ECE276B: Planning & Learning in Robotics
Lecture 11: Model-free Prediction

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From Optimal Control To Reinforcement Learning

- **Stochastic Optimal Control**: MDP with known motion model $p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u})$ and cost function $\ell(\mathbf{x}, \mathbf{u})$
  - **Model-based Prediction**: computes the value function V^π of a given policy π (policy evaluation theorem)
  - **Model-based Control**: optimizes the value function V^π to obtain an improved policy π' (policy improvement theorem)
- **Reinforcement Learning**: MDP with unknown motion model $p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u})$ and cost function $\ell(\mathbf{x}, \mathbf{u})$ but access to examples of system transitions and incurred costs
  - **Model-free Prediction**: estimates the value function V^π of a given policy π:
    - Monte-Carlo (MC) Prediction
    - Temporal-Difference (TD) Prediction
  - **Model-free Control**: optimizes the value function:
    - On-policy MC Control: ϵ-greedy
    - On-policy TD Control: SARSA
    - Off-policy MC Control: Importance Sampling
    - Off-policy TD Control: Q-Learning

Bellman Backup Operators

- Operators for policy-specific value functions:
  - **Policy Evaluation Backup Operator**:

$$\mathcal{T}_\pi[V](\mathbf{x}) := H[\mathbf{x}, \pi(\mathbf{x}), V(\cdot)] = \ell(\mathbf{x}, \pi(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x}, \pi(\mathbf{x}))} [V(\mathbf{x}')]$$

  - **Policy Q-Evaluation Backup Operator**:

$$\mathcal{T}_\pi[Q](\mathbf{x}, \mathbf{u}) := \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x}, \mathbf{u})} [Q(\mathbf{x}', \pi(\mathbf{x}'))]$$

- Operators for the optimal value function:
  - **Value Iteration Backup Operator**:

$$\mathcal{T}_*[V](\mathbf{x}) := \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} H[\mathbf{x}, \mathbf{u}, V(\cdot)] = \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x}, \mathbf{u})} [V(\mathbf{x}')] \right\}$$

  - **Q-Value Iteration Backup Operator**:

$$\mathcal{T}_*[Q](\mathbf{x}, \mathbf{u}) := \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x}, \mathbf{u})} \left[\min_{\mathbf{u}' \in \mathcal{U}(\mathbf{x}')} Q(\mathbf{x}', \mathbf{u}') \right]$$

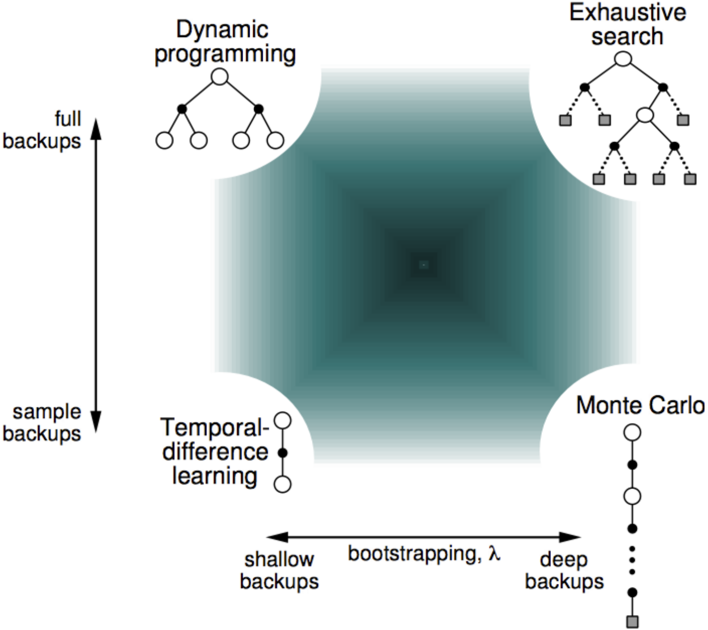
Model-free Prediction

- The main idea of model-free prediction is to approximate the Policy Evaluation backup operators $\mathcal{T}_\pi[V]$ and $\mathcal{T}_\pi[Q]$ using samples instead of computing the expectation over $\mathbf{x}'$ exactly:
  - Monte-Carlo (MC) methods:
    - The expected long-term cost can be approximated by a sample average over whole system trajectories (only applies to the First-Exit and Finite-Horizon settings)
  - Temporal-Difference (TD) methods:
    - The expected long-term cost can be approximated by a sample average over a single system transition and an estimate of the expected long-term cost at the new state (bootstrapping)

