ECE276B: Planning & Learning in Robotics
Lecture 16: Linear Quadratic Control

Instructor:

Nikolay Atanasov: natanasov@ucsd.edu

Teaching Assistants:

Zhichao Li: zhl355@eng.ucsd.edu

Ehsan Zobeidi: ezobeidi@eng.ucsd.edu

Ibrahim Akbar: iakbar@eng.ucsd.edu

UC San Diego

JACOBS SCHOOL OF ENGINEERING

Electrical and Computer Engineering

Globally Optimal Closed-Loop Control

- **Deterministic finite-horizon continuous-time optimal control**:

$$\begin{aligned} \min_{\pi \in PC^0([0,T],\mathcal{U})} \quad & V^\pi(0,x_0) := \int_0^T \ell(x(t),\pi(t,x(t)))dt + \mathfrak{q}(x(T)) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t),u(t)), \;\; x(0) = x_0 \\ & x(t) \in \mathcal{X}, \; \pi(t,x(t)) \in \mathcal{U} \end{aligned}$$

- **Hamiltonian**: $H(x,u,p) := \ell(x,u) + p^T f(x,u)$

HJB PDE: Sufficient Conditions for Optimality

If $V(t,x)$ satisfies the HJB PDE:

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u \in \mathcal{U}} H(x(t),u,\nabla_x V(t,\cdot)), \;\; V(T,x) = \mathfrak{q}(x), \;\; \forall x \in \mathcal{X}, t \in [0,T]$$

then it is the optimal value function and the policy $\pi(t,x)$ that attains the minimum is an optimal policy.

Locally Optimal Open-Loop Control

- **Deterministic finite-horizon continuous-time optimal control**:

$$\begin{aligned} \min_{\pi \in PC^0([0,T],\mathcal{U})} \quad & V^\pi(0, x_0) := \int_0^T \ell(x(t), \pi(t, x(t)))dt + \mathfrak{q}(x(T)) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t)), \;\; x(0) = x_0 \\ & x(t) \in \mathcal{X}, \; \pi(t, x(t)) \in \mathcal{U} \end{aligned}$$

- **Hamiltonian**: $H(x, u, p) := \ell(x, u) + p^T f(x, u)$

PMP ODE: Necessary Conditions for Optimality

If $(x^*(t), u^*(t))$ for $t \in [0, T]$ is a trajectory from an optimal policy $\pi^*(t, x)$, then it satisfies:

$$\begin{aligned} & \dot{x}^*(t) = f(x^*(t), u^*(t)), && x^*(0) = x_0 \\ & \dot{p}^*(t) = -\nabla_x \ell(x^*(t), u^*(t)) - [\nabla_x f(x^*(t), u^*(t))]^T p^*(t), && p^*(T) = \nabla_x \mathfrak{q}(x^*(T)) \\ & u^*(t) = \arg\min_{u \in \mathcal{U}} H(x^*(t), u, p^*(t)), && \forall t \in [0, T] \\ & H(x^*(t), u^*(t), p^*(t)) = constant, && \forall t \in [0, T] \end{aligned}$$

Tractable Problems

- Consider a deterministic finite-horizon problem with dynamics and cost:

$$\dot{x} = a(x) + Bu \qquad \ell(x,u) = q(x) + \frac{1}{2}u^T R u \qquad R \succ 0$$

- **Hamiltonian**:

$$H(x,u,p) = q(x) + \frac{1}{2}u^T R u + p^T a(x) + p^T B u$$

$$\nabla_u H(x,u,p) = Ru + B^T p \qquad \nabla_u^2 H(x,u,p) = R$$

- **HJB PDE**: obtains globally optimal value and policy:

$$\pi^*(t,x) = \arg\min_{u \in \mathcal{U}} H(x,u,V_x(t,x)) = -R^{-1}B^T V_x(t,x), \qquad t \in [0,T], x \in \mathcal{X}$$

$$V(T,x) = \mathfrak{q}(x), \qquad x \in \mathcal{X}$$

$$-V_t(t,x) = q(x) + a^T V_x(t,x) - \frac{1}{2}V_x(t,x)^T B R^{-1} B^T V_x(t,x), \qquad t \in [0,T], x \in \mathcal{X}$$

