ECE276B: Planning & Learning in Robotics
Lecture 1: Markov Chains

Lecturer:

Nikolay Atanasov: natanasov@ucsd.edu

Teaching Assistants:

Tianyu Wang: tiw161@eng.ucsd.edu

Yongxi Lu: yol070@eng.ucsd.edu

UC San Diego

JACOBS SCHOOL OF ENGINEERING

Electrical and Computer Engineering

What is this class about?

- In ECE276A, we studied the fundamental problems of sensing and state estimation:
  - how to model a robot's motion and observations
  - how to estimate (a distribution of) the robot state x_t from the history of observations $z_{0:t}$ and control inputs $u_{0:t-1}$
- In ECE276B, we will focus on the fundamental problems of planning and decision making:
  - how to model tasks such as navigate to a goal without crashing or improve the state estimate by choosing informative observations
  - how to select the controls $u_{0:t-1}$ that achieve these tasks
- References (not required):
  - Dynamic Programming and Optimal Control: Bertsekas
  - Planning Algorithms: LaValle (`http://planning.cs.uiuc.edu`)
  - Reinforcement Learning: Sutton & Barto (`http://incompleteideas.net/book/the-book.html`)
  - Calculus of Variations and Optimal Control Theory: Liberzon (`http://liberzon.csl.illinois.edu/teaching/cvoc.pdf`)

Logistics

- Course website: `https://natanaso.github.io/ece276b`
- Includes links to (sign up!):
  - **Piazza**: discussion – it is your responsibility to check Piazza regularly because class announcements, updates, etc., will be posted there
  - **GradeScope**: homework submissions and grades
- Four assignments (roughly 25% each, detailed rubric online) including:
  - theoretical homework
  - programming assignments in **python**
  - project report
- Grading:
  - Letter grades will be assigned based on the class performance, i.e., there will be a "curve"
  - **Late policy**: there will be a 10% penalty for submitting your work up to 1 week late. Work submitted more than a week late will receive 0 credit.

Prerequisites

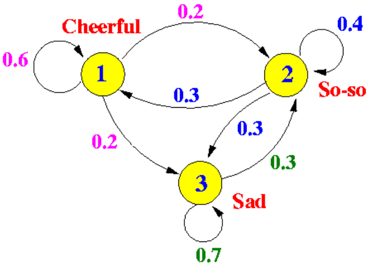
- **Probability theory**: random vectors, probability density functions, expectation, covariance, total probability, conditioning, Bayes rule
- **Linear algebra/systems**: eigenvalues, positive definiteness, linear systems of ODEs, matrix exponential
- **Optimization**: gradient descent, linear constraints, convex functions
- **Programming**: python/C++/Matlab, classes/objects, data structures (e.g., queue, list), data input/output, plotting
- It is up to you to judge if you are ready for this course!
  - Consult with your classmates who took ECE276A
  - Take a look at the ECE276A material: `https://natanaso.github.io/ece276a/schedule.html`
  - If the first assignment in ECE276B seems hard, the rest will be hard as well

Syllabus Snapshot

Date	Lecture	Materials	Assignments
Jan 09	Introduction, Markov Chains		
Jan 11	Markov Decision Processes	Bertsekas 1.1-1.2	
Jan 16	Dynamic Programming	Bertsekas 1.3-1.4	P1
Jan 18	Deterministic Shortest Path	Bertsekas 2.1-2.3	
Jan 23	Configuration Space	LaValle 4.3, 6.2-6.3	
Jan 25	Search-based Planning I	LaValle 2.1-2.3	
Jan 30	Search-based Planning II		P2
Feb 01	Sampling-based Planning I	LaValle 5.5-5.6	
Feb 06	Sampling-based Planning II		
Feb 08	TBD (Collision Checking, Non-holonomic Planning)		
Feb 13	Stochastic Shortest Path	Bertsekas 7.1-7.3	
Feb 15	Bellman Equations I	Sutton-Barto 4.1-4.4	P3
Feb 20	Bellman Equations II	Sutton-Barto 4.5-4.8	
Feb 22	Continuous-time Optimal Control	Bertsekas 3.1-3.2	
Feb 27	Linear Quadratic Control	Bertsekas 4.1	
Mar 01	Pontryagin's Maximum Principle	Bertsekas 3.3-3.4	
Mar 06	Model-free Prediction	Sutton-Barto 6-1-6.3	P4
Mar 08	Model-free Control	Sutton-Barto 6.4-6.7	
Mar 13	Value Function Approximation		
Mar 15	TBD (Exploration vs Exploitation)		

Markov Chain

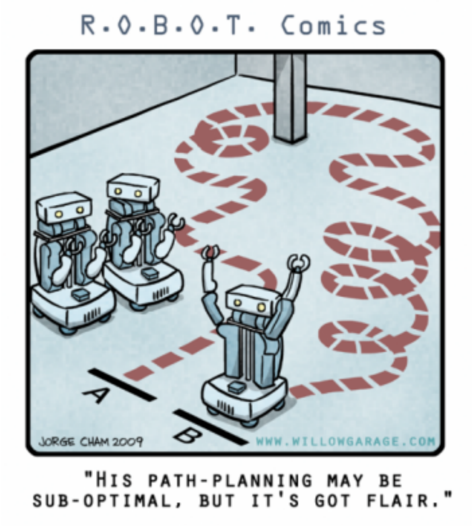
- A **Markov Chain** is a probabilistic model used to represent the evolution of a robot system
- The state $x_t \in \{1, 2, 3\}$ is fully observed (unlike HMM and Bayes filtering settings)
- The transitions are random, determined by a transition kernel but uncontrolled (just like in the HMM and Bayes filtering settings, the control input is known)
- A **Markov Decision Process** (MDP) is a Markov chain, whose transitions are controlled



$$P = \begin{bmatrix} 0.6 & 0.3 & 0 \\ 0.2 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.7 \end{bmatrix}$$