- **Sampling**: value estimates rely on samples:
  - DP does not sample
  - MC samples
  - TD samples

- **Bootstrapping**: value estimates rely on other value estimates:
  - DP bootstraps
  - MC does not bootstrap
  - TD bootstraps

Unified View of Reinforcement Learning



Monte-Carlo Policy Evaluation

- **Assumption**: MC policy evaluation applies only to the First-Exit (terminating) formulation

- **Episode**: a random sequence ρ_τ of states and controls from the start x_τ, following the system dynamics under policy π:

$$\rho_\tau := \mathbf{x}_\tau, \mathbf{u}_\tau, \mathbf{x}_{\tau+1}, \mathbf{u}_{\tau+1}, \ldots, \mathbf{x}_{T-1}, \mathbf{u}_{T-1}, \mathbf{x}_T \sim \pi$$

- **Long-term Cost**: $L_\tau(\rho_\tau) := \gamma^{T-\tau}\mathfrak{q}(\mathbf{x}_T) + \sum_{t=\tau}^{T-1} \gamma^{t-\tau}\ell(\mathbf{x}_t, \mathbf{u}_t)$

- **Goal**: approximate $V^\pi(\mathbf{x}_0)$ from several episodes $\rho_0^{(k)} \sim \pi$

- **MC Policy Evaluation**: uses the empirical mean of long-term costs obtained from different episodes $\rho_t^{(k)}$ to approximate the value of π, i.e., the expected long-term cost:

$$V^\pi(\mathbf{x}) = \mathbb{E}_{\rho\sim\pi}[L_\tau(\rho) \mid \mathbf{x}_\tau = \mathbf{x}] \approx \frac{1}{K}\sum_{k=1}^{K} L_\tau(\rho_t^{(k)})$$

First-visit Monte-Carlo Policy Evaluation

- **First-visit MC Policy Evaluation**:
  - for each state $\mathbf{x}$ and episode $\rho^{(k)}$, find the **first** time step t that state $\mathbf{x}$ is visited in $\rho^{(k)}$ and increment:
    - the number of visits to $\mathbf{x}$: $N(\mathbf{x}) \leftarrow N(\mathbf{x}) + 1$
    - the long-term cost starting from $\mathbf{x}$: $C(\mathbf{x}) \leftarrow C(\mathbf{x}) + L_t(\rho^{(k)})$
  - Approximate the value function of π: $V^{\pi}(\mathbf{x}) \approx \frac{C(\mathbf{x})}{N(\mathbf{x})}$
- **Every-visit MC Policy Evaluation**: same idea but the long-term costs are accumulated following **every** time step t that state $\mathbf{x}$ is visited in $\rho^{(k)}$

First-visit MC Policy Evaluation

Algorithm 1 First-visit MC Policy Evaluation

```
Initialize V^π(x), π(x), C(x) ← 0, N(x) ← 0
loop
    Generate ρ := x_0, u_0, x_1, u_1, ..., x_{T−1}, u_{T−1}, x_T from π
    for x ∈ ρ do
        L ← return following first appearance of x in ρ
        N(x) ← N(x) + 1
        C(x) ← C(x) + L
        V^π(x) ← C(x) / N(x)
```

- Every-visit MC would add to $C(\mathbf{x})$ not a single return L but the returns $\{L\}$ following all appearances of $\mathbf{x}$ in ρ