- **PMP**: both necessary and sufficient for a local minimum:

$$u(t) = \arg\min_{u \in \mathcal{U}} H(x,u,p) = -R^{-1}B^T p(t), \qquad t \in [0,T]$$

$$\dot{x} = a(x) - BR^{-1}B^T p, \qquad x(0) = x_0$$

$$\dot{p} = -q_x(x)^T - a_x(x)^T p, \qquad p(T) = \nabla_x \mathfrak{q}(x(T))$$

Example: Pendulum

$$\dot{x} = \begin{bmatrix} x_2 \\ k\sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(0) = x_0$$

$$a_x(x) = \begin{bmatrix} 0 & 1 \\ k\cos(x_1) & 0 \end{bmatrix}$$

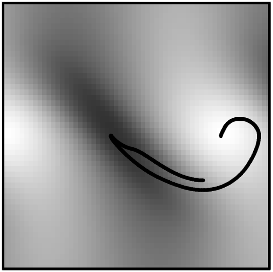
- Cost:

$$\ell(x,u) = 1 - e^{-2x_1^2} + \frac{r}{2}u^2 \text{ and } \mathfrak{q}(x) = 0$$

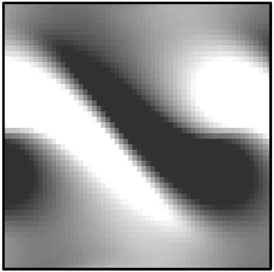
- PMP: locally optimal policy:

$$\begin{aligned} u(t) &= -r^{-1}p_2(t), & t &\in [0, T] \\ \dot{x}_1 &= x_2, & x_1(0) &= 0 \\ \dot{x}_2 &= k\sin(x_1) - r^{-1}p_2, & x_2(0) &= 0 \\ \dot{p}_1 &= -4e^{-2x_1^2}x_1 - p_2, & p_1(T) &= 0 \\ \dot{p}_2 &= -k\cos(x_1)p_1, & p_2(T) &= 0 \end{aligned}$$

- Cost-to-go and trajectories:



- Optimal policy (from HJB):



Linear Quadratic Control

- The key assumptions that allowed us to minimize the Hamiltonian analytically were:
  - The system dynamics are linear in the control u
  - The stage-cost is quadratic in the control u

- Let us study the simplest such setting in which a deterministic time-invariant linear system needs to minimize a quadratic cost over a finite horizon:

$$\min_{\pi \in PC^0([0,T],\mathbb{R}^m)} V^\pi(0, x_0) := \int_0^T \underbrace{\frac{1}{2}x(t)^T Q x(t) + \frac{1}{2}u(t)^T R u(t)}_{\ell(x(t),u(t))} dt + \underbrace{\frac{1}{2}x(T)^T Q_T x(T)}_{\mathfrak{q}(x(T))}$$

$$\text{s.t.} \quad \dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$x(t) \in \mathbb{R}^n, \; u(t) = \pi(t, x(t)) \in \mathbb{R}^m$$

where $Q = Q^T \succeq 0$, $Q_T = Q_T^T \succeq 0$, and $R = R^T \succ 0$

- This problem is called the **Linear Quadratic Regulator** (LQR)

LQR via the PMP

- Hamiltonian: $H(x,u,p) = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + p^T Ax + p^T Bu$
- Canonical equations with boundary conditions:

$$\dot{x} = \nabla_p H(x,u,p) = Ax + Bu, \qquad x(0) = x_0$$
$$\dot{p} = -\nabla_x H(x,u,p) = -Qx - A^T p, \qquad p(T) = \nabla_x \mathfrak{q}(x(T)) = Q_T x(T)$$