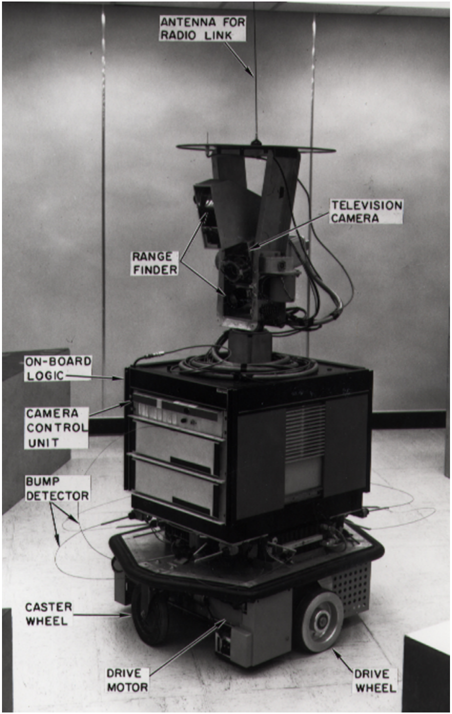
$$P_{ij} = \mathbb{P}(x_{t+1} = i \mid x_t = j)$$

Motion Planning

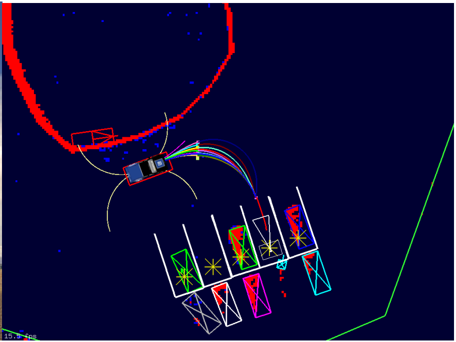


A* Search

- Invented by Hart, Nilsson and Raphael of Stanford Research Institute in 1968 for the Shakey robot
- Video: `https://youtu.be/qXdn6ynwpiI?t=3m55s`

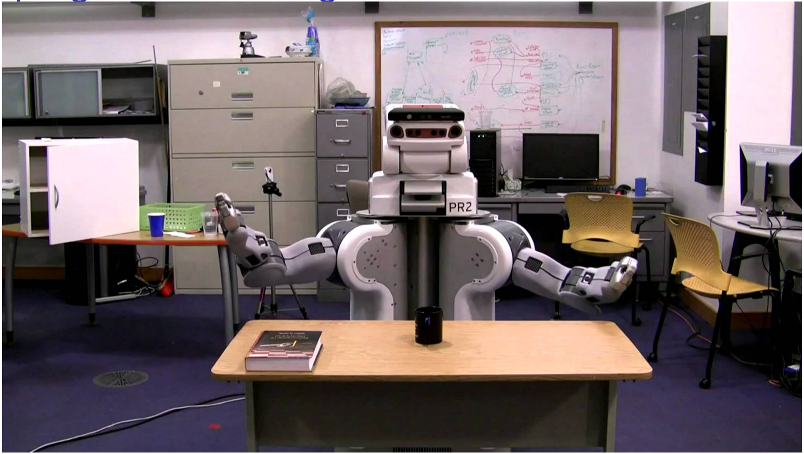


Search-based Planning



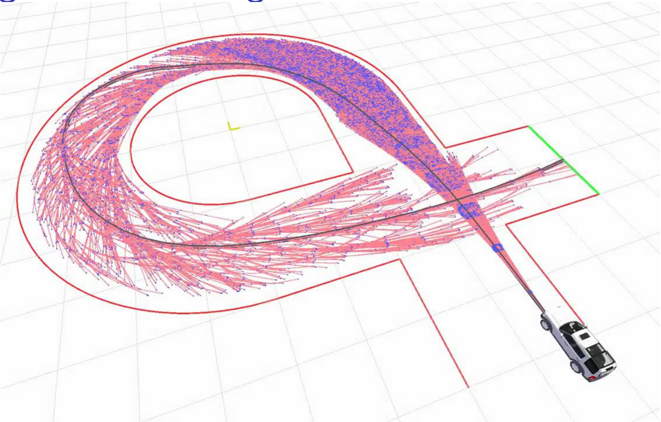
- CMU's autonomous car used search-based planning in the DARPA Urban Challenge in 2007
- Likhachev and Ferguson, "Planning Long Dynamically Feasible Maneuvers for Autonomous Vehicles," IJRR'09
- Video: `https://www.youtube.com/watch?v=4hFhl0Oi8KI`
- Video: `https://www.youtube.com/watch?v=qXZt-B7iUyw`
- Paper: `http://journals.sagepub.com/doi/pdf/10.1177/0278364909340445`