Running Sample Average

- Consider a sequence $x_1, x_2, \ldots,$ of samples from a random variable
- Usual way of computing the sample mean: $\mu_{k+1} = \frac{1}{k+1}\sum_{j=1}^{k+1} x_j$
- **Running sample average**:

$$\mu_{k+1} = \frac{1}{k+1}\sum_{j=1}^{k+1} x_j = \frac{1}{k+1}\left(x_{k+1} + \sum_{j=1}^{k} x_j\right) = \frac{1}{k+1}\left(x_{k+1} + k\mu_k\right)$$
$$= \mu_k + \frac{1}{k+1}(x_{k+1} - \mu_k)$$

- **Recency-weighted average**: update μ_k using a step-size $\alpha \neq \frac{1}{k+1}$:

$$\mu_{k+1} = \mu_k + \alpha(x_{k+1} - \mu_k) = (1-\alpha)^k x_1 + \sum_{j=1}^{k} \alpha(1-\alpha)^{k-j} x_{j+1}$$

- **Robbins-Monro Step Sizes**: convergence to the true mean is guaranteed almost surely under the following conditions:

$$\left(\begin{array}{c}\text{independence from}\\ \text{initial conditions}\end{array}\right) \quad \sum_{k=1}^{\infty} \alpha_k = \infty \qquad \sum_{k=1}^{\infty} \alpha_k^2 < \infty \ \text{(ensures convergence)}$$

First-visit MC Policy Evaluation

Algorithm 2 First-visit MC Policy Evaluation

1: Initialize $V^\pi(\mathbf{x})$, $\pi(\mathbf{x})$
2: **loop**
3: Generate $\rho := \mathbf{x}_0, \mathbf{u}_0, \mathbf{x}_1, \mathbf{u}_1, \dots, \mathbf{x}_{T-1}, \mathbf{u}_{T-1}, \mathbf{x}_T$ from π
4: **for** $\mathbf{x} \in \rho$ **do**
5: $L \leftarrow$ return following first appearance of $\mathbf{x}$ in ρ
6: $V^\pi(\mathbf{x}) \leftarrow V^\pi(\mathbf{x}) + \alpha(L - V^\pi(\mathbf{x}))$ ▷ usual choice: $\alpha := \frac{1}{N(\mathbf{x})+1}$

- The recency-weighted updates can be useful to track the value average in non-stationary problems (e.g., forgeting old episodes)

Temporal-Difference Policy Evaluation

- **Bootstrapping**: the value estimate of state $\mathbf{x}$ relies on the value estimate of another state

- TD combines the sampling of MC with the bootstrapping of DP:

$$
\begin{aligned}
V^{\pi}(\mathbf{x}) &= \mathbb{E}_{\rho\sim\pi}[L_{\tau}(\rho) \mid \mathbf{x}_{\tau} = \mathbf{x}] \\
&\overset{MC}{=\!=} \mathbb{E}_{\rho\sim\pi}\left[\gamma^{T-\tau}\mathfrak{q}(\mathbf{x}_T) + \sum_{t=\tau}^{T-1}\gamma^{t-\tau}\ell(\mathbf{x}_t,\mathbf{u}_t) \mid \mathbf{x}_{\tau} = \mathbf{x}\right] \\
&= \mathbb{E}_{\rho\sim\pi}\left[\ell(\mathbf{x}_{\tau},\mathbf{u}_{\tau}) + \gamma\left(\gamma^{T-\tau-1}\mathfrak{q}(\mathbf{x}_T) + \sum_{t=\tau+1}^{T-1}\gamma^{t-\tau-1}\ell(\mathbf{x}_t,\mathbf{u}_t)\right) \mid \mathbf{x}_{\tau} = \mathbf{x}\right] \\
&\underset{\text{bootstrap}}{\overset{TD(0)}{=\!=\!=\!=}} \mathbb{E}_{\rho\sim\pi}\left[\ell(\mathbf{x}_{\tau},\mathbf{u}_{\tau}) + \gamma V^{\pi}(\mathbf{x}_{\tau+1}) \mid \mathbf{x}_{\tau} = \mathbf{x}\right] \\
&\underset{\text{bootstrap}}{\overset{TD(n)}{=\!=\!=\!=}} \mathbb{E}_{\rho\sim\pi}\left[\sum_{t=\tau}^{\tau+n}\gamma^{t-\tau}\ell(\mathbf{x}_t,\mathbf{u}_t) + \gamma^{n+1}V^{\pi}(\mathbf{x}_{\tau+n+1}) \mid \mathbf{x}_{\tau} = \mathbf{x}\right]
\end{aligned}
$$

