120 matrix)

2-D Odometry from Bearing Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: bearing measurements $\mathbf{z}_{t,i} \in \mathbb{R}^2$ and $\mathbf{z}_{t+1,i} \in \mathbb{R}^2$ (unit vectors) at consecutive time steps to n **unknown** landmarks
- The measurements are related as follows:

$$d_{t,i}\mathbf{b}_{t,i} = {}_t\mathbf{p}_{t+1} + d_{t+1,i}R({}_t\theta_{t+1})\mathbf{b}_{t+1,i}, \qquad i = 1, \ldots, n$$

  where $d_{t,i}, d_{t+1,i}$ are the unknown distances to $\mathbf{m}_i$.
- There are $2n$ equations and $2n+3$ unknowns, which means that this problem is **not solvable**.

3-D Odometry from Bearing Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^3$ and ${}_tR_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t
- **Given**: pixel coordinates $\underline{\mathbf{z}}_{t,i} \in \mathbb{R}^3$ and $\underline{\mathbf{z}}_{t+1,i} \in \mathbb{R}^3$ at consecutive time steps to n **unknown** landmarks ($n \geq 5$) with **known** camera calibration matrices K_t and K_{t+1}
- Without loss of generality, assume that the first camera frame coincides with the world frame and denote $\mathbf{p} = {}_t\mathbf{p}_{t+1}$ and $R = {}_tR_{t+1}$
- Let $\underline{\mathbf{y}}_{t,i} := K_t^{-1}\underline{\mathbf{z}}_{t,i}$ and $\underline{\mathbf{y}}_{t+1,i} := K_{t+1}^{-1}\underline{\mathbf{z}}_{t+1,i}$ be the normalized pixel coordinates so that:

$$\lambda_{t,i}\underline{\mathbf{y}}_{t,i} = \mathbf{m}_i, \qquad \lambda_{t,i} = \mathbf{e}_3^\top \mathbf{m}_i = \text{unknown depth}$$

$$\lambda_{t+1,i}\underline{\mathbf{y}}_{t+1,i} = R^\top(\mathbf{m}_i - \mathbf{p}), \qquad \lambda_{t+1,i} = \mathbf{e}_3^\top R^\top(\mathbf{m}_i - \mathbf{p}) = \text{unknown depth}$$

Epipolar Constraint and Essential Matrix

- The pixel projections of landmark $\mathbf{m}_i$ in the two images satisfy:

$$\lambda_{t,i}\underline{\mathbf{y}}_{t,i} = \lambda_{t+1,i}R\underline{\mathbf{y}}_{t+1,i} + \mathbf{p}$$

- To eliminate the unknown depths $\lambda_{t,i}$, $\lambda_{t+1,i}$, pre-multiply by $\hat{\mathbf{p}}$ and note that $\hat{\mathbf{p}}\underline{\mathbf{y}}_{t,i}$ is perpendicular to $\underline{\mathbf{y}}_{t,i}$:

$$\underbrace{\lambda_{t,i}\underline{\mathbf{y}}_{t,i}^\top\hat{\mathbf{p}}\underline{\mathbf{y}}_{t,i}}_{0} = \lambda_{t+1,i}\underline{\mathbf{y}}_{t,i}^\top\hat{\mathbf{p}}R\underline{\mathbf{y}}_{t+1,i} + \underbrace{\underline{\mathbf{y}}_{t,i}^\top\hat{\mathbf{p}}\mathbf{p}}_{0}$$

- **Epipolar constraint**: the normalized pixel coordinates $\underline{\mathbf{y}}_{t,i} = K_t^{-1}\underline{\mathbf{z}}_{t,i}$ and $\underline{\mathbf{y}}_{t+1,i} = K_{t+1}^{-1}\underline{\mathbf{z}}_{t+1,i}$ of the same point $\mathbf{m}_i$ in two calibrated cameras with relative pose $(R, \mathbf{p})$ of cam 2 in the frame of cam 1 satisfy:

$$0 = \underline{\mathbf{y}}_{t,i}^\top\left(\hat{\mathbf{p}}R\right)\underline{\mathbf{y}}_{t+1,i} = \underline{\mathbf{y}}_{t,i}^\top E\underline{\mathbf{y}}_{t+1,i}$$

  where $E := \hat{\mathbf{p}}R \in \mathbb{R}^{3\times 3}$ is the **essential matrix**