Sampling-based Planning



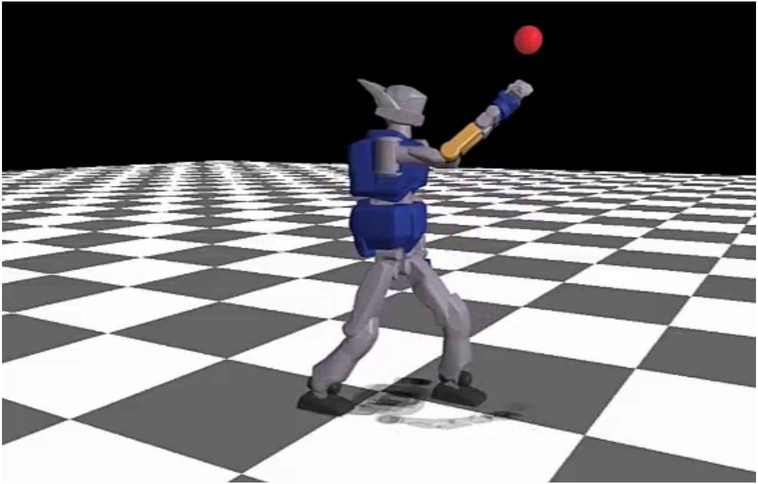
- RRT algorithm on the PR2 – planning with both arms (12 DOF)
- Karaman and Frazzoli, "Sampling-based algorithms for optimal motion planning," IJRR'11
- Video: `https://www.youtube.com/watch?v=vW74bC-Ygb4`
- Paper: `http://journals.sagepub.com/doi/pdf/10.1177/0278364911406761`

Sampling-based Planning



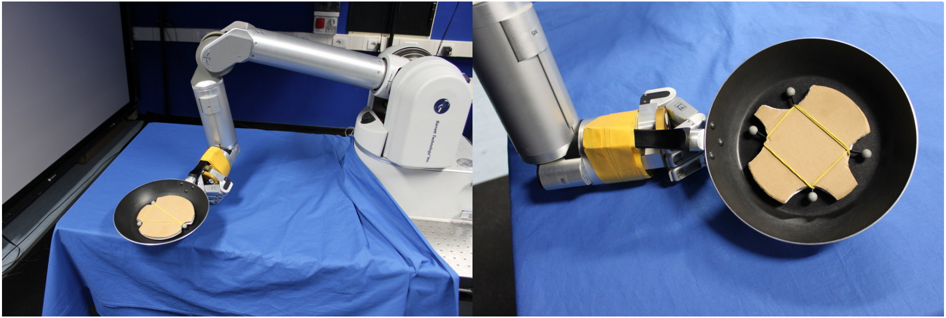
- RRT* algorithm on a race car – 270 degree turn
- Karaman and Frazzoli, "Sampling-based algorithms for optimal motion planning," IJRR'11
- Video: `https://www.youtube.com/watch?v=p3nZHnOWhrg`
- Video: `https://www.youtube.com/watch?v=LKL5qRBiJaM`
- Paper: `http://journals.sagepub.com/doi/pdf/10.1177/0278364911406761`

Dynamic Programming and Optimal Control



- Tassa, Mansard and Todorov, "Control-limited Differential Dynamic Programming," ICRA'14
- Video: `https://www.youtube.com/watch?v=tCQSSkBH2NI`
- Paper: `http://ieeexplore.ieee.org/document/6907001/`

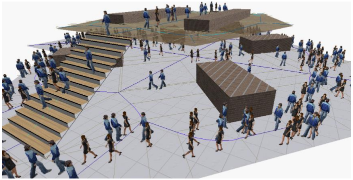
Model-free Reinforcement Learning



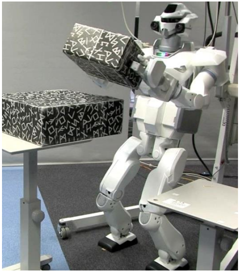
- Robot learns to flip pancakes
- Kormushev, Calinon and Caldwell, "Robot Motor Skill Coordination with EM-based Reinforcement Learning," IROS'10
- Video: `https://www.youtube.com/watch?v=W_gxLKSsSIE`
- Paper: `http://www.dx.doi.org/10.1109/IROS.2010.5649089`

Applications of Optimal Control & Reinforcement Learning

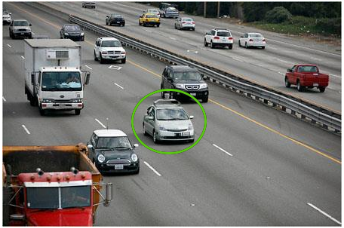
(a) Games



(b) Character Animation

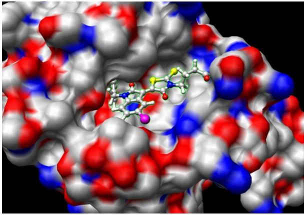


(c) Robotics



(d) Autonomous Driving

(e) Marketing



(f) Computational Biology

Problem Formulation

- **Motion model**: specifies how a dynamical system evolves

$$x_{t+1} = f(x_t, u_t, w_t) \sim p_f(\cdot \mid x_t, u_t), \quad t = 0, \ldots, T-1$$

  - discrete time $t \in \{0, \ldots, T\}$
  - state $x_t \in \mathcal{X}$
  - control $u_t \in \mathcal{U}(x_t)$ and $\mathcal{U} := \bigcup_{x \in \mathcal{X}} \mathcal{U}(x)$
  - motion noise w_t (random vector) with known probability density function (pdf) and assumed conditionally independent of other disturbances w_τ for $\tau \neq t$ for given x_t and u_t
  - the motion model is specified by the nonlinear function f or equivalently by the pdf p_f of x_{t+1} conditioned on x_t and u_t