Temporal-Difference Policy Evaluation

- **Prediction**: estimate V^π from trajectory samples $\rho = \mathbf{x}_0, \mathbf{u}_0, \mathbf{x}_1, \mathbf{u}_1, \ldots, \mathbf{x}_{T-1}, \mathbf{u}_{T-1}, \mathbf{x}_T \sim \pi$
- **MC Policy Evaluation**: updates the value estimate $V^\pi(\mathbf{x}_t)$ towards the long-term cost $L_t(\rho_t)$:

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha(L_t(\rho_t) - V^\pi(\mathbf{x}_t))$$

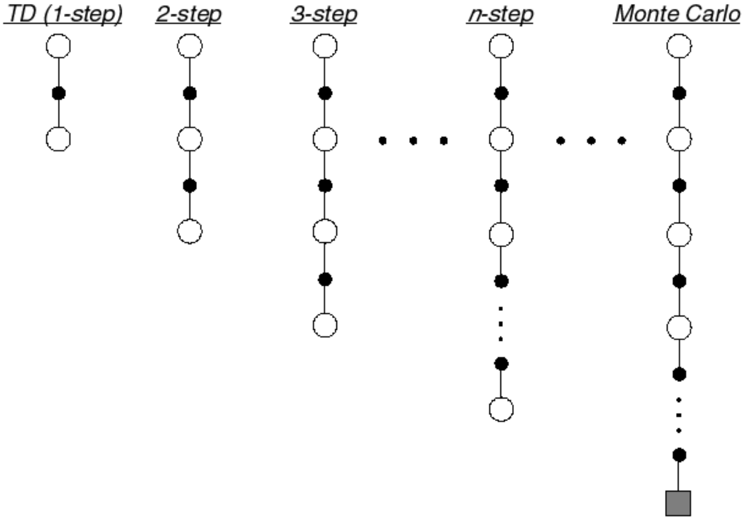
- **TD(0) Policy Evaluation**: updates the value estimate $V^\pi(\mathbf{x}_t)$ towards an *estimated* long-term cost $\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V^\pi(\mathbf{x}_{t+1})$:

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha(\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V^\pi(\mathbf{x}_{t+1}) - V^\pi(\mathbf{x}_t))$$

- **TD(n) Policy Evaluation**: updates the value estimate $V^\pi(\mathbf{x}_t)$ towards an *estimated* long-term cost $\sum_{\tau=t}^{t+n} \gamma^{\tau-t}\ell(\mathbf{x}_\tau, \mathbf{u}_\tau) + \gamma^{n+1} V^\pi(\mathbf{x}_{t+n+1})$:

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha\left(\sum_{\tau=t}^{t+n} \gamma^{\tau-t}\ell(\mathbf{x}_\tau, \mathbf{u}_\tau) + \gamma^{n+1} V^\pi(\mathbf{x}_{t+n+1}) - V^\pi(\mathbf{x}_t)\right)$$

TD(n) Prediction



MC and TD Errors

- **TD Error**: measures the difference between the estimated value $V^\pi(\mathbf{x}_t)$ and the better estimate $\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V^\pi(\mathbf{x}_{t+1})$:

$$\delta_t := \ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V^\pi(\mathbf{x}_{t+1}) - V^\pi(\mathbf{x}_t)$$