- **Essential matrix characterization**: a non-zero $E \in \mathbb{R}^{3\times 3}$ is an essential matrix iff its singular value decomposition is $E = U\mathbf{diag}(\sigma, \sigma, 0)V^\top$ for some $\sigma \geq 0$ and $U, V \in SO(3)$

3-D Odometry from Bearing Measurements (8-Pt Alg)

- The epipolar constraint $0 = \underline{\mathbf{y}}_{t,i}^\top E \underline{\mathbf{y}}_{t+1,i}$ is linear in the elements of E:

$$0 = \bar{\mathbf{y}}_i^\top \mathbf{e}$$

  where $\bar{\mathbf{y}}_i := \text{vec}(\underline{\mathbf{y}}_{t,i}\underline{\mathbf{y}}_{t+1,i}^\top) \in \mathbb{R}^9$, $\mathbf{e} := \text{vec}(E) \in \mathbb{R}^9$, and $\text{vec}(\cdot)$ is the vectorization of a matrix, which stacks its columns into a vector

- Stacking $\bar{\mathbf{y}}_i$ from all 8 observations together, we obtain an 8×9 matrix $\bar{Y} := \begin{bmatrix} \bar{\mathbf{y}}_1 & \cdots & \bar{\mathbf{y}}_8 \end{bmatrix}^\top$ leading to the following equation for $\mathbf{e}$:

$$\bar{Y}\mathbf{e} = 0$$

- Thus, $\mathbf{e}$ is a singular vector of $\bar{Y}$ associated to a singular value that equals zero

- If at least 8 linearly independent vectors $\bar{\mathbf{y}}_i$ are used to construct $\bar{Y}$, then the singular vector is unique (up to scalar multiplication) and $\mathbf{e}$ and E can be determined

3-D Odometry from Bearing Measurements (5-Pt Alg)

- The essential matrix E can be recovered from $\bar{Y}\mathbf{e} = 0$, even if only 5 linearly independent vectors $\bar{\mathbf{y}}_i$ are available using the fact that:

$$0 = EE^\top E - \frac{1}{2}\operatorname{tr}(EE^\top)E$$

- Stacking $\bar{\mathbf{y}}_i$'s together, we obtain a 5×9 matrix $\bar{Y} := \begin{bmatrix}\bar{\mathbf{y}}_1 & \cdots & \bar{\mathbf{y}}_5\end{bmatrix}^\top$

- The right nullspace of $\bar{Y}$ has dimension 4 and the vectors that span the nullspace (obtained from SVD or QR decomposition) correspond to 3×3 matrices N_i, $i = 1, \ldots, 4$ such that

$$E = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + \alpha_4 N_4, \qquad \alpha_i \in \mathbb{R}$$

- Since the measurements are scale-invariant, we can arbitrarily fix $\alpha_4 = 1$

- Substituting $E = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + N_4$, we obtain 9 cubic-in-$\alpha_i$ equations and can recover up to 10 solutions for E

3-D Odometry from Bearing Measurements

- Once E is recovered, $\mathbf{p}$ and R can be computed from the singular value decomposition of E
- **Pose recovery from the essential matrix**: there are exactly two relative poses corresponding to a non-zero essential matrix $E = U\mathbf{diag}(\sigma, \sigma, 0)V^\top$:

$$(\hat{\mathbf{p}}, R) = \left(UR_z\left(\frac{\pi}{2}\right)\mathbf{diag}(\sigma,\sigma,0)U^\top, UR_z^\top\left(\frac{\pi}{2}\right)V^\top \right)$$
$$(\hat{\mathbf{p}}, R) = \left(UR_z\left(-\frac{\pi}{2}\right)\mathbf{diag}(\sigma,\sigma,0)U^\top, UR_z^\top\left(-\frac{\pi}{2}\right)V^\top \right)$$