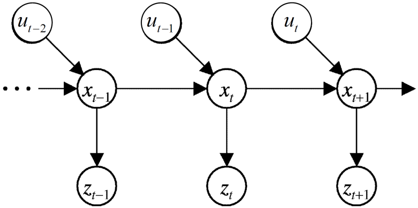
- **Observation model**: the state x_t might not be observable but perceived through measurements:

$$z_t = h(x_t, v_t) \sim p_h(\cdot \mid x_t), \quad t = 0, \ldots, T$$

  - measurement noise v_t (random vector) with known pdf and conditionally independent of other disturbances v_τ for $\tau \neq t$ for given x_t and w_t for all t
  - the observation model is specified by the nonlinear function h or equivalently by the pdf p_h of z_t conditioned on x_t

Problem Formulation

- **Markov Assumptions**
  - The state x_{t+1} only depends on the previous time input u_t and state x_t
  - The observation z_t only depends on the state x_t



- **Joint distribution**:

$$p(x_{0:T}, z_{0:T}, u_{0:T-1}) = \underbrace{p_{0|0}(x_0)}_{\text{prior}} \prod_{t=0}^{T} \underbrace{p_h(z_t \mid x_t)}_{\text{observation model}} \prod_{t=1}^{T} \underbrace{p_f(x_t \mid x_{t-1}, u_{t-1})}_{\text{motion model}}$$

- **The Problem of Acting Optimally**: Given a model p_f of the system evolution and direct observations of its state x_t (or prior pdf $p_{0|0}$ and observation model p_h) determine control inputs $u_{0:T-1}$ to minimize (maximize) a scalar-valued additive cost (reward) function:

$$J_0^{u_{0:T-1}}(x_0) := \mathbb{E}_{x_{1:T}} \left[\underbrace{g_T(x_T)}_{\text{terminal cost}} + \sum_{t=0}^{T-1} \underbrace{g(x_t, u_t)}_{\text{stage cost}} \,\middle|\, x_0, u_{0:T-1} \right]$$

Problem Solution: Control Policy

- The problem of acting optimally is called:
  - **Optimal Control** (OC): when the models p_f, p_h are known
  - **Reinforcement Learning** (RL): when the models are unknown but samples can be obtained from them
  - **Inverse RL/OC**: when the cost (reward) functions g are unknown

- The solution to an OC/RL problem is a **policy** π
  - Let $\pi_t(x_t)$ map a state $x_t \in \mathcal{X}$ to a feasible control input $u_t \in \mathcal{U}(x_t)$
  - The sequence $\pi := \{\pi_0(\cdot), \pi_1(\cdot), \ldots, \pi_{T-1}(\cdot)\} = \pi_{0:T-1}$ of functions π_t is called an **admissible control policy**
  - The cost (reward) of a policy $\pi \in \Pi$ (set of all admissible policies) is:

$$J_0^\pi(x_0) := \mathbb{E}_{x_{1:T}}\left[g_T(x_T) + \sum_{t=0}^{T-1} g(x_t, \pi_t(x_t)) \;\middle|\; x_0 \right]$$

  - a policy $\pi^* \in \Pi$ is an **optimal policy** if $J_0^{\pi^*}(x_0) \le J_0^\pi(x_0)$ for all $\pi \in \Pi$ and its cost will be denoted $J_0^*(x_0) := J_0^{\pi^*}(x_0)$

- Conventions differ in the control and machine learning communities:
  - **OC**: minimization, cost, state x, control u, policy μ
  - **RL**: maximization, reward, state s, action a, policy π
  - **ECE276B**: minimization, cost, state x, control u, policy π

Further Observations

- Goal: select controls to minimize long-term cumulative costs
  - Controls may have long-term consequences, e.g., delayed reward
  - It may be better to sacrifice immediate reward to gain long-term rewards:
    - A financial investment may take months to mature
    - Refueling a helicopter might prevent a crash in several hours
    - Blocking opponent moves might help winning chances many moves from now

- **Information state**: a sequence (history) of observations and control inputs $i_t := z_0, u_0, \ldots, z_{t-1}, u_{t-1}, z_t$ used in the partially observable setting to estimate the (pdf of the) state x_t

- A policy fully defines the behavior of the robot/agent by specifying, at any given point in time, which controls to apply. Policies can be:
  - **stationary** ($\pi \equiv \pi_0 \equiv \pi_1 \equiv \cdots$) $\subseteq$ **non-stationary** (time-dependent)
  - **deterministic** ($u_t = \pi_t(x_t)$) $\subseteq$ **stochastic** ($u_t \sim \pi_t(\cdot \mid x_t)$)
  - **open-loop** (a sequence $u_{0:T-1}$ regardless of x_t or i_t) $\subseteq$ **closed-loop** (π_t depends on x_t or i_t)

Problem Variations

- **deterministic** (no noise v_t, w_t) vs **stochastic**
- **fully observable** (no noise v_t and $z_t = x_t$) vs **partially observable**
  - **fully observable**: Markov Decision Process (MDP)
  - **partially observable**: Partially Observable Markov Decision Process (POMDP)
- **stationary** vs **nonstationary** (time-dependent $p_{f,t}$, $p_{h,t}$, g_t)
- **finite** vs **continuous** state space $\mathcal{X}$
  - tabular approach vs function approximation (linear, SVM, neural nets,...)
- **finite** vs **continuous** control space $\mathcal{U}$:
  - tabular approach vs optimization problem to select next-best control
- **discrete** vs **continuous** time:
  - finite-horizon discrete time: dynamic programming
  - infinite-horizon ($T \to \infty$) discrete time: Bellman equation (first-exit vs discounted vs average-reward)
  - continuous time: Hamilton-Jacobi-Bellman (HJB) Partial Differential Equation (PDE)
- reinforcement learning (p_f, p_h are unknown) variants:
  - **Model-based RL**: explicitly approximate models from experience and use optimal control algorithms
  - **Model-free RL**: directly learn a control policy without approximating the motion/observation models