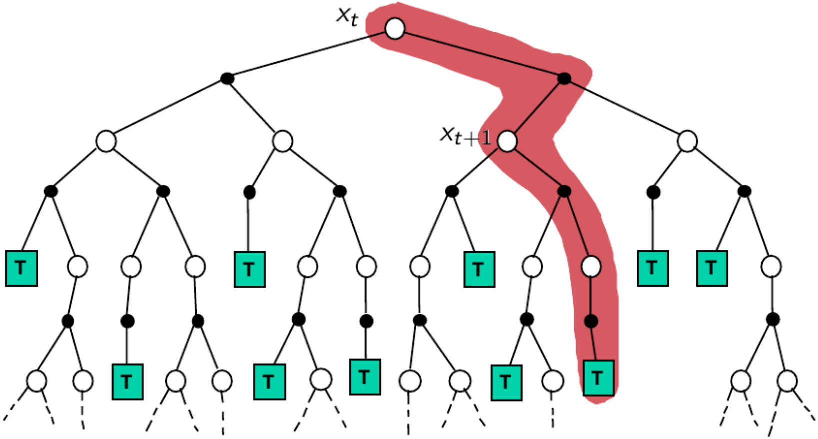
- **MC Error**: a sum of TD errors:

$$\begin{aligned} L_t(\rho_t) - V^\pi(\mathbf{x}_t) &= \ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma L_{t+1}(\rho_{t+1}) - V^\pi(\mathbf{x}_t) \\ &= \delta_t + \gamma \left(L_{t+1}(\rho_{t+1}) - V^\pi(\mathbf{x}_{t+1}) \right) \\ &= \delta_t + \gamma\delta_{t+1}\gamma^2 \left(L_{t+2}(\rho_{t+2}) - V^\pi(\mathbf{x}_{t+2}) \right) \\ &= \sum_{n=0}^{T-t-1} \gamma^n \delta_{t+n} \end{aligned}$$

- **MC and TD converge**: $V^\pi(\mathbf{x})$ approaches the true value function of π as the number of sampled episodes $\to \infty$ as long as α_k is a Robbins-Monro sequence and $\mathcal{X}$ is finite (needed for TD convergence)

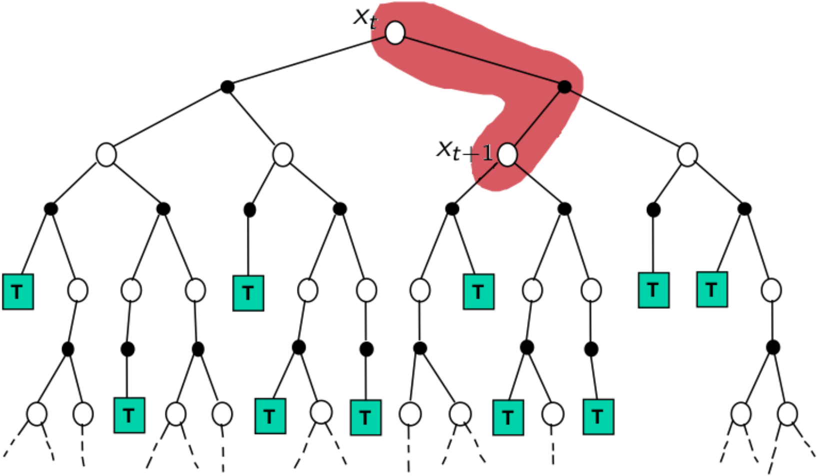
Monte-Carlo Backup

$$V^{\pi}(\mathbf{x}_t) \leftarrow V^{\pi}(\mathbf{x}_t) + \alpha(L_t(\rho_t) - V^{\pi}(\mathbf{x}_t))$$