- Only one of these will place the points in front of both cameras
- The ambiguity can be resolved by intersecting the measurements of a single point and verifying which solution places it on the positive optical z-axis of both cameras

Bearing Measurement Triangulation

- **Goal**: determine the coordinates of a point $\mathbf{m} \in \mathbb{R}^3$ observed by two cameras in the reference frame of the first camera

- **Given**: pixel coordinates $\mathbf{z}_1 \in \mathbb{R}^2$ and $\mathbf{z}_2 \in \mathbb{R}^2$ obtained from two calibrated cameras with known relative transformation $\mathbf{p} \in \mathbb{R}^3$ and $R \in SO(3)$ of cam 2 in the frame of cam 1:

$$\lambda_1 \underline{\mathbf{z}}_1 = \mathbf{m}, \qquad \lambda_1 = \mathbf{e}_3^\top \mathbf{m} = \text{unknown depth}$$

$$\lambda_2 \underline{\mathbf{z}}_2 = R^\top (\mathbf{m} - \mathbf{p}), \qquad \lambda_2 = \mathbf{e}_3^\top R^\top (\mathbf{m} - \mathbf{p}) = \text{unknown depth}$$

- We can determine $\mathbf{m} = \lambda_1 \underline{\mathbf{z}}_1$ by solving for the unknown depth λ_1 using the second measurement equation

- Note that $\lambda_2 = \lambda_1 \mathbf{e}_3^\top \mathbf{R}^\top \underline{\mathbf{z}}_1 - \mathbf{e}_3^\top \mathbf{R}^\top \mathbf{p}$ and thus:

$$\left(\lambda_1 \mathbf{e}_3^\top \mathbf{R}^\top \underline{\mathbf{z}}_1 - \mathbf{e}_3^\top \mathbf{R}^\top \mathbf{p}\right) \underline{\mathbf{z}}_2 = \lambda_1 \mathbf{R}^\top \underline{\mathbf{z}}_1 - \mathbf{R}^\top \mathbf{p}$$

$$\underbrace{\left(\mathbf{R}^\top \mathbf{p} - \mathbf{e}_3^\top \mathbf{R}^\top \mathbf{p} \underline{\mathbf{z}}_2\right)}_{\mathbf{a}} \frac{1}{\lambda_1} = \underbrace{\left(\mathbf{R}^\top \underline{\mathbf{z}}_1 - \mathbf{e}_3^\top \mathbf{R}^\top \underline{\mathbf{z}}_1 \underline{\mathbf{z}}_2\right)}_{\mathbf{b}}$$

$$\frac{1}{\lambda_1} = \frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}} \qquad \Rightarrow \qquad \mathbf{m} = \frac{\mathbf{a}^\top \mathbf{a}}{\mathbf{a}^\top \mathbf{b}} \underline{\mathbf{z}}_1$$

Summary: Bearing Measurements $\underline{\mathbf{z}}_i = \frac{1}{\lambda_i} R^\top(\mathbf{m}_i - \mathbf{p})$

- **2-D Localization**: given $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ and $z_1, z_2 \in [-\pi, \pi]$
  1. 2-D bearing: $\frac{1}{\lambda_i} R^\top(\theta)(\mathbf{m}_i - \mathbf{p}) = R(z_i)\mathbf{e}_1$
  2. Eliminate θ:

$$0 = \lambda_1 \mathbf{e}_1^\top R(\theta) R\left(\frac{\pi}{2}\right) R(\theta)\mathbf{e}_1 \lambda_2 = (\mathbf{m}_1 - \mathbf{p})^\top R(z_1) R\left(\frac{\pi}{2}\right) R^\top(z_2)(\mathbf{m}_2 - \mathbf{p})$$

