ECE276B: Planning & Learning in Robotics
Lecture 6: Configuration Space

Nikolay Atanasov

natanasov@ucsd.edu

UC San Diego
JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

Outline

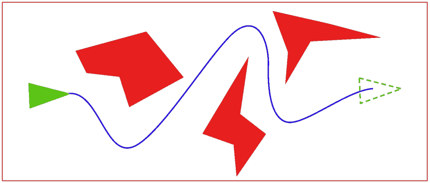
Motion Planning

Configuration Space

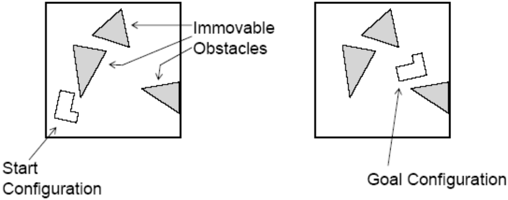
Graph Construction for Motion Planning

Motion Planning

- **Motion planning** is a deterministic shortest path (DSP) problem with continuous state space (infinite number of nodes) and state constraints (missing edges) introduced by obstacles

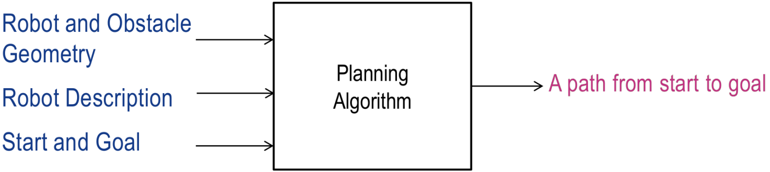


- The problem is also known as the **Piano Movers Problem**

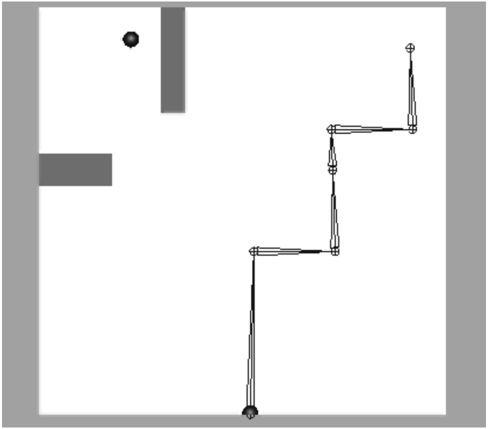


Motion Planning

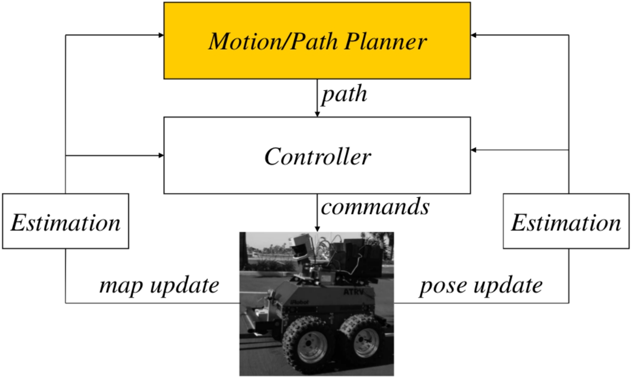
- Objective: find a feasible and cost-minimal path from an initial state to a goal region
- Cost function: distance, time, energy, risk, etc.
- Constraints:
  - environment constraints (e.g., obstacles)
  - kinematics/dynamics of the robot



Example: Six-Joint Robot Arm



Planning vs Control



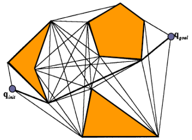
- Distinction between planning and control
  - **Planning**: automatic generation of global collision-free trajectories (global reasoning)
  - **Control**: automatic generation of control inputs for local reactive trajectory tracking (local reasoning)

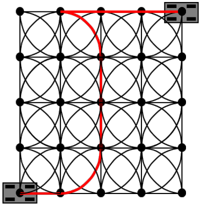
Analyzing Motion Planning Algorithms

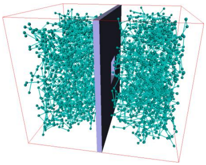
- **Completeness**: a planning algorithm is called complete if it:
  - returns a feasible solution, if one exists,
  - returns FAIL in finite time, otherwise.
- **Optimality**:
  - a planning algorithm is optimal if it returns a path with shortest length J^* among all possible paths from start to goal
  - a planning algorithm is ϵ-**suboptimal** if it returns a path with length $J \leq \epsilon J^*$ for $\epsilon \geq 1$ where J^* is the optimal length
- **Efficiency**: a planning algorithm is efficient if it finds a solution with the least possible computation operations across all inputs
- **Generality**: a planning algorithm is general if it can handle high-dimensional robots or environments and various obstacle or kinematic/dynamic constraints