- Minimum principle:

$$\nabla_u H(x,u,p) = Ru + B^T p = 0 \quad \Rightarrow \quad u^*(t) = -R^{-1}B^T p(t)$$
$$\nabla_u^2 H(x,u,p) = R \succ 0 \quad \Rightarrow \quad u^*(t) \text{ is a minimum}$$

- **Hamiltonian matrix**: the canonical equations can now be simplified to a linear time-invariant (LTI) system with two-point boundary conditions:

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \quad \begin{matrix} x(0) = x_0 \\ p(T) = Q_T x(T) \end{matrix}$$

LQR via the PMP

- **Claim**: There exists a matrix $M(t) = M(t)^T \succeq 0$ such that $p(t) = M(t)x(t)$ for all $t \in [0, T]$
- We can solve the LTI system described by the Hamiltonian matrix backwards in time:

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \underbrace{e^{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}(t-T)}}_{\Phi(t,T)} \begin{bmatrix} x(T) \\ Q_T x(T) \end{bmatrix}$$

$$x(t) = (\Phi_{11}(t, T) + \Phi_{12}(t, T)Q_T)x(T)$$
$$p(t) = (\Phi_{21}(t, T) + \Phi_{22}(t, T)Q_T)x(T)$$

- It turns out that $D(t, T) := \Phi_{11}(t, T) + \Phi_{12}(t, T)Q_T$ is invertible for $t \in [0, T]$ and thus:

$$p(t) = \underbrace{(\Phi_{21}(t, T) + \Phi_{22}(t, T)Q_T)D^{-1}(t, T)}_{=:M(t)} x(t), \quad \forall t \in [0, T]$$

LQR via the PMP

- From $x(0) = D(0, T)x(T)$, we obtain an **open-loop control policy**:

$$u(t) = -R^{-1}B^T(\Phi_{21}(t, T) + \Phi_{22}(t, T)Q_T)D(0, T)^{-1}x_0$$

- From the claim that $p(t) = M(t)x(t)$, however, we can also obtain a **linear state feedback** control policy:

$$u(t) = -R^{-1}B^T M(t)x(t)$$

- We can obtain a better description of $M(t)$ by differentiating $p(t) = M(t)x(t)$ and using the canonical equations:

$$\begin{aligned}
\dot{p}(t) &= \dot{M}(t)x(t) + M(t)\dot{x}(t) \\
-Qx(t) - A^T p(t) &= \dot{M}(t)x(t) + M(t)Ax(t) - M(t)BR^{-1}B^T p(t) \\
-\dot{M}(t)x(t) &= Qx(t) + A^T M(t)x(t) + M(t)Ax(t) - M(t)BR^{-1}B^T M(t)x(t)
\end{aligned}$$

which needs to hold for all $x(t)$ and $t \in [0, T]$ and satisfy the boundary condition $p(T) = M(T)x(T) = Q_T x(T)$

LQR via the PMP (Summary)

- A unique candidate $u(t) = -R^{-1}B^T M(t)x(t)$ satsifies the necessary conditions of the PMP for optimality
- The candidate policy is linear in the state and the matrix $M(t)$ satisfies a quadratic **Riccati differential equation** (RDE):

$$-\dot{M}(t) = Q + A^T M(t) + M(t)A - M(t)BR^{-1}B^T M(t), \quad M(T) = Q_T$$

- Other tools (e.g., the HJB PDE) are needed to decide whether $u(t)$ is a globally optimal policy

LQR via the HJB PDE

- Hamiltonian: $H(x, u, p) = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + p^T Ax + p^T Bu$
- HJB PDE:

$$\pi^*(t,x) = \arg\min_{u \in \mathcal{U}} H(x, u, V_x(t,x)) = -R^{-1}B^T V_x(t,x), \qquad t \in [0,T], x \in \mathcal{X}$$
$$-V_t(t,x) = \frac{1}{2}x^T Qx + x^T A^T V_x(t,x) - \frac{1}{2}V_x(t,x)^T BR^{-1}B^T V_x(t,x), \qquad t \in [0,T], x \in \mathcal{X}$$
$$V(T,x) = \frac{1}{2}x^T Q_T x$$