  3. The position $\mathbf{p}$ in on one of two circles containing $\mathbf{m}_1$ and $\mathbf{m}_2$ and we need a third bearing measurement z_3 to disambiguate it
  4. Find β, γ such that $R^\top(z_1) + \beta R^\top(z_2) + \gamma R^\top(z_3) = 0$ and combine $R^\top(z_i)(\mathbf{m}_i - \mathbf{p}) = \lambda_i \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ to solve for θ
  5. Orientation: $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \frac{\mathbf{u}}{\|\mathbf{u}\|_2}$ for $\mathbf{u} = R^\top(z_1)\mathbf{m}_1 + \beta R^\top(z_2)\mathbf{m}_2 + \gamma R^\top(z_3)\mathbf{m}_3$

- **3-D Localization (P3P)**: $\mathbf{m}_i \in \mathbb{R}^3$, $\underline{\mathbf{z}}_i \in \mathbb{R}^3$ (homogeneous), $i = 1, 2, 3$
  1. Convert P3P to relative position localization by determining the depths $\lambda_1, \lambda_2, \lambda_3$ via Grunert's method
  2. Define angles γ_{ij} among normalized $\underline{\mathbf{z}}_1, \underline{\mathbf{z}}_2, \underline{\mathbf{z}}_3$ and apply the law of cosines: $\lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j \cos(\gamma_{ij}) = \|\mathbf{m}_1 - \mathbf{m}_j\|_2^2$
  3. Let $\lambda_2 = u\lambda_1$ and $\lambda_3 = v\lambda_1$ and combine the 3 equations to get a fourth order polynomial: $a_4 v^4 + a_3 v^3 + a_2 v^2 + a_1 v + a_0 = 0$

Summary: Bearing Measurements $\underline{\mathbf{z}}_i = \frac{1}{\lambda_i} R^\top (\mathbf{m}_i - \mathbf{p})$

- **3-D Localization (PnP)**
  1. Rewrite $\lambda_i \underline{\mathbf{z}}_i = R^\top (\mathbf{m}_i - \mathbf{p})$ in matrix form and solve for $\mathbf{x} := (\lambda_1, \ldots, \lambda_n, -R^\top \mathbf{p})^\top$ in terms of R
  2. The equations for λ_i and $-R^\top \mathbf{p}$ turn out to be linear in R so we are left with one equation with 3 unknowns (the 3 degrees of freedom of R)
  3. Obtain a fourth order polynomial $J(\mathbf{g})$ in terms of the Cayley-Gibbs-Rodrigues rotation parameterization $\mathbf{g}$
  4. Compute a Macaulay matrix of the coefficients of $J(\mathbf{g})$ symbolically once. Online, determine the roots of $J(\mathbf{g})$ via an eigen-decomposition of the Schur complement of the Macaulay matrix.

- **2-D Odometry**: not solvable

- **3-D Odometry**: 5-point or 8-point algorithm:
  1. Obtain E from the epipolar constraint: $0 = \text{vec}\left(\underline{\mathbf{y}}_{t,i} \underline{\mathbf{y}}_{t+1,i}^\top\right)^\top \text{vec}(E)$, $i = 1, \ldots, 5$ and the property $0 = EE^\top E - \frac{1}{2}\text{tr}(EE^\top)E$
  2. Recover two possible camera poses based on $SVD(E) = U\mathbf{diag}(\sigma, \sigma, 0)V^\top$ and choose the one that places the measurements in front of both cameras