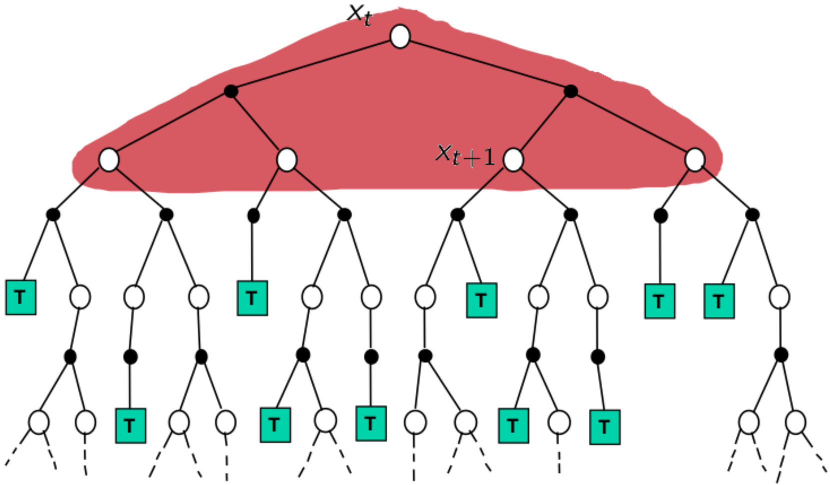
Temporal-Difference Backup

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha(\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V^\pi(\mathbf{x}_{t+1}) - V^\pi(\mathbf{x}_t))$$



Dynamic-Programming Backup

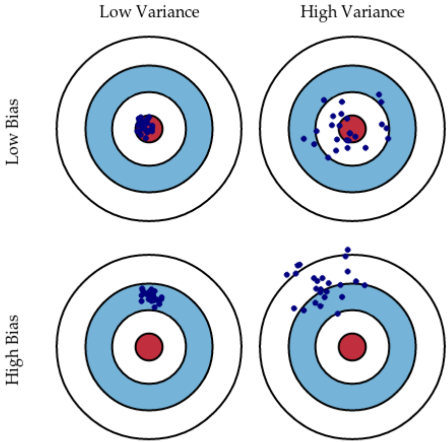
$$V^{\pi}(\mathbf{x}_t) \leftarrow \ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma \mathbb{E}_{\mathbf{x}_{t+1} \sim p_f(\cdot|\mathbf{x}_t, \mathbf{u}_t)} \left[V^{\pi}(\mathbf{x}_{t+1}) \right]$$



MC vs TD Policy Evaluation

- MC:
  - Must wait until the end of an episode before updating $V^\pi(\mathbf{x})$
  - The value estimates are **zero bias but high variance** (long-term cost depends on *many* random transitions)
  - Not very sensitive to initialization
  - Has good convergence properties even with function approximation (infinite state space)
- TD:
  - Can update $V^\pi(\mathbf{x})$ before knowing the complete episode and hence can learn online, after each transition, regardless of subsequent controls
  - The value estimates are **biased but low variance** (the TD(0) target depends on *one* random transition)
  - More sensitive to initialization than MC
  - May not converge with function approximation (infinite state space)

Bias-Variance Trade-off



Batch MC and TD Policy Evaluation

- **Batch setting**: given finite experience $\{\rho^{(k)}\}_{k=1}^{K}$
  - Accumulate value function updates according to MC or TD for $k = 1, \ldots, K$
  - Apply the update to the value function **only** after a complete pass through the data
  - Repeat until the value function estimate converges

- **Batch MC**: converges to V^π that best fits the observed costs:

$$V^\pi(\mathbf{x}) = \arg\min_V \sum_{k=1}^{K} \sum_{t=0}^{T_k} \left(L_t(\rho^{(k)}) - V \right)^2 \mathbb{1}\{\mathbf{x}_t^{(k)} = \mathbf{x}\}$$

- **Batch TD(0)**: converges to V^π of the maximum likelihood MDP model that best fits the observed data

$$\hat{p}_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) = \frac{1}{N(\mathbf{x}, \mathbf{u})} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbb{1}\{\mathbf{x}_t^{(k)} = \mathbf{x}, \mathbf{u}_t^{(k)} = \mathbf{u}, \mathbf{x}_{t+1}^{(k)} = \mathbf{x}'\}$$

$$\hat{\ell}(\mathbf{x}, \mathbf{u}) = \frac{1}{N(\mathbf{x}, \mathbf{u})} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbb{1}\{\mathbf{x}_t^{(k)} = \mathbf{x}, \mathbf{u}_t^{(k)} = \mathbf{u}\} \ell(\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)})$$