- Guess a solution to the HJB PDE based on the intuition from the PMP:

$$\pi(t,x) = -R^{-1}B^T M(t)x$$
$$V(t,x) = \frac{1}{2}x^T M(t)x$$
$$V_t(t,x) = \frac{1}{2}x^T \dot{M}(t)x$$
$$V_x(t,x) = M(t)x$$

LQR via the HJB PDE

- Substituting the candidate $V(t,x)$ into the HJB PDE leads to the same **RDE** as before and we know that $M(t)$ satisfies it!

$$\frac{1}{2}x^T M(T)x = \frac{1}{2}x^T Q_T x$$
$$-\frac{1}{2}x^T \dot{M}(t)x = \frac{1}{2}x^T Qx + x^T A^T M(t)x - \frac{1}{2}x^T M(t)BR^{-1}B^T M(t)x,\ t \in [0,T], x \in \mathcal{X}$$

- **Conclusion**: Since $M(t)$ satisfies the RDE, $V(t,x) = x^T M(t)x$ is the unique solution to the HJB PDE and is the optimal value function for the linear quadratic problem with an associated optimal policy $\pi(t,x) = -R^{-1}B^T M(t)x$.

- **General Strategy for Continuous-time Optimal Control Problems**:
  1. Identify a candidate policy using the PMP
  2. Use intuition from 1. to guess a candidate value function
  3. Verify that the candidate policy and value function satisfy the HJB PDE

Continuous-time Finite-horizon LQG

- **Linear Quadratic Gaussian** (LQG) regulation problem:

$$\min_{\pi \in PC^0([0,T],\mathbb{R}^m)} V^\pi(0,x_0) := \frac{1}{2}\mathbb{E}\left\{ \int_0^T e^{-\frac{t}{\gamma}} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q & P^T \\ P & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + e^{-\frac{T}{\gamma}} x(T)^T Q_T x(T) \right\}$$

$$\begin{aligned} \text{s.t.} \quad & dx = (Ax + Bu)dt + Cd\omega, \;\; x(0) = x_0 \\ & x(t) \in \mathbb{R}^n, \; u(t) = \pi(t, x(t)) \in \mathbb{R}^m \end{aligned}$$

- **Discount factor**: $\gamma \in [0, \infty]$

- **Optimal value**: $V^*(t,x) = \frac{1}{2} x^T M(t) x + m(t)$

- **Optimal policy**: $\pi^*(t,x) = -R^{-1}(P + B^T M(t))x$

- **Riccati Equation**:

$$-\dot{M}(t) = Q + A^T M(t) + M(t)A - (P + B^T M(t))^T R^{-1}(P + B^T M(t)) - \frac{1}{\gamma}M(t), \qquad M(T) = Q_T$$

$$-\dot{m} = \frac{1}{2}\operatorname{tr}(CC^T M(t)) - \frac{1}{\gamma}m(t), \qquad m(T) = 0$$

- $M(t)$ is independent of the noise amplitude C, which implies that the optimal policy $\pi^*(t,x)$ is **the same for the stochastic (LQG) and deterministic (LQR) problems**!

Continuous-time Infinite-horizon LQG

- **Linear Quadratic Gaussian** (LQG) regulation problem:

$$\min_{\pi \in PC^0(\mathbb{R}^n, \mathbb{R}^m)} V^\pi(x_0) := \frac{1}{2}\mathbb{E}\left\{ \int_0^\infty e^{-\frac{t}{\gamma}} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q & P^T \\ P & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \right\}$$