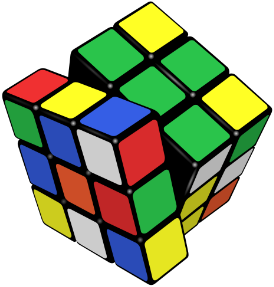
Example: Inventory Control

- Consider the problem of keeping an item stocked in a warehouse:
  - If there is too little, we will run out of it soon (not preferred).
  - If there is too much, the storage cost will be high (not preferred).

- We can model this scenario as a discrete-time system:
  - $x_t \in \mathbb{R}$: stock available in the warehouse at the beginning of the t-th time period
  - $u_t \in \mathbb{R}_{\geq 0}$: stock ordered and immediately delivered at the beginning of the t-th time period (supply)
  - w_t: (random) demand during the t-th time period with known pdf. Note that excess demand is back-logged, i.e., corresponds to negative stock x_t
  - **Motion model**: $x_{t+1} = x_t + u_t - w_t$
  - **Cost function**: $\mathbb{E}\left[R(x_T) + \sum_{t=0}^{T-1} \left(r(x_t) + cu_t - pw_t\right)\right]$ where
    - pw_t: revenue
    - cu_t: cost of items
    - $r(x_t)$: penalizes too much stock or negative stock
    - $R(x_T)$: remaining items we cannot sell or demand that we cannot meet

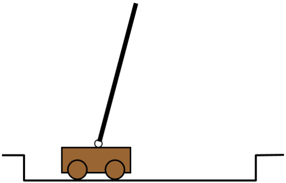
Example: Rubik's Cube

- Invented in 1974 by Ernõ Rubik
- Formalization
  - State space: $\sim 4.33 \times 10^{19}$
  - Actions: 12
  - Reward: -1 for each time step
  - Deterministic, Fully Observable
- The cube can be solved in 20 or fewer moves



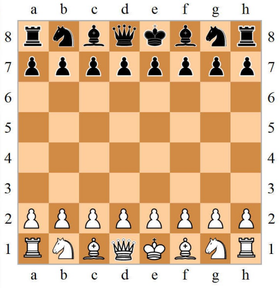
Example: Pole Balancing

- Move the cart left and right in order to keep the pole balanced
- Formalization
  - State space: 4-D continuous $(x, \dot{x}, \theta, \dot{\theta})$
  - Actions: $\{-N, N\}$
  - Reward:
    - 0 when in the goal region
    - -1 when outside the goal region
    - -100 when outside the feasible region
  - Deterministic, Fully Observable



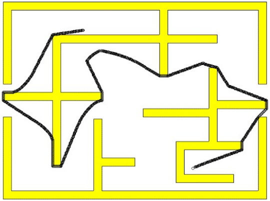
Example: Chess

- Formalization
  - State space: $\sim 10^{47}$
  - Actions: from 0 to 218
  - Reward: 0 each step, $\{-1, 0, 1\}$ at the end of the game
  - Deterministic, opponent-dependent state transitions (can be modeled as a game)
- The size of the game tree is 10^{123}



Example: Grid World Navigation

- Navigate to a goal without crashing into obstacles
- Formalization
  - State space: robot pose, e.g., 2-D position
  - Actions: allowable robot movement, e.g., $\{left, right, up, down\}$
  - Reward: -1 until the goal is reached; $-\infty$ if an obstacles is hit
  - Can be deterministic or stochastic; fully or partially observable



Definition of Markov Chain

- **Stochastic process**: an indexed collection of random variables $\{x_0, x_1, \ldots\}$ on a measurable space $(\mathcal{X}, \mathcal{F})$
  - example: time series of weekly demands for a product
- A temporally homogeneous **Markov chain** is a stochastic process $\{x_0, x_1, \ldots\}$ of $(\mathcal{X}, \mathcal{F})$-valued random variables such that:
  - $x_0 \sim p_{0|0}(\cdot)$ for a prior probability density function on $(\mathcal{X}, \mathcal{F})$
  - $\mathbb{P}(x_{t+1} \in A \mid x_{0:t}) = \mathbb{P}(x_{t+1} \in A \mid x_t) = \int_A p_f(x \mid x_t)dx$ for $A \in \mathcal{F}$ and a conditional pdf $p_f(\cdot \mid x_t)$ on $(\mathcal{X}, \mathcal{F})$
- Intuitive definition:
  - In a Markov Chain the distribution of $x_{t+1} \mid x_{0:t}$ depends only on x_t (a memoryless stochastic process)
  - The state captures all information about the history, i.e., once the state is known, the history may be thrown away
  - "The future is independent of the past given the present" (**Markov Assumption**)

Formal Definition of Markov Chain

- A measurable space $(\mathcal{X}, \mathcal{F})$ is called **nice** (or standard Borel space) if it is **isomorphic** to a compact metric space with the Borel σ-algebra (i.e., there exists a one-to-one map ϕ from $\mathcal{X}$ into $\mathbb{R}^n$ such that both ϕ and ϕ^{-1} are measurable)