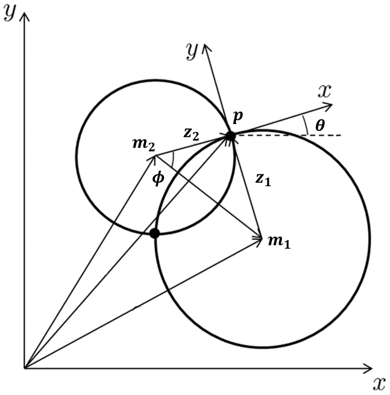
Outline

2-D Localization from Range Measurements

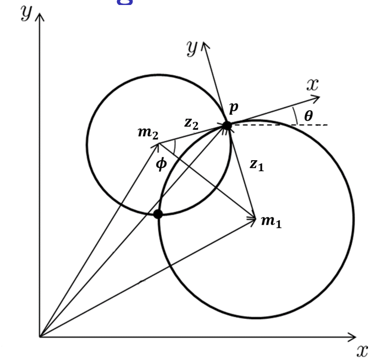
- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^2$ and orientation $\theta \in (-\pi, \pi]$
- **Given**: two landmark positions $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ (world frame) and **range** measurements (body frame):

$$z_i = \|\mathbf{m}_i - \mathbf{p}\|_2 \in \mathbb{R}, \quad i = 1, 2$$

- Because all possible positions whose distance to $\mathbf{m}_1$ is z_1 is a circle, the possible robot positions are given by the intersection of two circles



2-D Localization from Range Measurements



- Applying the law of cosines to the triangle gives:

$$z_2^2 = z_1^2 + \|\mathbf{m}_2 - \mathbf{m}_1\|_2^2 - 2z_1\|\mathbf{m}_2 - \mathbf{m}_1\|_2 \cos\phi$$

- Solving for ϕ and then the circle intersection points provides the possible robot positions:

$$\mathbf{p} = \mathbf{m}_2 + z_2 R(\pm\phi)\frac{\mathbf{m}_1 - \mathbf{m}_2}{\|\mathbf{m}_1 - \mathbf{m}_2\|_2}$$

- The orientation of the robot θ is **not identifiable**

2-D Localization from Range Measurements

- **Pose disambiguation**: the robot can make a move with known translation $\mathbf{p}_\Delta$ (measured in the frame at time t) and take two new range measurements
- There are 2 possible robot positions at each time frame for a total of 4 combinations but comparing $\|\mathbf{p}_{t+1} - \mathbf{p}_t\|_2$ to the known $\|\mathbf{p}_\Delta\|_2$ leaves only two valid options (and we cannot distinguish between them)
- To obtain the orientation, we use geometric constraints:

$$\mathbf{p}_{t+1} - \mathbf{p}_t = R(\theta_t)\mathbf{p}_\Delta = \begin{bmatrix} p_{\Delta,x} & -p_{\Delta,y} \\ p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} \begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix}$$

- As long as $\det \begin{bmatrix} p_{\Delta,x} & -p_{\Delta,y} \\ p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} = \|\mathbf{p}_\Delta\|_2^2 \neq 0$, we can compute:

$$\begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix} = \frac{1}{\|\mathbf{p}_\Delta\|_2^2} \begin{bmatrix} p_{\Delta,x} & p_{\Delta,y} \\ -p_{\Delta,y} & p_{\Delta,x} \end{bmatrix} (\mathbf{p}_{t+1} - \mathbf{p}_t)$$

$$\theta_t = \mathbf{atan2}(\sin\theta_t, \cos\theta_t)$$

3-D Localization from Range Measurements

- **Goal**: determine the robot position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$
- **Given**: three landmark positions $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 \in \mathbb{R}^3$ (world frame) and **range** measurements (body frame):

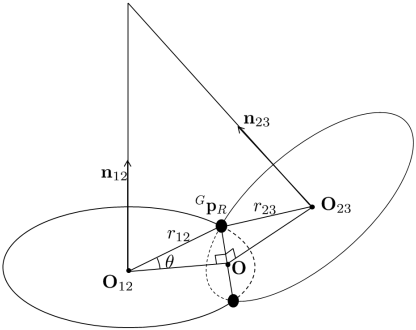
$$z_i = \|\mathbf{m}_i - \mathbf{p}\|_2 \in \mathbb{R}, \quad i = 1, 2, 3$$