$$\text{s.t.} \quad dx = (Ax + Bu)dt + Cd\omega, \;\; x(0) = x_0$$

$$x(t) \in \mathbb{R}^n, \; u(t) = \pi(x(t)) \in \mathbb{R}^m$$

- **Discount factor**: $\gamma \in [0, \infty)$
- **Optimal value**: $V^*(x) = \frac{1}{2}x^T Mx + m$
- **Optimal policy**: $\pi^*(x) = -R^{-1}(P + B^T M)x$
- **Riccati Equation** ('care' in Matlab):

$$\frac{1}{\gamma}M = Q + A^T M + MA - (P + B^T M)^T R^{-1}(P + B^T M)$$

$$m = \frac{\gamma}{2}\text{tr}(CC^T M)$$

- M is independent of the noise amplitude C, which implies that the optimal policy $\pi^*(x)$ is **the same for LQG and LQR**!

Discrete-time Linear Quadratic Control

Discrete-time Finite-horizon Linear Quadratic Regulator

- **Linear Quadratic Regulator** (LQR) problem:

$$\min_{\pi_{0:T-1}} \quad V_0^{\pi}(x) := \frac{1}{2}\left\{\sum_{t=0}^{T-1}\left(x_t^T Q x_t + u_t^T R u_t\right) + x_T^T Q_T x_T\right\}$$

$$\text{s.t.} \quad x_{t+1} = Ax_t + Bu_t, \;\; x_0 = x$$

$$x(t) \in \mathbb{R}^n, \; u_t = \pi_t(x_t) \in \mathbb{R}^m$$

- Since this is a discrete-time finite-horizon problem, we can use Dynamic Programming

- At $t = T$, there are no control choices and the value function is quadratic in x:

$$V_T^*(x) = \frac{1}{2}x^T M_T x := \frac{1}{2}x^T Q_T x, \qquad \forall x \in \mathbb{R}^n$$

- Iterate backwards in time $t = T-1, \ldots, 0$:

$$V_t^*(x) = \min_u \left\{\frac{1}{2}\left(x^T Q x + u^T R u\right) + V_{t+1}^*(Ax + Bu)\right\}$$

Discrete-time Finite-horizon Linear Quadratic Regulator

- At $t = T - 1$:

$$V^*_{T-1}(x) = \min_u \frac{1}{2} \left\{ x^T Q x + u^T R u + (Ax + Bu)^T M_T (Ax + Bu) \right\}$$

- $V^*_{T-1}(x)$ is a positive-definite quadratic function of u since $R \succ 0$
- Taking the gradient and setting it equal to 0:

$$\begin{aligned}
\pi^*_{T-1}(x) &= -\left(B^T Q_T B + R\right)^{-1} B^T Q_T A x \\
V^*_{T-1}(x) &= \frac{1}{2} x^T M_{T-1} x \\
M_{T-1} &= A^T M_T A + Q - A^T M_T B \left(B^T M_T B + R\right)^{-1} B^T M_T A
\end{aligned}$$

Discrete-time Finite-horizon Linear Quadratic Regulator

- At $t = T - 2$:

$$V^*_{T-2}(x) = \min_u \frac{1}{2} \left\{ x^T Q x + u^T R u + (Ax + Bu)^T M_{T-1} (Ax + Bu) \right\}$$

- $V^*_{T-2}(x)$ is a positive-definite quadratic function of u since $R \succ 0$

- Taking the gradient and setting it equal to 0:

$$\begin{aligned} \pi^*_{T-2}(x) &= -\left(B^T M_{T-1} B + R \right)^{-1} B^T M_{T-1} A x \\ V^*_{T-2}(x) &= \frac{1}{2} x^T M_{T-2} x \\ M_{T-2} &= A^T M_{T-1} A + Q - A^T M_{T-1} B \left(B^T M_{T-1} B + R \right)^{-1} B^T M_{T-1} A \end{aligned}$$

