ECE276B: Planning & Learning in Robotics
Lecture 16: Model-free Control

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Generalized Policy Iteration

- **Policy Evaluation**: Given π, compute V^π:

$$V^\pi(x) = g(x, \pi(x)) + \gamma \mathbb{E}_{x' \sim p_f(\cdot|x,\pi(x))} \left[V^\pi(x') \right], \quad \forall x \in \mathcal{X}$$

- **Policy Improvement**: Given V^π obtain a new policy π':

$$\pi'(x) = \arg\min_{u \in \mathcal{U}(x)} \left\{ g(x, u) + \gamma \mathbb{E}_{x' \sim p_f(\cdot|x,u)} \left[V^\pi(x') \right] \right\}, \quad \forall x \in \mathcal{X}$$

Policy Improvement Theorem

Let π and π' be deterministic policies such that $V^\pi(x) \geq Q^\pi(x, \pi'(x))$ for all $x \in \mathcal{X}$. Then, π' is at least as good as π, i.e., $V^\pi(x) \geq V^{\pi'}(x)$ for all $x \in \mathcal{X}$

- **Proof**:

$$\begin{aligned}
V^\pi(x) \geq Q^\pi(x, \pi'(x)) &= g(x, \pi'(x)) + \gamma \mathbb{E}_{x' \sim p_f(\cdot x, \pi'(x))} V^\pi(x') \\
&\geq g(x, \pi'(x)) + \gamma \mathbb{E}_{x' \sim p_f(\cdot x, \pi'(x))} Q^\pi(x', \pi'(x')) \\
&= g(x, \pi'(x)) + \gamma \mathbb{E}_{x' \sim p_f(\cdot x, \pi'(x))} \left\{ g(x', \pi'(x')) + \gamma \mathbb{E}_{x'' \sim p_f(\cdot x', \pi'(x'))} V^\pi(x'') \right\} \\
&\geq \cdots \geq \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t g(x_t, \pi'(x_t)) \middle| x_0 = x \right] = V^{\pi'}(x)
\end{aligned}$$

Model-free Generalized Policy Iteration

- **Policy Evaluation**: given π, GPI iterates $\mathcal{T}_\pi$ to compute V^π:

$$DP : \mathcal{T}_\pi[V](x) = g(x, \pi(x)) + \gamma \mathbb{E}_{x' \sim p_f(\cdot|x,\pi(x))} \left[V(x') \right]$$

$$TD : \mathcal{T}_\pi[V](x_t) \approx V(x_t) + \alpha \left[g(x_t, u_t) + \gamma V(x_{t+1}) - V(x_t) \right]$$

$$MC : \mathcal{T}_\pi[V](x_t) \approx V(x_t) + \alpha \left[\sum_{k=0}^{T-t-1} \gamma^k g(x_{t+k}, u_{t+k}) + \gamma^{T-t} g_T(x_T) - V(x_t) \right]$$

or alternatively Q^π:

$$DP : \mathcal{T}_\pi[Q](x, u) = g(x, u) + \gamma \mathbb{E}_{x' \sim p_f(\cdot|x,u)} \left[Q(x', \pi(x')) \right]$$

$$TD : \mathcal{T}_\pi[Q](x_t, u_t) \approx Q(x_t, u_t) + \alpha \left[g(x_t, u_t) + \gamma Q(x_{t+1}, u_{t+1}) - Q(x_t, u_t) \right]$$

$$MC : \mathcal{T}_\pi[Q](x_t, u_t) \approx Q(x_t, u_t) + \alpha \left[\sum_{k=0}^{T-t-1} \gamma^k g(x_{t+k}, u_{t+k}) + \gamma^{T-t} g_T(x_T) - Q(x_t, u_t) \right]$$