Motion Planning Approaches

- **Exact algorithms** in continuous space
  - Computationally expensive and unsuitable for high-dimensional spaces
- **Search-based planning algorithms**
  - discretize the state space into a regular grid
  - contruct a graph incrementally
  - solve a DSP problem via label correcting
  - inefficient in high-dim spaces without heuritic function guidance due to the regular discretization
  - resolution complete with finite-time (sub)optimality guarantees
- **Sampling-based planning algorithms**
  - discretize the state space irregularly by sampling states
  - construct a graph incrementally
  - solve a DSP problem via label correcting
  - efficient in high-dim spaces but problems with "narrow passages"
  - probabilistically complete with asmyptotic (sub)optimality guarantees







Outline

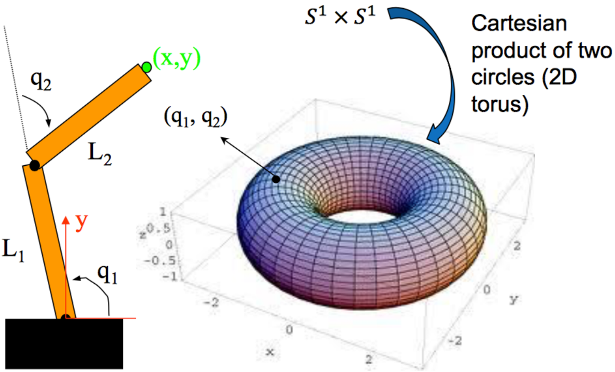
Configuration Space

- A **configuration** is a specification of the position of **every** point on a robot body
- A configuration $\mathbf{q}$ is expressed as a vector of the degrees of freedom (DOF) of the robot:

$$\mathbf{q} = (q_1, \ldots, q_n)$$

  - 3 DOF: differential drive robot $(x, y, \theta) \in \mathbb{R}^2 \times [-\pi, \pi)$
  - 6 DOF: rigid body with pose $T \in SE(3)$
  - 7 DOF: 7-link manipulator (humanoid arm): $(\theta_1, \ldots, \theta_7) \in [-\pi, \pi)^7$
- **Configuration space** C: set of all possible robot configurations
- $\dim(C)$: min DOF needed to completely specify a robot configuration
- **Work space** W: 2D or 3D Euclidean space where the robot operates

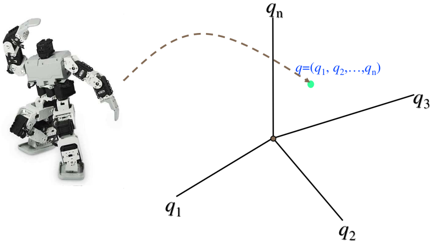
Example: C-Space of a Two Link Manipulator

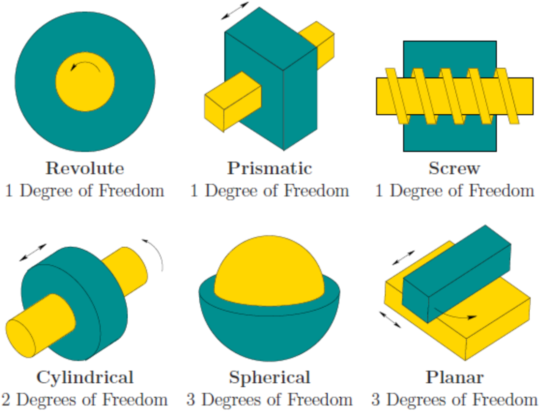


Topology of Configuration Space

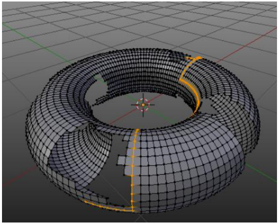
Degrees of Freedom for Robots with Joints

- An **articulated object** is a set of rigid bodies connected by joints.
- Examples of articulated robots: arms, humanoids

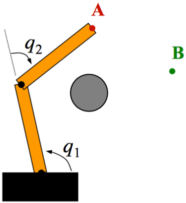




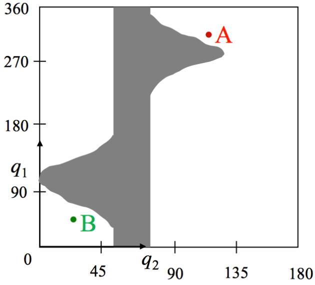
Obstacles in C-Space



- A configuration $\mathbf{q} \in C$ is collision-free, or **free**, if the robot placed at $\mathbf{q}$ does not intersect any obstacles in the work space W
- The **free space** $C_{free} \subseteq C$ is the set of all free configurations
- The **obstacle space** $C_{obs} \subseteq C$ is the set of all configurations in which the robot collides either with an obstacle or with itself (self-collision)



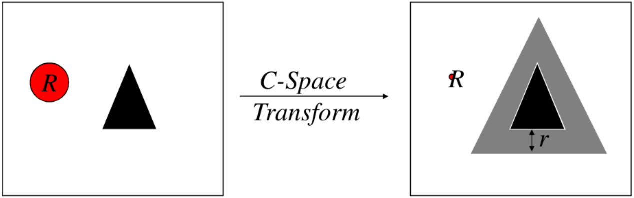
An obstacle in the robot's workspace



The C-space representation

How do we compute C_{obs}?