- All possible positions whose distance to $\mathbf{m}_1$ is z_1 is a sphere
- The possible robot positions are the intersections of three spheres
- To find the intersection of 3 spheres, we first find the intersection of sphere one and two (a circle) and of sphere two and three (a circle). The intersection of these two circles gives the possible robot positions.
- **Degenerate case**: all landmarks are on the same line – the intersection of the spheres is a circle with infinitely many possible robot positions

3-D Localization from Range Measurements

- **Intersecting circle of spheres with radii z_1 and z_2**: center $\mathbf{o}_{12}$, radius r_{12}, normal vector $\mathbf{n}_{12}$ (perpendicular to the circle plane)
- Law of Cosines: $z_2^2 = z_1^2 + \|\mathbf{m}_2 - \mathbf{m}_1\|_2^2 - 2z_1\|\mathbf{m}_2 - \mathbf{m}_1\|_2 \cos\theta_{12}$
- Geometric relationships:

$$\mathbf{o}_{12} = \mathbf{m}_1 + z_1 \cos\theta_{12}\mathbf{n}_{12}$$
$$r_{12} = z_1|\sin(\theta_{12})|$$
$$\mathbf{n}_{12} = \frac{\mathbf{m}_2 - \mathbf{m}_1}{\|\mathbf{m}_2 - \mathbf{m}_1\|_2}$$



- **Intersecting circle of spheres with radii z_2 and z_3**: center $\mathbf{o}_{23}$, radius r_{23}, normal vector $\mathbf{n}_{23}$ (perpendicular to the circle plane):

$$\mathbf{o}_{23} = \mathbf{m}_2 + z_2 \cos\theta_{23}\mathbf{n}_{23} \qquad r_{23} = z_2|\sin(\theta_{23})| \qquad \mathbf{n}_{23} = \frac{\mathbf{m}_3 - \mathbf{m}_2}{\|\mathbf{m}_3 - \mathbf{m}_2\|_2}$$

3-D Localization from Range Measurements

- The intersecting points of the two circles can be obtained from:

$$\begin{aligned} \mathbf{n}_{12}^\top(\mathbf{o}_{12} - \mathbf{o}) &= 0 \\ \mathbf{n}_{23}^\top(\mathbf{o}_{23} - \mathbf{o}) &= 0 \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top(\mathbf{o}_{12} - \mathbf{o}) &= 0 \end{aligned} \qquad \begin{bmatrix} \mathbf{n}_{12}^\top \\ \mathbf{n}_{23}^\top \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \end{bmatrix} \mathbf{o} = \begin{bmatrix} \mathbf{n}_{12}^\top \mathbf{o}_{12} \\ \mathbf{n}_{23}^\top \mathbf{o}_{23} \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \mathbf{o}_{12} \end{bmatrix}$$

- As long as the three landmarks are not on the same line, we can uniquely solve for **o**:

$$\det \begin{bmatrix} \mathbf{n}_{12}^\top \\ \mathbf{n}_{23}^\top \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \end{bmatrix} \neq 0 \qquad \Leftrightarrow \qquad \mathbf{n}_{12} \text{ and } \mathbf{n}_{23} \text{ not colinear}$$

- The two possible robot positions are:

$$\mathbf{p} = \mathbf{o}_{12} + r_{12} R(\mathbf{n}_{12}, \pm\theta) \frac{\mathbf{o} - \mathbf{o}_{12}}{\|\mathbf{o} - \mathbf{o}_{12}\|_2} \qquad \cos\theta = \frac{\|\mathbf{o} - \mathbf{o}_{12}\|_2}{r_{12}}$$