- **Policy Improvement**: given V^π or Q^π compute improved π':

$$Q^\pi(x, u) = g(x, u) + \gamma \mathbb{E}_{x' \sim p_f(\cdot|x,u)} \left[V^\pi(x') \right]$$

$$\pi'(x) = \arg\min_{u \in \mathcal{U}(x)} Q^\pi(x, u)$$

Model-free Generalized Policy Iteration

- **Policy Evaluation**: use MC or TD to estimate Q^π instead of V^π so that the policy improvement step is model-free, i.e., can compute $\min_u Q^\pi(x, u)$ without knowing p_f
- **Policy Improvement**: the fact that Q^π is approximation still causes some problems:
  - Picking the "best" control according to the current estimate Q^π might not be the actual best control
  - If a deterministic policy is used for Evaluation/Improvement, one will observe returns for only one of the possible controls at each state and also might not visit many states. Hence, estimating Q^π will not be possible at those never-visited states and controls

Example: Greedy Control Selection (David Silver)

- There are two doors in front of you
- You open the left door and get reward 0
  $V(left) = +0$
- You open the right door and get reward $+1$
  $V(right) = +1$
- You open the right door and get reward $+3$
  $V(right) = +3$
- You open the right door and get reward $+2$
  $V(right) = +2$
- Are you sure you have chosen the best door?

"Behind one door is tenure - behind the other is flipping burgers at McDonald's."

Model-free Control

- Two ideas to ensure that you do not commit to the (wrong) controls too early and continue exploring the state and control spaces:

  1. **Exploring Starts**: in each episode $\rho^{(k)} \sim \pi$, choose initial state-control pairs with non-zero probability among all possible pairs $\mathcal{X} \times \mathcal{U}$
  2. ϵ-**Soft Policy**: use a **stochastic policy** under which every control has a non-zero probability of being chosen and hence every reachable state will have non-zero probability of being encountered

First-visit MC Policy Iteration with Exploring Starts

Algorithm 1 MC Policy Iteration with Exploring Starts

Init: $Q(x,u), \pi(x)$ for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$
loop
  Choose $(x_0, u_0) \in \mathcal{X} \times \mathcal{U}$ randomly ▷ exploring starts!
  Generate an episode $\rho = x_0, u_0, x_1, u_1, \ldots, x_{T-1}, u_{t-1}, x_T$ from π
  for each x, u in ρ **do**
    $G \leftarrow$ return following the first occurrence of x, u
    $Q(x,u) \leftarrow Q(x,u) + \alpha\,(G - Q(x,u))$
  for each x in ρ **do**
    $\pi(x) \leftarrow \arg\min_u Q(x,u)$

ϵ-Greedy Exploration

- As an alternative to exploring starts, to ensure continual exploration it must be possible to encounter all $|\mathcal{U}(x)|$ controls at state x with non-zero probability

- **ϵ-Greedy Policy** a stochastic policy that picks the best control according to $Q(x, u)$ in the policy improvement step but ensures that all other controls are selected with a small (non-zero) probability:

$$\pi(u \mid x) = \begin{cases} 1 - \epsilon + \frac{\epsilon}{|\mathcal{U}(x)|} & \text{if } u = \arg\min_{u' \in \mathcal{U}(x)} Q(x, u') \\ \frac{\epsilon}{|\mathcal{U}(x)|} & \text{otherwise} \end{cases}$$

ϵ-Greedy Policy Improvement

Theorem: ϵ-Greedy Policy Improvement

For any ϵ-soft policy π, the ϵ-greedy policy π' with respect to Q^π is an improvement, i.e., $V^{\pi'}(x) \leq V^\pi(x)$ for all $x \in \mathcal{X}$