- A **Markov transition kernel** is a function $\mathbb{P}_f : (\mathcal{X}, \mathcal{F}) \to [0, 1]$ on a nice space $(\mathcal{X}, \mathcal{F})$ such that:
  - $\mathbb{P}_f(x, \cdot)$ is a probability measure on $(\mathcal{X}, \mathcal{F})$ for all $x \in S$
  - $\mathbb{P}_f(\cdot, A)$ is measurable for all $A \in \mathcal{F}$

- A temporally homogeneous **Markov chain** is a sequence $\{x_0, x_1, \ldots\}$ of $(\mathcal{X}, \mathcal{F})$-valued random variables such that:
  - $x_0 \sim \mathbb{P}_{0|0}(\cdot)$ for a prior probability measure on $(\mathcal{X}, \mathcal{F})$
  - $x_{t+1} \mid x_{0:t} \sim \mathbb{P}_f(x_t, \cdot)$ for a Markov transition kernel $\mathbb{P}_f$ on $(\mathcal{X}, \mathcal{F})$, i.e., the distribution of $x_{t+1} \mid x_{0:t}$ depends only on x_t so that:

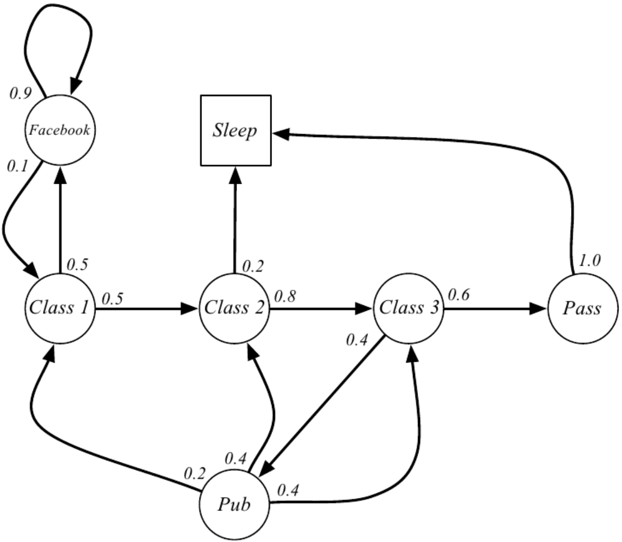
"the future is conditionally independent of the past, given the present"

Markov Chain

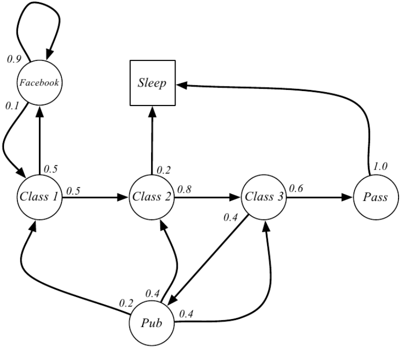
A **Markov Chain** is a stochastic process defined by a tuple $(\mathcal{X}, p_{0|0}, p_f)$:

- $\mathcal{X}$ is discrete/continuous set of states
- $p_{0|0}$ is a prior pmf/pdf defined on $\mathcal{X}$
- $p_f(\cdot \mid x_t)$ is a conditional pmf/pdf defined on $\mathcal{X}$ for given $x_t \in \mathcal{X}$ that specifies the stochastic process transitions. In the finite-dimensional case, the transition pmf is summarized by a matrix $P_{ij} := \mathbb{P}(x_{t+1} = i \mid x_t = j) = p_f(i \mid x_t = j)$

Example: Student Markov Chain



Example: Student Markov Chain



- Sample paths:
  - C1 C2 C3 Pass Sleep
  - C1 FB FB C1 C2 Sleep
  - C1 C2 C3 Pub C2 C3 Pass Sleep
  - C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 Sleep

- Transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{array}{c} FB \\ C1 \\ C2 \\ C3 \\ Pub \\ Pass \\ Sleep \end{array}$$

Chapman-Kolmogorov Equation

- n-**step transition probabilities** of a time-homogeneous Markov chain on $\mathcal{X} = \{1, \dots, N\}$

$$P_{ij}^{(n)} := \mathbb{P}(X_{t+n} = i \mid X_t = j) = \mathbb{P}(X_n = i \mid X_0 = j)$$

- **Chapman-Kolmogorov**: the n-step transition probabilities can be obtained recursively from the 1-step transition probabilities:

$$P_{ij}^{(m+n)} = \sum_{k=1}^{N} P_{ik}^{(n)} P_{kj}^{(m)}, \quad 0 \leq t \leq n$$

$$P^{(n)} = \underbrace{P \cdots P}_{n \text{ times}} = P^n$$

- Given the transition matrix P and a vector $p_{0|0}$ of prior probabilities, the vector of probabilities after t steps is:

$$\boxed{p_{t|t} = P^t p_{0|0}}$$

Example: Student Markov Chain

$$P = \begin{array}{c} \begin{bmatrix} 0.9 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} FB \\ C1 \\ C2 \\ C3 \\ Pub \\ Pass \\ Sleep \end{array}$$

$$P^2 = \begin{array}{c} \begin{bmatrix} 0.86 & 0.45 & 0 & 0 & 0.1 & 0 & 0 \\ 0.09 & 0.05 & 0 & 0.08 & 0 & 0 & 0 \\ 0.05 & 0 & 0 & 0.16 & 0.1 & 0 & 0 \\ 0 & 0.4 & 0 & 0.16 & 0.32 & 0 & 0 \\ 0 & 0 & 0.32 & 0 & 0.16 & 0 & 0 \\ 0 & 0 & 0.48 & 0 & 0.24 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0.6 & 0.08 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} FB \\ C1 \\ C2 \\ C3 \\ Pub \\ Pass \\ Sleep \end{array}$$

$$P^{100} = \begin{array}{c} \begin{bmatrix} 0.01 & 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.99 & 0.99 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} FB \\ C1 \\ C2 \\ C3 \\ Pub \\ Pass \\ Sleep \end{array}$$