- As in the 2-D case, the robot orientation R is **not identifiable**

3-D Localization from Range Measurements

- **Pose disambiguation**: the robot can make a move with known translation $\mathbf{p}_\Delta \in \mathbb{R}^3$ and rotation $R_\Delta \in SO(3)$ and take three new range measurements
- As in the 2-D case, after eliminating the impossible robot positions, we should be left with only two options for $\mathbf{p}_t$ and $\mathbf{p}_{t+1}$
- Given $\mathbf{p}_t$, $\mathbf{p}_{t+1}$, $\mathbf{p}_\Delta$, and R_Δ, we can now obtain R_t

$$\mathbf{p}_{t+1} = \mathbf{p}_t + R_t \mathbf{p}_\Delta$$

- This is not sufficient because the rotation about $\mathbf{p}_\Delta$ is not identifiable
- The robot needs to **move a second time** to a third pose $\mathbf{p}_{t+2}, R_{t+2}$ with known translation $\mathbf{p}_{\Delta,2} \in \mathbb{R}^3$ and take three more range measurements to the three landmarks:

$$\mathbf{p}_{t+2} = \mathbf{p}_{t+1} + R_{t+1}\mathbf{p}_{\Delta,2} = \mathbf{p}_{t+1} + R_t R_\Delta \mathbf{p}_{\Delta,2}$$

3-D Localization from Range Measurements

- Putting the previous two equations together:

$$\begin{aligned}\mathbf{p}_{t+1} - \mathbf{p}_t &= R_t \mathbf{p}_\Delta \\ \mathbf{p}_{t+2} - \mathbf{p}_{t+1} &= R_t R_\Delta \mathbf{p}_{\Delta,2}\end{aligned}$$

- Taking a cross product between the two:

$$(\mathbf{p}_{t+1} - \mathbf{p}_t) \times (\mathbf{p}_{t+2} - \mathbf{p}_{t+1}) = R_t(\mathbf{p}_\Delta \times R_\Delta \mathbf{p}_{\Delta,2})$$

- As long as $U := [\mathbf{p}_\Delta,\ R_\Delta \mathbf{p}_{\Delta,2},\ \mathbf{p}_\Delta \times R_\Delta \mathbf{p}_{\Delta,2})]$ is nonsingular, i.e., $\mathbf{p}_\Delta$ and $R_\Delta \mathbf{p}_{\Delta,2}$ are not co-linear or equivalently **the three robot positions are not on the same line**, we can determine the robot orientation:

$$R_t = [(\mathbf{p}_{t+1} - \mathbf{p}_t),\ (\mathbf{p}_{t+2} - \mathbf{p}_{t+1}),\ (\mathbf{p}_{t+1} - \mathbf{p}_t) \times (\mathbf{p}_{t+2} - \mathbf{p}_{t+1})]U^{-1}$$

2-D Odometry from Range Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^2$ and ${}_t\theta_{t+1} \in (-\pi, \pi]$ between two robot frames at time $t+1$ and t
- **Given**: range measurements $z_{t,i} \in \mathbb{R}$ and $z_{t+1,i} \in \mathbb{R}$ at consecutive time steps to n **unknown** landmarks
- Let $\mathbf{m}_{t+1,i}$ be the relative position to the i-th landmark at $t+1$ so that:

$$
\begin{aligned}
z_{t+1,i} &= \|\mathbf{m}_{t+1,i}\|_2 \\
z_{t,i} &= \|{}_t\mathbf{p}_{t+1} + R({}_t\theta_{t+1})\mathbf{m}_{t+1,i}\|_2
\end{aligned}
$$

- Squaring and combining these equations, we get:

$$
[{}_t\mathbf{p}_{t+1}]^\top {}_t\mathbf{p}_{t+1} + 2\mathbf{m}_{t+1,i}^\top R^\top({}_t\theta_{t+1}){}_t\mathbf{p}_{t+1} = z_{t,i}^2 - z_{t+1,i}^2, \quad i = 1, \ldots, n
$$

- We have n equations with $n+3$ unknowns (3 for the relative pose and n for the unknown directions to the landmarks at $t+1$), which is **not solvable**.