Discrete-time Finite-horizon Linear Quadratic Regulator

- **Batch Approach**: instead of using the DPA, express the system evolution as a large matrix system

$$\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_T \end{bmatrix}}_{\mathbf{s}} = \underbrace{\begin{bmatrix} I \\ A \\ \vdots \\ A^T \end{bmatrix}}_{\mathcal{A}} x_0 + \underbrace{\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A^{T-1}B & \cdots & \cdots & B \end{bmatrix}}_{\mathcal{B}} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{T-1} \end{bmatrix}}_{\mathbf{v}}$$

- Write the objective function in terms of **s** and **v**:

$$V_0^\pi(x_0) = \frac{1}{2}\left(\mathbf{s}^T \mathcal{Q} \mathbf{s} + \mathbf{v}^T \mathcal{R} \mathbf{v}\right) \qquad \begin{aligned} \mathcal{Q} &:= \mathbf{diag}(Q, \ldots, Q, Q_T) \succeq 0 \\ \mathcal{R} &:= \mathbf{diag}(R, \ldots, R) \succ 0 \end{aligned}$$

Discrete-time Finite-horizon Linear Quadratic Regulator

- Express $V_0^\pi(x_0)$ only in terms of the initial condition x_0 and the control sequence $\mathbf{v}$ by using the batch dynamics $\mathbf{s} = \mathcal{A}x_0 + \mathcal{B}\mathbf{v}$:

$$V_0^\pi(x_0) = \frac{1}{2}\left(\mathbf{v}^T\left(\mathcal{B}^T\mathcal{Q}\mathcal{B} + \mathcal{R}\right)\mathbf{v} + 2x_0^T\left(\mathcal{A}^T\mathcal{Q}\mathcal{A}\right)\mathbf{v} + x_0^T\mathcal{A}^T\mathcal{Q}\mathcal{A}x_0\right)$$

- $V_0^\pi(x_0)$ is a positive-definite quadratic function of $\mathbf{v}$ since $\mathcal{R} \succ 0$

- Taking the gradient wrt $\mathbf{v}$ and setting it equal to 0:

$$\mathbf{v}^* = -\left(\mathcal{B}^T\mathcal{Q}\mathcal{B} + \mathcal{R}\right)^{-1}\mathcal{B}^T\mathcal{Q}\mathcal{A}x_0$$

$$V_0^*(x_0) = \frac{1}{2}x_0^T\left(\mathcal{A}^T\mathcal{Q}\mathcal{A} - \mathcal{A}^T\mathcal{Q}\mathcal{B}\left(\mathcal{B}^T\mathcal{Q}\mathcal{B} + \mathcal{R}\right)^{-1}\mathcal{B}^T\mathcal{Q}\mathcal{A}\right)x_0$$

- The optimal sequence of control inputs $u_{0:T-1}^*$ is a linear function of x_0

- The optimal value function $V_0^*(x_0)$ is a quadratic function of x_0

Discrete-time Finite-horizon LQG

- **Linear Quadratic Gaussian** (LQG) regulation problem:

$$\min_{\pi_{0:T-1}} \quad V_0^{\pi}(x) := \frac{1}{2}\mathbb{E}\left\{ \sum_{t=0}^{T-1} \gamma^t \left(x_t^T Q x_t + 2u_t^T P x_t + u_t^T R u_t \right) + \gamma^T x_T^T Q_T x_T \right\}$$

$$\text{s.t.} \quad x_{t+1} = Ax_t + Bu_t + Cw_t, \;\; x_0 = x, \; w_t \sim \mathcal{N}(0, I)$$
$$x(t) \in \mathbb{R}^n, \; u_t = \pi_t(x_t) \in \mathbb{R}^m$$

- **Discount factor**: $\gamma \in [0, 1]$
- **Optimal value**: $V_t^*(x) = \frac{1}{2}x^T M_t x + m_t$
- **Optimal policy**: $\pi_t^*(x) = -(R + \gamma B^T M_{t+1} B)^{-1}(P + \gamma B^T M_{t+1} A)x$
- **Riccati Equation**:

$$M_t = Q + \gamma A^T M_{t+1} A - (P + \gamma B^T M_{t+1} A)^T (R + \gamma B^T M_{t+1} B)^{-1} (P + \gamma B^T M_{t+1} A), \quad M_T = Q_T$$
$$m_t = \gamma m_{t+1} + \gamma \frac{1}{2} \operatorname{tr}(CC^T M_{t+1}), \qquad m_T = 0$$

- M_t is independent of the noise amplitude C, which implies that the optimal policy $\pi_t^*(x)$ is **the same for LQG and LQR**!