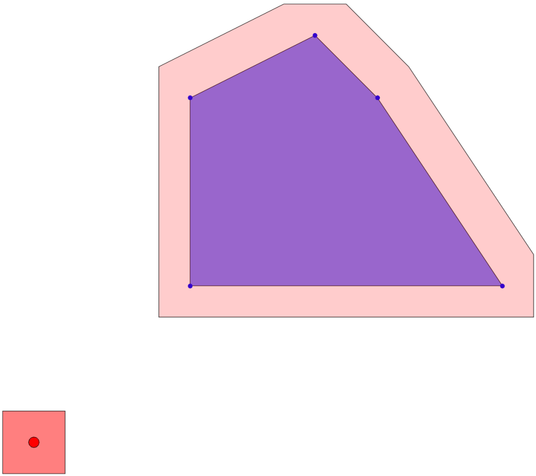
- **Input**: polygonal robot body R and polygonal obstacle O in environment
- **Output**: polygonal obstacle CO in configuration space
- **Assumption**: the robot translates only
- **Idea**:
  - Circular robot: expand all obstacles by the radius of the robot
  - Symmetric robot: Minkowski (set) sum
  - Asymmetric robot: Minkowski (set) difference



C_{obs} for Symmetric Robots

- The obstacle CO in C-Space is obtained via the **Minkowski sum** of the obstacle set O and the robot set R:

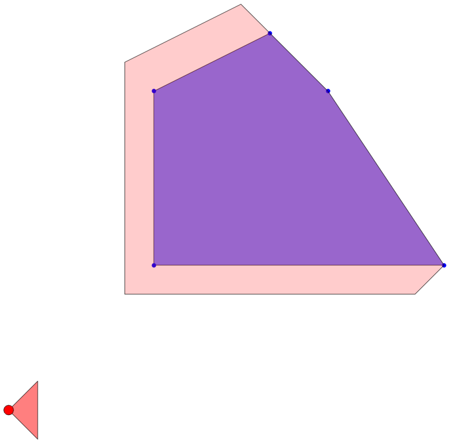
$$CO = O \oplus R := \{a + b \mid a \in O, b \in R\}$$



C_{obs} for Asymmetric Robots

- When the robot is not symmetric about the origin, we need to flip the robot set R before adding it to the obtacle set O:

$$CO = O \oplus (-R) = \{a - b \mid a \in O, b \in R\}$$

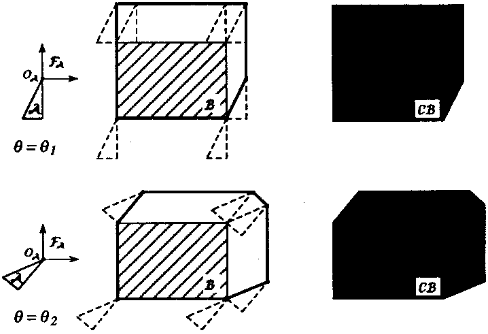


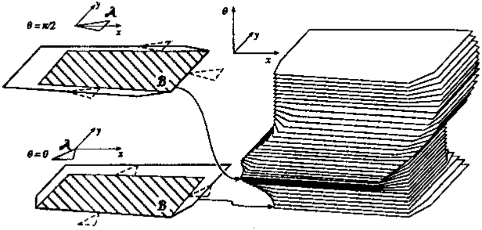
Properties of C_{obs}

- Properties of C_{obs}
  - If O and R are **convex**, then C_{obs} is **convex**
  - If O and R are **closed**, then C_{obs} is **closed**
  - If O and R are **compact**, then C_{obs} is **compact**
  - If O and R are **algebraic**, then C_{obs} is **algebraic**
  - If O and R are **connected**, then C_{obs} is **connected**
- After a C-Space transform, planning can be done for a point robot
  - **Advantage**: collision checking for a point robot is very efficient
  - **Disadvantage**: need to transform the obstacles every time the map is updated (e.g., $O(n)$ methods exist to compute distance transforms for circular robots)
  - **Disadvantage**: expensive to compute in higher dimensions
  - **Alternative**: plan in the original space and only check configurations of interest for collisions