3-D Odometry from Range Measurements

- **Goal**: determine the relative transformation ${}_t\mathbf{p}_{t+1} \in \mathbb{R}^3$ and ${}_tR_{t+1} \in SO(3)$ between two robot frames at time $t+1$ and t

- **Given**: range measurements $z_{t,i} \in \mathbb{R}$ and $z_{t+1,i} \in \mathbb{R}$ at consecutive time steps to n **unknown** landmarks

- Following the same derivation as in the 2-D case, we obtain:

$$\left[{}_t\mathbf{p}_{t+1}\right]^\top {}_t\mathbf{p}_{t+1} + 2\mathbf{m}_{t+1,i}^\top \left[{}_tR_{t+1}\right]^\top {}_t\mathbf{p}_{t+1} = z_{t,i}^2 - z_{t+1,i}^2, \quad i = 1, \ldots, n$$

- We have n equations with $2n+6$ unknowns (6 for the relative pose and $2n$ for the unknown directions to the landmarks at $t+1$), which is **not solvable**.

Summary: Range Measurements $z_i = \|\mathbf{m}_i - \mathbf{p}\|_2$

- **2-D Localization**: given $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^2$ and $z_1, z_2 \in \mathbb{R}$
  1. Law of Cosines: $z_2^2 = z_1^2 + \|\mathbf{m}_2 - \mathbf{m}_1\|_2^2 - 2z_1\|\mathbf{m}_2 - \mathbf{m}_1\|_2 \cos\theta$
  2. Position: $\mathbf{p} = \mathbf{m}_2 + z_2 R(\pm\theta)\frac{\mathbf{m}_1 - \mathbf{m}_2}{\|\mathbf{m}_1 - \mathbf{m}_2\|_2}$
  3. Move with known $\mathbf{p}_\Delta, \theta_\Delta$ (in frame t)
  4. Orientation: $(\mathbf{p}_{t+1} - \mathbf{p}_t) = R(\theta_t)\mathbf{p}_\Delta$

- **3-D Localization**: given $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 \in \mathbb{R}^3$ and $z_1, z_2, z_3 \in \mathbb{R}$
  1. Intersection of 2 circles with centers $\mathbf{o}_{12}, \mathbf{o}_{23}$, radii r_{12}, r_{23}, normals $\mathbf{n}_{12}, \mathbf{n}_{23}$ obtained via Law of Cosines and point $\mathbf{o}$ on intersecting line:

$$\begin{bmatrix} \mathbf{n}_{12}^\top \\ \mathbf{n}_{23}^\top \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \end{bmatrix} \mathbf{o} = \begin{bmatrix} \mathbf{n}_{12}^\top \mathbf{o}_{12} \\ \mathbf{n}_{23}^\top \mathbf{o}_{23} \\ (\mathbf{n}_{12} \times \mathbf{n}_{23})^\top \mathbf{o}_{12} \end{bmatrix}$$

  2. Position: $\mathbf{p} = \mathbf{o}_{12} + r_{12} R(\mathbf{n}_{12}, \pm\theta)\frac{\mathbf{o} - \mathbf{o}_{12}}{\|\mathbf{o} - \mathbf{o}_{12}\|_2}$, where $\cos\theta = \frac{\|\mathbf{o} - \mathbf{o}_{12}\|_2}{r_{12}}$
  3. Move twice with known $\mathbf{p}_\Delta, R_\Delta, \mathbf{p}_{\Delta,2}, R_{\Delta,2}$
  4. Orientation: as long as $U := [\mathbf{p}_\Delta,\ R_\Delta \mathbf{p}_{\Delta,2},\ \mathbf{p}_\Delta \times R_\Delta \mathbf{p}_{\Delta,2})]$ is nonsingular:

$$R_t = [(\mathbf{p}_{t+1} - \mathbf{p}_t),\ (\mathbf{p}_{t+2} - \mathbf{p}_{t+1}),\ (\mathbf{p}_{t+1} - \mathbf{p}_t) \times (\mathbf{p}_{t+2} - \mathbf{p}_{t+1})]U^{-1}$$

- **Odometry**: not solvable