- **Proof**:

$$\begin{aligned}
\mathbb{E}_{u' \sim \pi'(\cdot|x)}\left[Q^\pi(x,u')\right] &= \sum_{u' \in \mathcal{U}(x)} \pi'(u' \mid x) Q^\pi(x,u') \\
&= \frac{\epsilon}{|\mathcal{U}(x)|} \sum_{u' \in \mathcal{U}(x)} Q^\pi(x,u') + (1-\epsilon) \min_{u \in \mathcal{U}(x)} Q^\pi(x,u) \\
&\leq \frac{\epsilon}{|\mathcal{U}(x)|} \sum_{u' \in \mathcal{U}(x)} Q^\pi(x,u') + (1-\epsilon) \sum_{u \in \mathcal{U}(x)} \frac{\pi(u \mid x) - \frac{\epsilon}{|\mathcal{U}(x)|}}{1-\epsilon} Q^\pi(x,u) \\
&= \sum_{u \in \mathcal{U}(x)} \pi(u \mid x) Q^\pi(x,u) = V^\pi(x)
\end{aligned}$$

- Then, from the policy improvement theorem, $V^{\pi'}(x) \leq V^\pi(x)$, $\forall x \in \mathcal{X}$

First-visit MC Policy Iteration with ϵ-Greedy Improvement

Algorithm 2 First-visit MC Policy Iteration with ϵ-Greedy Improvement

Init: $Q(x,u)$, $\pi(u|x)$ (ϵ-soft policy) for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$
loop
    Generate an episode $\rho := x_0, u_0, x_1, u_1, \ldots, x_{T-1}, u_{t-1}, x_T$ from π
    for each x, u in ρ **do**
        $G \leftarrow$ return following the first occurrence of x, u
        $Q(x,u) \leftarrow Q(x,u) + \alpha\,(G - Q(x,u))$
    for each x in ρ **do**
        $u^* \leftarrow \arg\min_u Q(x,u)$
        $\pi(u|x) \leftarrow \begin{cases} 1 - \epsilon + \frac{\epsilon}{|\mathcal{U}(x)|} & \text{if } u = u^* \\ \frac{\epsilon}{|\mathcal{U}(x)|} & \text{if } u \neq u^* \end{cases}$

Temporal-Difference Control

- TD prediction has several advantages over MC prediction:
  - Works with incomplete episodes
  - Lower variance
  - Can perform online updates after every transition
- To use TD instead of MC in a complete policy iteration algorithm, we still need to trade-off exploration and exploitation:
  - Apply TD to $Q(x, u)$ for policy evaluation
  - Can update $Q(x, u)$ after every transition within an episode
  - Use an ϵ-greedy policy for policy improvement

TD Policy Iteration with ϵ-Greedy Improvement (SARSA)

- **SARSA**: estimates the action-value function Q^{π} using TD updates after every $S_t, A_t, R_{t+1}, S_{t+1}, A_{t+1}$ transition:

$$Q(x_t, u_t) \leftarrow Q(x_t, u_t) + \alpha \left[g(x_t, u_t) + \gamma Q(x_{t+1}, u_{t+1}) - Q(x_t, u_t) \right]$$

- Ensures exploration via an ϵ-greedy policy in the policy improvement step

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Algorithm 3 SARSA

Init: $Q(x, u)$ for all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$
loop
  $\pi \leftarrow \epsilon$-greedy policy derived from Q
  Generate episode $\rho := (x_{0:T}, u_{0:T-1})$ from π
  for $(x, u, x', u') \in \rho$ **do**
    $Q(x, u) \leftarrow Q(x, u) + \alpha \left[g(x, u) + \gamma Q(x', u') - Q(x, u) \right]$

Convergence of Model-free Policy Iteration

- **Greedy in the Limit with Infinite Exploitation** (GLIE):
  - All state-control pairs are explored infinitely many times: $\lim_{k \to \infty} N(x, u) = \infty$
  - The ϵ-greedy policy converges to a greedy policy

$$\lim_{k \to \infty} \pi_k(u \mid x) = \mathbb{1}\{u = \arg\min_{u' \in \mathcal{U}(x)} Q(x, a')\}$$