Minkowski Sums in Higher Dimensions

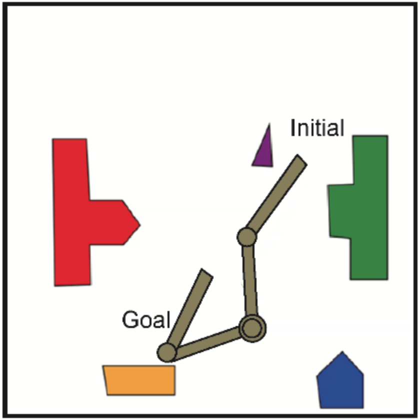
- The configuration space for a rigid non-circular robot in a 2D world is 3 dimensional (2D position + orientation)



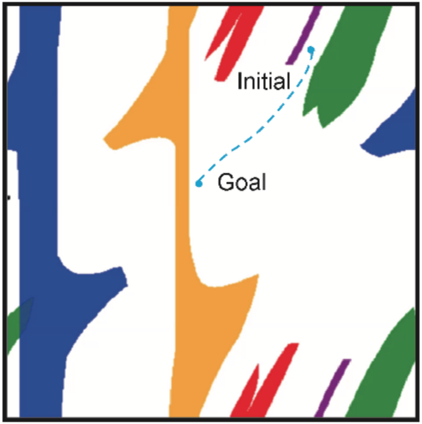


Configuration Space for Articulated Robots

- The configuration space for a N-DOF robot arm is N-dimensional
- Computing exact C-Space obstacles becomes complicated



Workspace



Configuration space

Interactive visualization: https://robotics.cs.unc.edu/C-space/

Outline

Motion Planning

Configuration Space

Graph Construction for Motion Planning

Motion Planning as Graph Search Problem

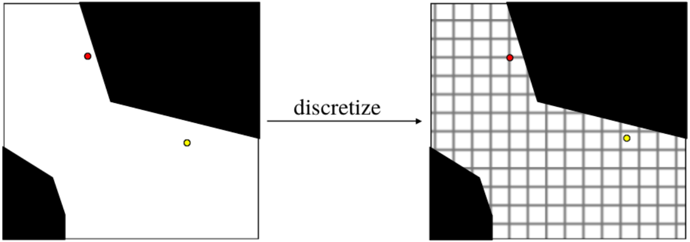
- Motion planning as a deterministic shortest path problem on a graph:
  1. Decide:
     a) pre-compute the C-Space (e.g., inflate the obstacles with the robot radius)
     b) perform collision checking on the fly
  2. Construct a graph representing the planning problem
  3. Search the graph for a (close-to) optimal path
- Often collision checking, graph construction, and planning are all interleaved and performed on the fly

Graph Construction Methods

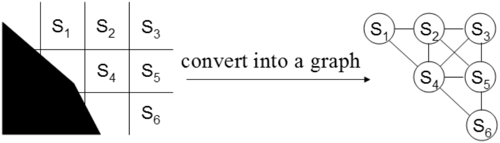
- **Cell decomposition**: decompose the free space into simple cells and represent its connectivity by the adjacency graph of these cells
  - X-connected grids
  - Tree decompositions
  - Lattice-based graphs
- **Skeletonization**: represent the connectivity of free space by a network of 1-D curves:
  - Visibility graphs
  - Generalized Voronoi diagrams
  - Other Roadmaps

X-Connected Grid

1. Overlay a uniform grid over the C-space

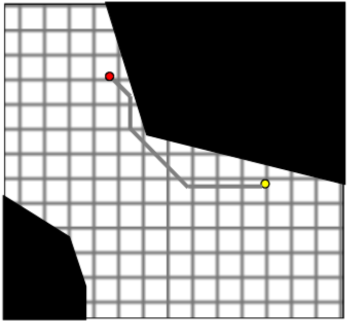


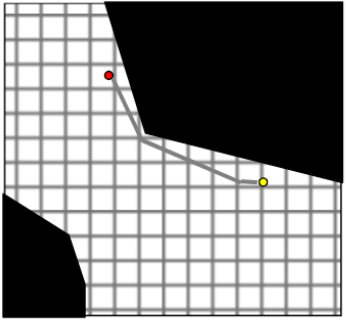
2. Convert the grid into a graph:



X-Connected Grid

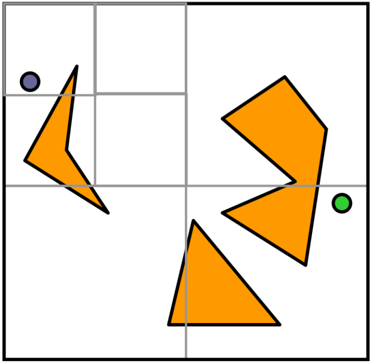
- How many neighbors?
  - 8-connected grid: paths restricted to 45° turns
  - 16-connected grid: paths restricted to 22.5° turns
  - 3-D (x, y, θ) discretization of $SE(2)$