Averaging n-Step Returns

- Define the n-step return:

$$
\begin{aligned}
L_t^{(n)}(\rho) &:= \ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma \ell(\mathbf{x}_{t+1}, \mathbf{u}_{t+1}) + \ldots + \gamma^n \ell(\mathbf{x}_{t+n}, \mathbf{u}_{t+n}) + \gamma^{n+1} V^\pi(\mathbf{x}_{t+n+1}) \\
L_t^{(0)}(\rho) &= \ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V^\pi(\mathbf{x}_{t+1}) && (TD(0)) \\
L_t^{(1)}(\rho) &= \ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma \ell(\mathbf{x}_{t+1}, \mathbf{u}_{t+1}) + \gamma^2 V^\pi(\mathbf{x}_{t+2}) \\
&\vdots \\
L_t^{(\infty)}(\rho) &= \ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma \ell(\mathbf{x}_{t+1}, \mathbf{u}_{t+1}) + \ldots + \gamma^{T-t-1} \ell(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) + \gamma^{T-t} \mathfrak{q}(\mathbf{x}_T) && (MC)
\end{aligned}
$$

- **TD(n)**:

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha(L_t^{(n)}(\rho) - V^\pi(\mathbf{x}_t))$$

- **Averaged-return TD**: combines bootstrapping from several states:

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha \left(\frac{1}{2} L_t^{(2)}(\rho) + \frac{1}{2} L_t^{(4)}(\rho) - V^\pi(\mathbf{x}_t) \right)$$

- Can we combine information from all time-steps?

Forward-view $TD(\lambda)$

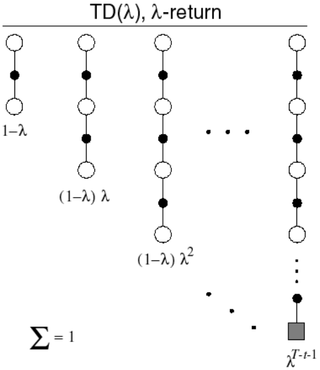
- **λ-return**: combines all n-step returns:

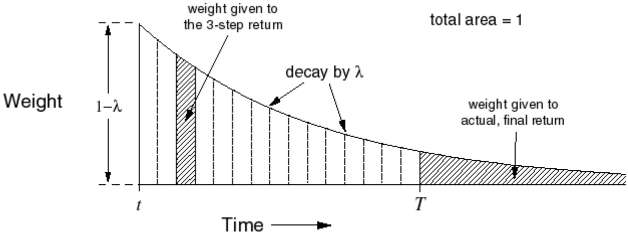
$$L_t^{\lambda}(\rho) = (1-\lambda)\sum_{n=0}^{\infty}\lambda^n L_t^{(n)}(\rho)$$

- **Forward-view** $TD(\lambda)$:

$$V^{\pi}(\mathbf{x}_t) \leftarrow V^{\pi}(\mathbf{x}_t) + \alpha\left(L_t^{\lambda}(\rho) - V^{\pi}(\mathbf{x}_t)\right)$$

- Like MC, the L_t^{λ} return can only be computed from complete episodes



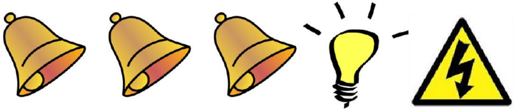


Backward-view $TD(\lambda)$

- Forward-view $TD(\lambda)$ is equivalent to $TD(0)$ for $\lambda = 0$ and to every-visit MC for $\lambda = 1$

- Backward-view $TD(\lambda)$ allows online updates from incomplete episodes

- **Credit assignment problem**: did the bell or the light cause the shock?



  - **Frequency heuristic**: assigns credit to the most frequent states
  - **Recency heuristic**: assigns credit to the most recent states
  - **Eligibility trace**: combines both heuristics

$$e_t(\mathbf{x}) = \gamma \lambda e_{t-1}(\mathbf{x}) + \mathbb{1}\{\mathbf{x} = \mathbf{x}_t\}$$

- **Backward-view** $TD(\lambda)$: updates in proportion to the **TD error** δ_t and the **eligibility trace** $e_t(\mathbf{x})$:

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha \left(\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V^\pi(\mathbf{x}_{t+1}) - V^\pi(\mathbf{x}_t) \right) e_t(\mathbf{x}_t)$$