- Example: If $\epsilon_k = \frac{1}{k}$, then ϵ-greedy is GLIE

Theorem: Convergence of MC Policy Iteration

GLIE MC Policy Iteration converges to the optimal action-value function, $Q(x, u) \to Q^*(x, u)$ as the number of episodes $k \to \infty$

Theorem: Convergence of TD Policy Iteration

SARSA converges to the optimal action-value function, $Q(x, u) \to Q^*(x, u)$ as $k \to \infty$ as long as:

- The sequence of ϵ-greedy policies $\pi_k(u \mid x)$ is GLIE
- The sequence of step sizes α_k is Robbins-Monro

On-Policy vs Off-Policy Learning

- **On-policy Prediction**: estimate V^π or Q^π using experience from π
- **Off-policy Prediction**: estimate V^π or Q^π using experience from μ
- On-policy methods:
  - evaluate or improve the policy that is used to make decisions
  - require well-designed exploration functions
  - empirically successful with function approximation
- Off-policy methods:
  - evaluate or improve a different policy from the (behavior) policy used to generate data
  - can use an effective exploratory policy to generate data while learning about an optimal policy
  - can learn from observing humans or other agents
  - can re-use experience from old policies $\pi_1, \pi_2, \ldots, \pi_{n-1}$
  - can learn about multiple policies while following one policy
  - have problems with function approximation and eligibility traces

Importance Sampling for Off-policy Learning

- To use returns generated from μ to evaluate π, we need to re-weight the stage-costs according to the similarity (i.e., likelihood) of states encountered by the two different policies

- **Importance Sampling**: estimates the expectation of a function with respect to a different distribution:

$$\begin{aligned}\mathbb{E}_{x\sim p(\cdot)}[f(x)] &= \int p(x)f(x)dx \\ &= \int q(x)\frac{p(x)}{q(x)}f(x)dx = \mathbb{E}_{x\sim q(\cdot)}\left[\frac{p(x)}{q(x)}f(x)\right]\end{aligned}$$

Importance Sampling for Off-policy MC Learning

- To use returns generated from μ to evaluate π via MC, weight the long-term cost G_t via importance-sampling corrections along the whole episode:

$$G_t^{\pi/\mu} = \frac{\pi(u_t|x_t)}{\mu(u_t|x_t)} \frac{\pi(u_{t+1}|x_{t+1})}{\mu(u_{t+1}|x_{t+1})} \cdots \frac{\pi(u_{T-1}|x_{T-1})}{\mu(u_{T-1}|x_{T-1})} G_t$$

- Update the value estimate towards the *corrected return*:

$$V(x_t) \leftarrow V(x_t) + \alpha \left(G_t^{\pi/\mu} - V(x_t) \right)$$

- Importance sampling in MC can dramatically increase the variance and cannot be used if μ is zero when π is non-zero

Importance Sampling for Off-policy TD Learning

- To use returns generated from μ to evaluate π via TD, weight the TD target $g(x,u) + \gamma V(x')$ by importance sampling:

$$V(x_t) \leftarrow V(x_t) + \alpha \left(\frac{\pi(u_t \mid x_t)}{\mu(u_t \mid x_t)} \left(g(x_t, u_t) + \gamma V(x_{t+1})\right) - V(x_t) \right)$$

- Importance sampling in TD is much lower variance than in MC and the policies need to be similar (i.e., μ should not be zero when π is non-zero) over a single step only

Off-policy TD Control without Importance Sampling

- **Q-Learning** (Watkins, 1989): one of the early breakthroughs in reinforcement learning was the development of an off-policy TD algorithm that does not use importance sampling

- Q-Learning approximates $\mathcal{T}_*[Q](x,u)$ directly using samples:

$$Q(x_t,u_t) \leftarrow Q(x_t,u_t) + \alpha \left[g(x_t,u_t) + \gamma \min_{u \in \mathcal{U}(x_{t+1})} Q(x_{t+1},u) - Q(x_t,u_t) \right]$$

- The learned Q function eventually approximates Q^* **regardless of the policy being followed**!