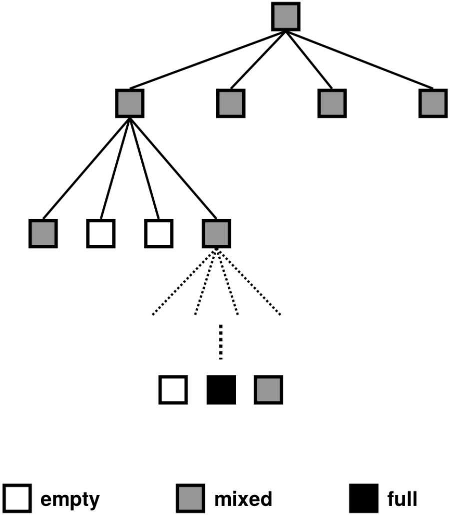




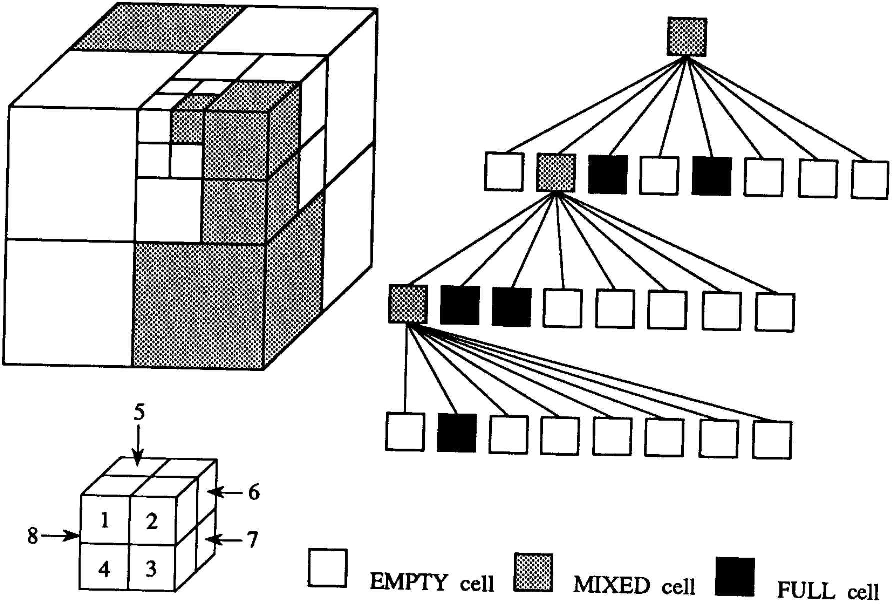
- Problems:
  1. What should we do with partially blocked cells?
  2. Discretization leads to a very dense graph in high dimensions and many of the transitions are difficult to execute due to dynamics constraints

Adaptive Quadtree Decomposition



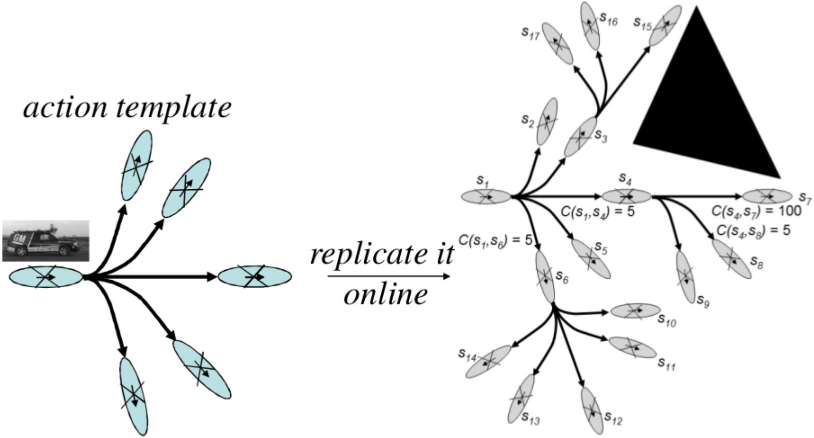


Adaptive Octree Decomposition



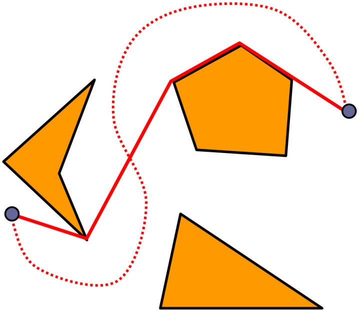
Lattice-Based Graph

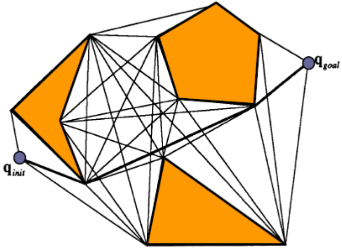
- Instead of dense discretization, construct a graph by a recursive application of a finite set of dynamically feasible motions (e.g., action template, motion primitive, movement primitive, macro action, etc.)
- **Pros**: sparse graph, feasible paths
- **Cons**: possibly incomplete