Discrete-time Infinite-horizon LQG

- **Linear Quadratic Gaussian** (LQG) regulation problem:

$$\min_{\pi} \; V^{\pi}(x) := \frac{1}{2}\mathbb{E}\left\{\sum_{t=0}^{\infty}\gamma^t\left(x_t^T Q x_t + 2u_t^T P x_t + u_t^T R u_t\right)\right\}$$

$$\text{s.t.} \quad x_{t+1} = Ax_t + Bu_t + Cw_t, \;\; x_{t_0} = x_0, \; w_t \sim \mathcal{N}(0, I)$$

$$x(t) \in \mathbb{R}^n, \; u_t = \pi(x_t) \in \mathbb{R}^m$$

- **Discount factor**: $\gamma \in [0, 1)$
- **Optimal value**: $V^*(x) = \frac{1}{2}x^T M x + m$
- **Optimal policy**: $\pi^*(x) = -(R + \gamma B^T M B)^{-1}(P + \gamma B^T M A)x$
- **Riccati Equation** ('dare' in Matlab):

$$M = Q + \gamma A^T M A - (P + \gamma B^T M A)^T (R + \gamma B^T M B)^{-1}(P + \gamma B^T M A)$$

$$m = \frac{\gamma}{2(1-\gamma)}\operatorname{tr}(CC^T M)$$

- M is independent of the noise amplitude C, which implies that the optimal policy $\pi^*(x)$ is **the same for LQG and LQR**!

Relation between Continuous- and Discrete-time LQR

- The continuous-time system:

$$\dot{x} = Ax + Bu$$
$$\ell(x,u) = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u$$

can be discretized with time step τ:

$$x_{t+1} = (I + \tau A)x_t + \tau B u_t$$
$$\tau\ell(x,u) = \frac{\tau}{2}x^T Q x + \frac{\tau}{2}u^T R u$$

- In the limit as $\tau \to 0$, the discrete-time Riccati equation reduces to the continuous one:

$$M = \tau Q + (I + \tau A)^T M (I + \tau A)$$
$$- (I + \tau A)^T M \tau B(\tau R + \tau B^T M \tau B)^{-1} \tau B^T M (I + \tau A)$$
$$M = \tau Q + M + \tau A^T M + \tau M A - \tau M B (R + \tau B^T M B)^{-1} B^T M + o(\tau^2)$$
$$0 = Q + A^T M + M A - M B (R + \tau B^T M B)^{-1} B^T M + \frac{1}{\tau} o(\tau^2)$$

Encoding Goals as Quadratic Costs

- In the finite-horizon case, the matrices A, B, Q, R can be time-varying which is useful for specifying reference trajectories x_t^* and for approximating non-LQG problems

- The cost $\|x_t - x_t^*\|^2$ can be captured in the LQG formulation by modifying the state and cost as follows:

$$\tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}, \text{etc.}$$

$$\frac{1}{2}\tilde{x}^T \tilde{Q}_t \tilde{x} = \frac{1}{2}\tilde{x}^T (D_t^T D_t)\tilde{x} \qquad D_t \tilde{x}_t := \begin{bmatrix} I & -x_t^* \end{bmatrix} \tilde{x}_t = x_t - x_t^*$$

- If the target/goal is stationary, we can instead include it in the state $\tilde{x}$ and use $D := \begin{bmatrix} I & -I \end{bmatrix}$. This has the advantage that the resulting policy is independent of x^* and can be used for any target x^*.