Theorem

Q-Learning converges almost surely to Q^* assuming that all state-control pairs continue to be updated and the learning rate α is chosen via the usual Robbins-Monro stochastic approximation condition

- C. J. Watkins and P. Dayan. "Q-learning," Machine learning, 1992.

Q-Learning: Off-policy TD Learning

Algorithm 4 Q-Learning

Init: $Q(x, u)$ for all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$
loop
    $\pi \leftarrow \epsilon$-greedy policy derived from Q
    Generate episode $\rho := (x_{0:T}, u_{0:T-1})$ from π
    for $(x, u, x') \in \rho$ **do**
        $Q(x, u) \leftarrow Q(x, u) + \alpha \left[g(x, u) + \gamma \min_{u'} Q(x', u') - Q(x, u)\right]$

Relationship Between DP and TD

Full Backups (DP)	**Sample Backups** (TD)
Policy Evaluation $V(x) \leftarrow \mathcal{T}_\pi[V](x) = g(x, \pi(x)) + \gamma \mathbb{E}_{x'}\left[V(x')\right]$	**TD Prediction** $V(x) \leftarrow V(x) + \alpha(g(x,u) + \gamma V(x') - V(x))$
Policy Q-Evaluation $Q(x,u) \leftarrow \mathcal{T}_\pi[Q](x,u) = g(x,u) + \gamma \mathbb{E}_{x'}\left[Q(x', \pi(x'))\right]$	**SARSA** $Q(x,u) \leftarrow Q(x,u) + \alpha(g(x,u) + \gamma Q(x',u') - Q(x,u))$
Value Iteration $V(x) \leftarrow \mathcal{T}_*[V](x) = \min_u \left\{g(x,u) + \gamma \mathbb{E}_{x'}\left[V(x')\right]\right\}$	**N/A**
Q-Value Iteration $Q(x,u) \leftarrow \mathcal{T}_*[Q](x,u) = g(x,u) + \gamma \mathbb{E}_{x'}\left[\min_{u'} Q(x',u')\right]$	**Q-Learning** $Q(x,u) \leftarrow Q(x,u) + \alpha\left(g(x,u) + \gamma \min_{u'} Q(x',u') - Q(x,u)\right)$

Batch Sampling-based Q-Value Iteration

Algorithm 5 Batch Sampling-based Q-Value Iteration

Init: $Q^{(0)}(x, u)$ for all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$
loop
    $\pi \leftarrow \epsilon$-greedy policy derived from Q
    Generate episodes $\{\rho^{(k)}\}_{k=1}^{K}$ from π
    for $(x, u) \in \mathcal{X} \times \mathcal{U}$ **do**
        $$Q^{(i+1)} = \frac{1}{K} \sum_{k=1}^{K} \frac{\sum_{t=0}^{T} \mathcal{T}_*[Q^{(i)}](x_t^{(k)}, u_t^{(k)}, x_{t+1}^{(k)}) \mathbb{1}\{(x_t^{(k)}, u_t^{(k)}) = (x, u)\}}{\sum_{t=0}^{T} \mathbb{1}\{(x_t^{(k)}, u_t^{(k)}) = (x, u)\}}$$

- Batch Sampling-based Q-Value Iteration behaves like $Q^{(i+1)} = \mathcal{T}_*[Q^{(i)}] + \text{noise}$. Does it actually converge?