Visibility Graph

- Visibility graphs introduced in Shakey Project, SRI [Nilsson, 1969]
- Also called **shortest path roadmap**
- **Shortest paths are like rubber-bands**: if there is a collision-free path between two points, then there is a piecewise linear path that bends only at the obstacle vertices
- **Visibility graph**:
  - **Nodes**: start, goal, and all obstacle vertices
  - **Edges**: between any two vertices that "see" each other, i.e., the edge does not intersect obstacles or is an obstacle edge





Visibility Graph Construction

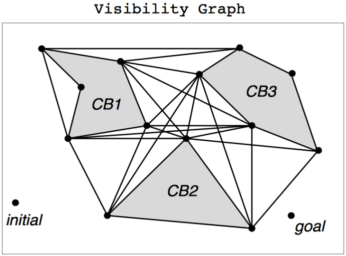
Algorithm Visibility Graph Construction

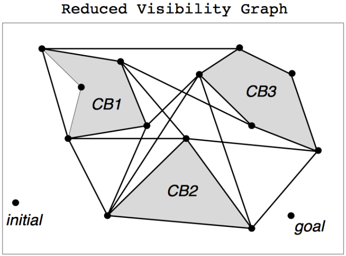
1: **Input**: $\mathbf{q}_I$, $\mathbf{q}_G$, polygonal obstacle vertices $\mathcal{P}$
2: **Output**: visibility graph G
3: **for** every pair of vertices u, v in $\mathcal{P} \cup \{\mathbf{q}_I, \mathbf{q}_G\}$ **do** ▷ $O(n^2)$
4:     **if** segment(u, v) is an obstacle edge **then** ▷ $O(n)$
5:         insert edge(u, v) into G
6:     **else**
7:         **for** every obstacle edge e **do** ▷ $O(n)$
8:             **if** segment(u, v) intersects e **then**
9:                 **break and go to line 3**
10:         insert edge(u, v) into G

- **Time complexity**: $O(n^3)$ but can be reduced to $O(n^2 \log n)$ with rotational sweep or even to $O(n^2)$ with an optimal algorithm
- **Space complexity**: $O(n^2)$

Reduced Visibility Graph

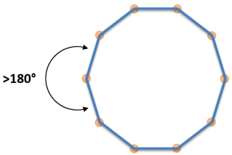
- Not all edges are needed
- **Reduced visibility graph** – keep only edges between consecutive **reflex vertices** and **bitangents**
- A vertex of a polygonal obstacle is **reflex** if the exterior angle (computed in C_{free}) is larger than π
- A **bitangent edge** must touch two **reflex vertices** that are mutually visible from each other, and the the line must extend outward past each of them without poking into C_{obs}



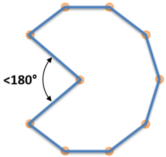


Reflex Vertices and Bitangents

- A vertex of a polygonal obstacle is **reflex** if the exterior angle (computed in C_{free}) is larger than π

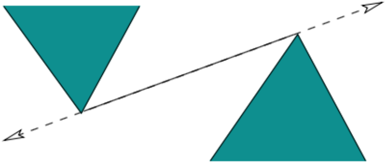


(a) 10 reflex vertices



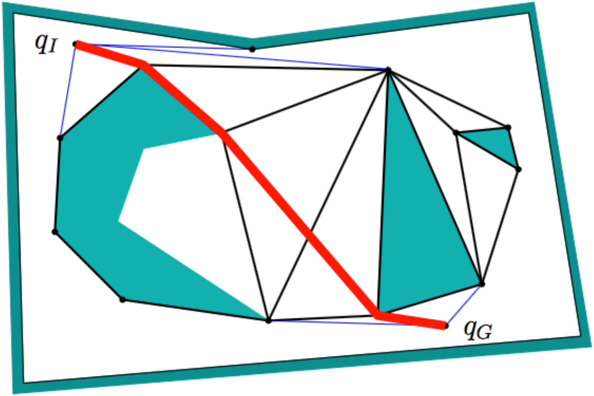
(b) 9 reflex vertices

- A **bitangent edge** must touch two **reflex vertices** that are mutually visible from each other, and the the line must extend outward past each of them without poking into C_{obs}



Reduced Visibility Graph

- Reduced visibility graph: includes edges between consecutive reflex vertices on C_{obs} and bitangent edges
- The shortest path in a reduced visibility graph is the shortest path between start $\mathbf{q}_I$ and goal $\mathbf{q}_G$