First Passage Time

- **First Passage Time**: the number of transitions necessary to go from x_0 to state i for the first time (random variable $\tau_i := \inf\{t \geq 1 \mid x_t = i\}$)
- **Recurrence Time**: the first passage time to go from $x_0 = i$ to i
- **Probability of first passage in n steps**: $\rho_{ij}^{(n)} := \mathbb{P}(\tau_i = n \mid x_0 = j)$

$$
\begin{aligned}
\rho_{ij}^{(1)} &= P_{ij} \\
\rho_{ij}^{(2)} &= [P^2]_{ij} - P_{ii}\rho_{ij}^{(1)} \qquad \text{(first time we visit } i \text{ should not be 1!)} \\
&\vdots \\
\rho_{ij}^{(n)} &= [P^n]_{ij} - [P^{n-1}]_{ii}\rho_{ij}^{(1)} - [P^{n-2}]_{ii}\rho_{ij}^{(2)} - \cdots - P_{ii}\rho_{ij}^{(n-1)}
\end{aligned}
$$

- **Probability of first passage**: $\rho_{ij} := \mathbb{P}(\tau_i < \infty \mid x_0 = j) = \sum_{n=1}^{\infty} \rho_{ij}^{(n)}$
- **Number of visits to i up to time n**:

$$
v_i^{(n)} := \sum_{t=0}^{n} \mathbb{1}\{x_t = i\} \qquad v_i := \lim_{n\to\infty} v_i^{(n)}
$$

Recurrence and Transience

- **Absorbing state**: a state i such that $P_{ii} = 1$
- **Transient state**: a state i such that $\rho_{ii} < 1$
- **Recurrent state**: a state i such that $\rho_{ii} = 1$
- **Positive recurrent state**: a recurrent state i with $\mathbb{E}\left[\tau_i \mid x_0 = i\right] < \infty$
- **Null recurrent state**: a recurrent state i with $\mathbb{E}\left[\tau_i \mid x_0 = i\right] = \infty$
- **Periodic state**: can only be visited at integer multiples of t
- **Ergodic state**: a positive recurrent state that is aperiodic

Recurrence and Transience

Total Number of Visits Lemma

$\mathbb{P}(v_i \geq k+1 \mid x_0 = i) = \rho_{ii}^k$ for all $k \geq 0$

Proof: By the (strong) Markov property and induction
$(\mathbb{P}(v_i \geq k+1 \mid x_0 = i) = \rho_{ii}\mathbb{P}(v_i \geq k \mid x_0 = i))$.

$0-1$ Law for Total Number of Visits

i is recurrent iff $\mathbb{E}\left[v_i \mid x_0 = i\right] = \infty$

Proof: Since v_i is discrete, we can write $v_i = \sum_{k=0}^{\infty} \mathbb{1}\{v_i > k\}$ and

$$\mathbb{E}\left[v_i \mid x_0 = i\right] = \sum_{k=0}^{\infty} \mathbb{P}\left(v_i \geq k+1 \mid x_0 = i\right) = \sum_{k=0}^{\infty} \rho_{ii}^k = \frac{\rho_{ii}}{1-\rho_{ii}}$$

Theorem: Recurrence is contagious

i is recurrent and $\rho_{ji} > 0 \quad \Rightarrow \quad j$ is recurrent and $\rho_{ij} = 1$

Classification of Markov Chains

- **Absorbing Markov Chain**: contains at least one absorbing state that can be reached from every other state (not necessarily in one step)
- **Irreducible Markov Chain**: it is possible to go from every state to every state (not necessarily in one step)
- **Ergodic Markov Chain**: an aperiodic, irreducible and positive recurrent Markov chain
- **Stationary distribution**: a vector $w \in \{p \in [0,1]^N \mid \mathbf{1}^T p = 1\}$ such that $Pw = w$
  - Absorbing chains have stationary distributions with nonzero elements only in absorbing states
  - Ergodic chains have a unique stationary distribution (Perron-Frobenius Theorem)
  - Some periodic chains only satisfy a weaker condition, where $w_i > 0$ only for recurrent states and w_i is the frequency $\frac{v_i^{(n)}}{n+1}$ of being in state i as $n \to \infty$

Absorbing Markov Chains

- Interesting questions:
  - Q1: On average, how mant times is the process in state i?
  - Q2: What is the probability that the state will eventually be absorbed?
  - Q3: What is the expected absorption time?
  - Q4: What is the probability of being absorbed by i given that we started in j?