Least-squares Backup Version

- $Q^{(i+1)}(x,u) = \textbf{mean}\left\{\mathcal{T}_*[Q^{(i)}](x_t^{(k)}, u_t^{(k)}, x_{t+1}^{(k)}),\ \forall k, t \text{ such that } (x_t^{(k)}, u_t^{(k)}) = (x,u)\right\}$
- $\textbf{mean}\left\{x^{(k)} = \arg\min_x \sum_{k=1}^{K} \|x^{(k)} - x\|^2\right\}$
- $Q^{(i+1)}(x,u) = \arg\min_q \sum_{k=1}^{K} \sum_{(x_t^{(k)}, u_t^{(k)})=(x,u)} \left\|\mathcal{T}_*[Q^{(i)}](x_t^{(k)}, u_t^{(k)}, x_{t+1}^{(k)}) - q\right\|^2$
- $Q^{(i+1)} = \arg\min_Q \sum_{k=1}^{K} \sum_{t=0}^{T} \left\|\mathcal{T}_*[Q^{(i)}](x_t^{(k)}, u_t^{(k)}, x_{t+1}^{(k)}) - Q(x_t^{(k)}, u_t^{(k)})\right\|^2$

Algorithm 6 Batch Least-squares Q-Value Iteration

Init: $Q^{(0)}(x,u)$ for all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$
loop
  $\pi \leftarrow \epsilon$-greedy policy derived from Q
  Generate episodes $\{\rho^{(k)}\}_{k=1}^{K}$ from π
  $Q^{(i+1)} = \arg\min_Q \sum_{k=1}^{K} \sum_{t=0}^{T} \left\|\mathcal{T}_*[Q^{(i)}](x_t^{(k)}, u_t^{(k)}, x_{t+1}^{(k)}) - Q(x_t^{(k)}, u_t^{(k)})\right\|^2$

Small Steps in the Backup Direction

- Full backup: $Q^{(i+1)} \leftarrow \mathcal{T}_*[Q^{(i)}]$
- Partial backup: $Q^{(i+1)} \leftarrow \epsilon \mathcal{T}_*[Q^{(i)}] + (1-\epsilon) Q^{(i)}$
- Equivalent to a gradient step on squared error:

$$\begin{aligned} Q^{(i+1)} \leftarrow \epsilon \mathcal{T}_*[Q^{(i)}] + (1-\epsilon) Q^{(i)} &= Q^{(i)} - \epsilon \left(Q^{(i)} - \mathcal{T}_*[Q^{(i)}] \right) \\ &= Q^{(i)} - \epsilon \left(\frac{1}{2} \nabla_Q \| Q^{(i)} - \mathcal{T}_*[Q^{(i)}] \|^2 \Big|_{Q=Q^{(i)}} + \text{noise} \right) \end{aligned}$$

- Behaves like stochastic gradient descent on $L(Q) := \frac{1}{2}\|\mathcal{T}_*[Q] - Q\|^2$ but the objective is changing, i.e., $\mathcal{T}_*[Q^{(i)}]$ is a moving target. Does it converge?
- **Stochastic Approximation Theory**: a "partial update" to ensure contraction + appropriate step size ϵ implies convergence to the contraction fixed point: $\lim_{i \to \infty} Q^{(i)} = Q^*$
- T. Jaakkola, M. Jordan, S. Singh, "On the convergence of stochastic iterative dynamic programming algorithms," Neural computation, 1994.

Least-squares Partial Backup Version

Algorithm 7 Batch Gradient Least-squares Q-Value Iteration

1: **Init**: $Q^{(0)}(x,u)$ for all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$
2: **loop**
3: $\quad \pi \leftarrow \epsilon$-greedy policy derived from Q
4: $\quad$ Generate episodes $\{\rho^{(k)}\}_{k=1}^{K}$ from π
5: $\quad Q^{(i+1)}(x,u) \leftarrow Q^{(i)}(x,u) + \frac{\epsilon}{2}\nabla_Q \sum_{k=1}^{K}\sum_{t=0}^{T-1} \|\mathcal{T}_*[Q^{(i)}](x_t^{(k)}, u_t^{(k)}, x_{t+1}^{(k)}) - Q(x_t^{(k)}, u_t^{(k)})\|^2$

- Watkins Q-learning is a special case with $T = 1$