Reduced Visibility Graph

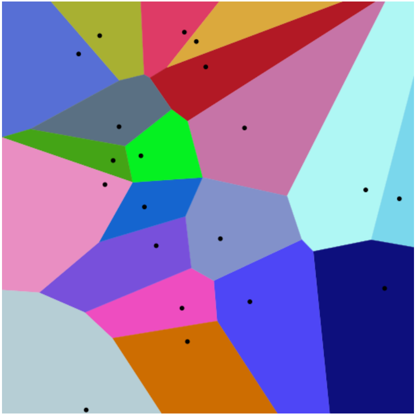
- What do we need to construct a reduced visibility graph?
  - Subroutine to check if a vertex is reflex
  - Subroutine to check if two vertices are visible
  - Subroutine to check if there exists a bitangent
- Pros:
  - independent of the size of the environment
  - can make multiple shortest path queries for the same graph, i.e., the environment remains the same but the start and goal change
- Cons:
  - shortest paths always graze the obstacles
  - hard to deal with a non-uniform cost function
  - hard to deal with non-polygonal obstacles
  - can get expensive in high dimensions with a lot of obstacles

Voronoi Diagram

- Suppose there are n point obstacles $\mathbf{o}_k$ for $k = 1, \ldots, n$
- **Voronoi diagram**: a collection of Voronoi cells V_k for $k = 1, \ldots, n$
- **Voronoi cell of $\mathbf{o}_k$**: a set V_k of points $\mathbf{x}$ such that:

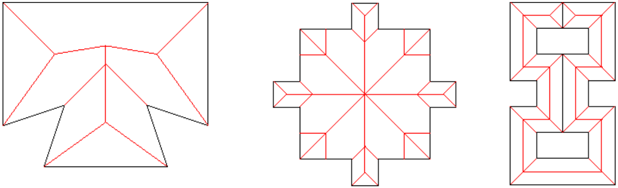
$$d(\mathbf{x}, \mathbf{o}_k) \leq d(\mathbf{x}, \mathbf{o}_j), \text{for all } j \neq k$$

- Example: the points may represent fire stations and the Voronoi cells specify their serving areas

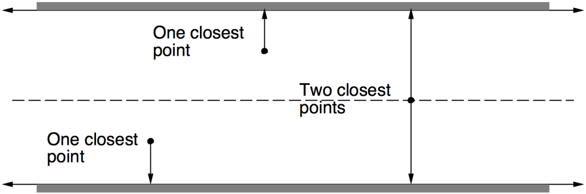


Maximum Clearance Roadmap

- Maximize clearance instead of minimizing travel distance
- Maintains a set of points that are equidistant to two nearest obstacles



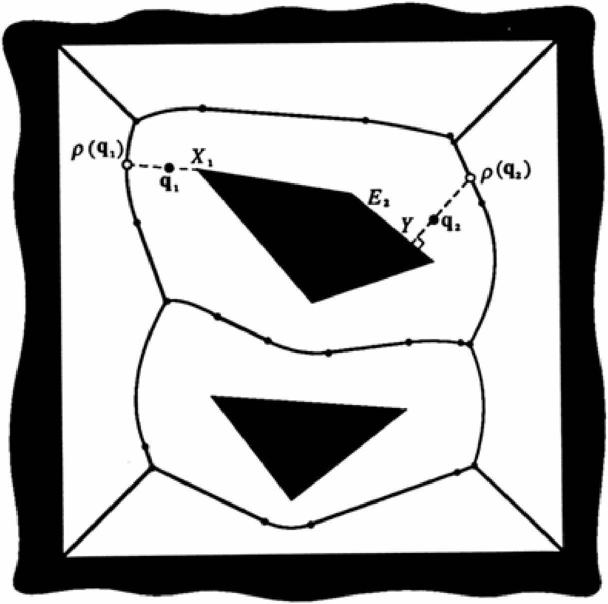
- Suppose we have just two line obstacles. What is the set of points that keeps the robots as far away from the obstacles as possible?



Maximum Clearance Roadmap

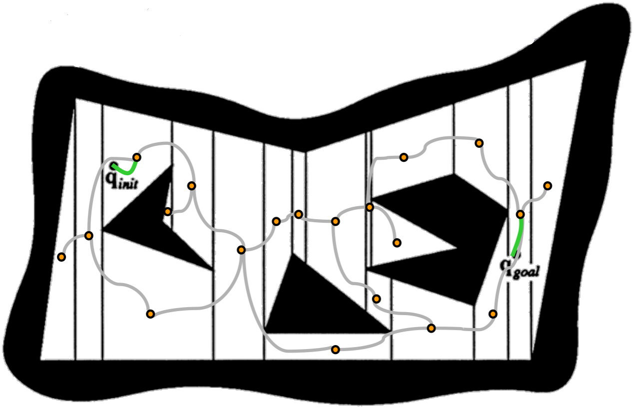
- Construction:
  - Naive implementation: take every pair of obstacle features, compute locus of equally spaced points, and take the intersection
  - Efficient algorithms available, e.g., CGAL, distance transform + skeletonization (e.g., Zhang-Suen or Guo-Hall algorithms)
- Motion Planning:
  - Add a shortest path from start to the nearest segment of the diagram
  - Add a shortest path from goal to the nearest segment of the diagram
- Complexity:
  - Time complexity for n points in $\mathbb{R}^d$: $O(n \log n + n^{\lceil d/2 \rceil})$
  - Space complexity: $O(n)$
- Pros:
  - paths tend to stay away from obstacles
  - independent of the size of the environment
- Cons:
  - difficult to construct in higher dimensions
  - can result in highly suboptimal paths