Absorbing Markov Chains

- **Canonical form**: reorder the states so that the transient ones come first: $P = \begin{bmatrix} Q & 0 \\ R & I \end{bmatrix}$

- One can show that $P^n = \begin{bmatrix} Q^n & 0 \\ * & I \end{bmatrix}$ and $Q^n \to 0$ as $n \to \infty$

  Proof: If i is transient, then $\rho_{ij} < \infty$ and from the 0-1 Law:

$$\infty > \mathbb{E}\left[v_i \mid x_0 = j\right] = \mathbb{E}\left[\sum_{n=0}^{\infty} \mathbb{1}\{x_n = i\} \;\middle|\; x_0 = j\right] = \sum_{n=0}^{\infty} [P^n]_{ij}$$

- **Fundamental matrix**: $Z^A = (I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n$ exists for an absorbing Markov chain
  - Expected number of times the chain is in state i: $Z^A_{ij} = \mathbb{E}\left[v_i \mid x_0 = j\right]$
  - Expected absorption time when starting from state j: $\sum_i Z^A_{ij}$
  - Let $B = RZ^A$. The probability of reaching absorbing state i starting from state j is B_{ij}

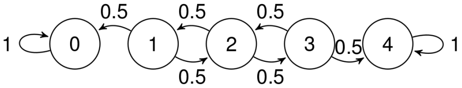
Example: Drunkard's Walk

- Transition matrix:

$$P = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

- Canonical form:

$$P = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0 & 1 \end{bmatrix}$$



- Fundamental matrix:

$$Z^A = (I - Q)^{-1} = \begin{bmatrix} 1.5 & 1 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 1 & 1.5 \end{bmatrix}$$

Perron-Frobenius Theorem

Theorem

Let P be the transition matrix of an irreducible, aperiodic, finite, time-homogeneous Markov chain with stationary distribution w. Then

- 1 is the eigenvalue of max modulus, i.e., $|\lambda| < 1$ for all other eigenvalues
- 1 is a simple eigenvalue, i.e., the associated eigenspace and left-eigenspace have dimension 1
- The left eigenvector is $\mathbf{1}^T$, the unique eigenvector w is nonnegative and

$$\lim_{n\to\infty} P^n = w\mathbf{1}^T$$

Hence, w is the unique stationary distribution for the Markov chain and any initial distribution converges to it.

Fundamental Matrix for Ergodic Chains

- We can try to get a fundamental matrix as in the absorbing case but $(I - P)^{-1}$ does not exist because $\mathbf{1}^T P = \mathbf{1}^T$ (Perron-Frobenius)
- $I + Q + Q^2 + \ldots = (I - Q)^{-1}$ converges because $Q^n \to 0$
- Try $I + (P - w\mathbf{1}^T) + (P^2 - w\mathbf{1}^T) + \ldots$ because $P^n \to w\mathbf{1}^T$ (Perron-Frobenius)
- Note that $Pw\mathbf{1}^T = w\mathbf{1}^T$ and $(w\mathbf{1}^T)^2 = w\mathbf{1}^T w\mathbf{1}^T = w\mathbf{1}^T$

$$(P - w\mathbf{1}^T)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} P^{n-i} (w\mathbf{1}^T)^i = P^n + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (w\mathbf{1}^T)^i$$

$$= P^n + \underbrace{\left[\sum_{i=1}^{n} (-1)^i \binom{n}{i} \right]}_{(1-1)^n - 1} (w\mathbf{1}^T) = P^n - w\mathbf{1}^T$$

- Thus, the following inverse exists:

$$I + \sum_{n=1}^{\infty} (P^n - w\mathbf{1}^T) = I + \sum_{n=1}^{\infty} (P - w\mathbf{1}^T)^n = (I - P + w\mathbf{1}^T)^{-1}$$

Fundamental Matrix for Ergodic Chains

- **Fundamental matrix**: $Z^E := (I - P + w\mathbf{1}^T)^{-1}$ where P is the transition matrix and w is the stationary distribution.
- **Properties**: $Z^E w = w$, $\mathbf{1}^T Z^E = \mathbf{1}^T$, and $(I - P)Z^E = I - w\mathbf{1}^T$
- **Mean first passage time**: $m_{ij} := \mathbb{E}\left[\tau_i \mid x_0 = j\right] = \dfrac{Z^E_{ii} - Z^E_{ij}}{w_i}$

Example: Land of Oz

- Transition matrix:

$$P = \begin{bmatrix} 0.5 & 0.5 & 0.25 \\ 0.25 & 0 & 0.25 \\ 0.25 & 0.5 & 0.5 \end{bmatrix}$$

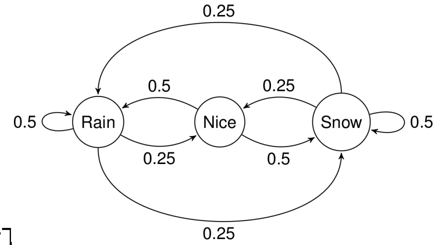
- Stationary distribution:

$$w = \begin{bmatrix} 0.4 & 0.2 & 0.4 \end{bmatrix}^T$$

- Fundamental matrix:

$$I - P + w\mathbf{1}^T = \begin{bmatrix} 0.9 & -0.1 & 0.15 \\ -0.05 & 1.2 & -0.05 \\ 0.15 & -0.1 & 0.9 \end{bmatrix}$$

$$Z^E = \begin{bmatrix} 1.147 & 0.08 & -0.187 \\ 0.04 & 0.84 & 0.04 \\ -0.187 & 0.08 & 1.147 \end{bmatrix}$$



- Mean first passage time:

$$m_{21} = \frac{Z^E_{22} - Z^E_{21}}{w_2} = \frac{0.84 - 0.04}{0.2} = 4$$