Maximum Clearance Roadmap



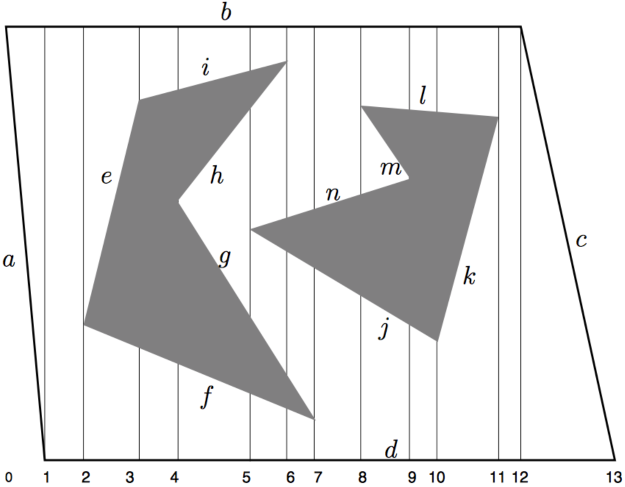
Trapezoidal Decomposition

- The free space C_{free} is represented by a collection of non-overlapping trapezoids whose union is exactly C_{free}
- Draw a vertical line from every vertex until you hit an obstacle
  - **Nodes**: trapezoid centroids and line midpoints
  - **Edges**: between every pair of nodes whose cells are adjacent

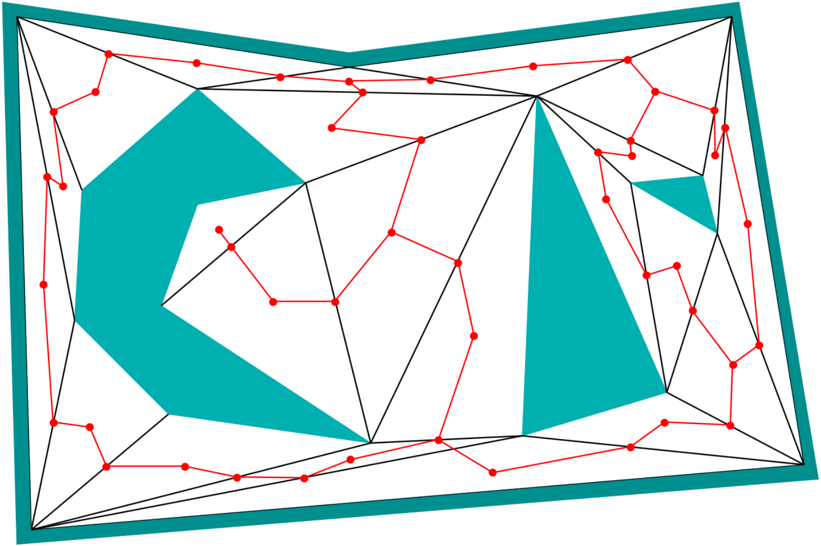


Cylindrical Decomposition

- Similar to trapezoidal decomposition, except the vertical lines continue after obstacles
- Generalizes better to high dimensions and complex configuration spaces

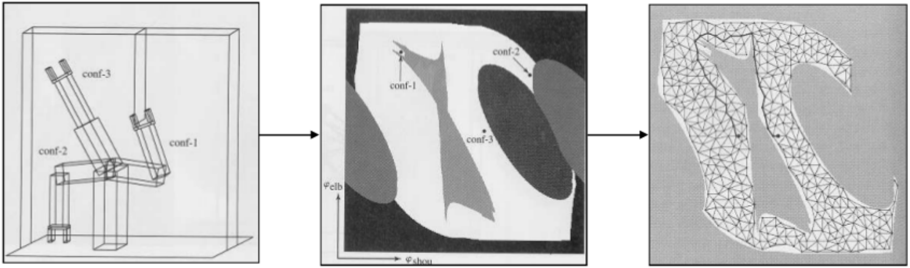


Triangular Decomposition



Probabilistic Roadmap

- Construction:
  - Randomly sample valid configurations
  - Add edges between samples that are easy to connect with a simple local controller (e.g., straight line controller)
  - Add start and goal configurations to the graph with appropriate edges



- Pros and Cons:
  - Simple and highly effective in high dimensions
  - Can result in suboptimal paths, no guarantees on suboptimality
  - Difficulty with narrow passages